CONSTANT NORM ALGORITHMS CLASS

Alban Goupil, Jacques Palicot[*]
FT R&D/DMR/DDH — CCETT
4 rue du Clos Courtel — BP 59
35512 Cesson-Sévigné CEDEX – France
{*alban.goupil, jacques.palicot*}*@rd.francetelecom.com*

ABSTRACT

In the context of blind equalization, a new class of Bussgang techniques called Constant Norm Algorithm (CNA), which contains the well-known CMA, is developed. From this class, two new cost functions designed for QAM modulation are derived. The first, named CQA for Constant sQuare Algorithm, is better adapted for QAM than the CMA. It results in a lower algorithm's noise without an increase of complexity. The second, is a weighting between the CMA and the CQA to get the advantages of both. The weighting coefficient is dynamically driven and justifies the name of Constant Dynamic Norm Algorithm (CDNA), which performs the same convergence speed as the CMA with a lower algorithm's noise.

1 Introduction

Equalization through filtering (fig. 1), attempts to find the source data a_n (supposedly i.i.d.), in the most efficient way according to a certain criterion (like MMSE, ZF,...), from an observation x_n, which is the result of the convolution of a_n by a finite impulse response channel H and disturbed by a white additive Gaussian noise b_n. In the framework of blind equalization, also called unsupervised or self-learning, the only available *a priori* knowledge is the statistics of the data a_n.

The resolution to this problem can be made by filtering the received data through a filter W. This filter is optimized in order to minimize a certain cost function $\mathcal{J}$. This minimization is made, for example, by a stochastic gradient algorithm (fig. 1). With a view to simplifying the notations, the time indexes will often be left out.

The aim, therefore, is to find a cost function $\mathcal{J}$ such as the perfect equalizer W_{opt} is the global minimum. We will now limit ourselves to cost functions which verify:

$$\begin{cases} W_{\text{opt}} = \underset{W}{\operatorname{argmin}}\, \mathcal{J}(z) \\ \mathcal{J}(z) = \mathbb{E}\, J(z) \end{cases} \tag{1}$$

[*]Jacques Palicot is now with IRISA – Campus universitaire de Beaulieu, Rennes Cedex, 35042 France – *jacques.palicot@irisa.fr*

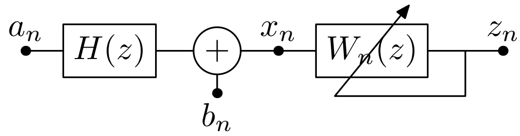

Figure 1: Blind equalization scheme

where $\mathbb{E}$ indicates expectation. From this cost function, it is possible to develop a stochastic gradient algorithm:

$$\begin{cases} W_{n+1} = W_n - \mu\phi(z)\overline{X_n}, \\ \phi(z) = \dfrac{\partial J(z)}{\partial \overline{z}}. \end{cases} \tag{2}$$

In section 2.1 we recall the cost function of the CMA. The following section presents a modification of CMA called CQA for QAM. Then, in 2.3, we develop a generalized form for these algorithms, called Constant Norm Algorithm (CNA). Finally, subsection 2.4 deals with a weighting algorithm between the CMA and the new CQA. Extensive simulations prove the efficiency of the two new schemes proposed in this paper.

2 Cost functions

2.1 Constant Modulus Algorithm

The $\text{CMA}^{p,q}$ has been developed by Godard [1] for constant modulus modulations (like the PSK). This is one of the most widely studied algorithms. The cost function can be written as

$$\mathcal{J}(z) = \frac{1}{pq}\mathbb{E}\left||z|^p - R\right|^q. \tag{3}$$

When $p = q = 2$, the algorithm takes the simple form

$$W_{n+1} = W_n - \mu\left(|z|^2 - R\right) z\,\overline{X_n}; \tag{4}$$

constant R is chosen so that the inverse of the channel is a minimum of CMA in a noiseless environment and for a doubly-infinite length equalizer. This is then found to be equal to $\mathbb{E}\,|a|^{2p} / \mathbb{E}\,|a|^p$.

The fact that this cost function, which was conceived for the PSK modulation, also works for QAM is quite surprising. However, in this case, the descent algorithm (2) generates a significant amount of noise.

2.2 Constant sQuare Algorithm

By noticing that the QAM type modulations are more "square" than "round", we can slightly modify the CMA, in order to reduce the noise from the algorithm (fig. 2). If, in place of constraining the equalizer's output to be on a circle, we actually consider a square, the residual noise would be lower, since the average distance between the symbols of the constellation and the square (ℓ_{CQA}), is shorter than that between the symbols and the circle (ℓ_{CMA}).

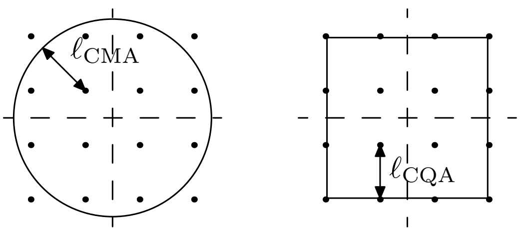

Figure 2: Principle of CMA and CQA.

This principle was already developed in the multi-modulus algorithm (MMA) [2]. But it is more a matter of the decomposition of the CMA on the in-phase and quadrature components, than the generalization of the CMA to QAM.

The modulus is a norm on the plane, from which we can derive the circle. Similarly, a simple norm exists, from which we can better derive the square corresponding to the "square" side of the QAM: infinite norm,

$$\|z\|_\infty = \max\big(|\Re z|\,, |\Im z|\big), \tag{5}$$

where $\Re z$ is the real and $\Im z$ the imaginary part of z. We can therefore write the cost function corresponding to the CQA as

$$\mathcal{J}(z) = \frac{1}{pq}\mathbb{E}\left|\|z\|_\infty^p - R\right|^q. \tag{6}$$

Of course, the constant R depends on p, q and the constellation, but it differs from that of the CMA. We will later show that, for $q = 2$, $R = \mathbb{E}\,\|a\|_\infty^{2p} / \mathbb{E}\,\|a\|_\infty^p$.

The CQA therefore attempts to correspond the equalizer's output to a square of "radius" R, and not to a circle. We also notice that the CMA only works on the basis of modulus. Yet, for a 16-QAM, there are three different modulus levels (fig. 2), while there are only two different infinite norm levels, which explains the smaller noise in the CQA, in comparison to the CMA.

The pseudo-error function, $\phi(z)$, used in the descent algorithm (2), for the CQA^{2,2} becomes

$$\begin{cases} \phi(z) = \left(\|z\|_\infty^2 - R\right)\|z\|_\infty F(z) \\ F(z) = \begin{cases} \operatorname{sgn}(\Re z) & \text{if } |\Re z| > |\Im z|, \\ i\cdot\operatorname{sgn}(\Im z) & \text{otherwise.}\end{cases}\end{cases} \tag{7}$$

2.3 Constant Norm Algorithm

The CMA and the CQA belong to an algorithm's class that we name the Constant Norm Algorithm (CNA). Indeed, if $n(\cdot)$ is a norm on $\mathbb{R}^2$, then we can write the CMA and CQA cost functions as a particular case of:

$$\mathcal{J}(z) = \frac{1}{pq}\mathbb{E}\left|n^p(z) - R\right|^q. \tag{8}$$

In order to respect (1), we can derive the constant R as presented below. Actually, R is fixed in such a way that the perfect equalizer (in the sense of the ZF criterion) is a minimum of $\mathcal{J}$ in a noiseless environment. The condition is expressed by the relation:

$$\left.\frac{\partial\mathcal{J}(\alpha a)}{\partial\alpha}\right|_{\alpha=1} = 0. \tag{9}$$

This means that for a perfect channel followed by a 1-tap equalizer, its coefficient α should converge to 1. In other words, the equalizer must at least recover the power of the signal. Now, since for $\alpha \in \mathbb{R}$, we have $n(\alpha z) = |\alpha|\,n(z)$, we find:

$$\frac{\partial\mathcal{J}}{\partial\alpha} = \mathbb{E}\left(n^p(\alpha a) - R\right)^{q-1}|\alpha|^{p-1}\operatorname{sgn}(\alpha) n^p(a). \tag{10}$$

And the necessary, but not sufficient condition becomes:

$$\left.\frac{\partial\mathcal{J}(\alpha a)}{\partial\alpha}\right|_{\alpha=1} = \mathbb{E}\left(n^p(a) - R\right)^{q-1} n^p(a) = 0. \tag{11}$$

In the particular case of the CNA^{p,2},

$$R = \frac{\mathbb{E}\,n^{2p}(a)}{\mathbb{E}\,n^p(a)}; \tag{12}$$

where we recognize the CMA^{p,2} and CQA^{p,2} constants.

2.4 Constant Dynamic Norm Algorithm

Contrary to the CMA, the phase recovery by the CQA seems to be a drawback, which makes it sensitive to a carrier residue. However, thanks to the CNA, it appears to be possible to keep the advantage of the CMA during the transient phase, and that of the CQA to achieve a better steady state. Actually, if the cost function belongs to the CNA class and if the involved norm is a weighted norm between the CMA and the CQA, we can define a new algorithm called CDNA, which offers the advantages of both the CMA and CQA. The norm used is therefore given by:

$$\|z\|_\lambda = \alpha\lambda\,\|z\|_\infty + (1-\lambda)\,|z|\,. \tag{13}$$

This norm depends on the weighting parameter λ. If $\lambda = 0$, the norm is equivalent to the modulus, and if $\lambda = 1$, the norm is equivalent to the infinite norm.

The coefficient $\alpha > 0$ is used in order to have an additional degree of freedom. It should also be noted that parameter λ, included in the norm, is also allowed to be adaptively modified by a stochastic gradient algorithm. In fact, we could write the function $\mathcal{J}$ as:

$$\mathcal{J}(z) = \frac{1}{pq}\mathbb{E}\left|\|z\|_\lambda^p - R(\lambda)\right|^q. \tag{14}$$

For $p = q = 2$, we therefore find the following updating algorithm,

$$\begin{cases} W_{n+1} = W_n - \mu_W \left(\|z\|_\lambda^2 - R(\lambda)\right) \|z\|_\lambda \\ \qquad (\alpha\lambda F(z) + (1-\lambda_n)\,\mathrm{csgn}(z))\,\overline{X_n}, \\ \lambda_{n+1} = \lambda_n - \frac{\mu_\lambda}{2}\left(\|z\|_{\lambda_n}^2 - R(\lambda_n)\right) \\ \qquad \left(2\,\|z\|_{\lambda_n}(\alpha\,\|z\|_\infty - |z|) - R'(\lambda_n)\right), \end{cases} \tag{15}$$

where $R(\lambda)$ is the rational function given by (12) and $R'(\lambda)$, its derivative.

3 Simulations

3.1 Comparison CMA/CQA

We have simulated the CMA and the CQA on the 16-QAM with an SNR set to 20 dB, and with channels taken from Proakis [3], Lee [4] and Altekar and Beaulieu [5]. The equalizers are transversal filters of 31 taps, initialized to have a single, center spike. For each channel, we carried out the simulations to have almost the same convergence speed. The comparison involves an average over 100 runs of ISI measurement defined in (16) (with c the global impulse response of the system), which is not sensitive to the phase recovery.

$$\mathrm{ISI} = \frac{\sum |c_i|^2 - \max |c_i|^2}{\max |c_i|^2}. \tag{16}$$

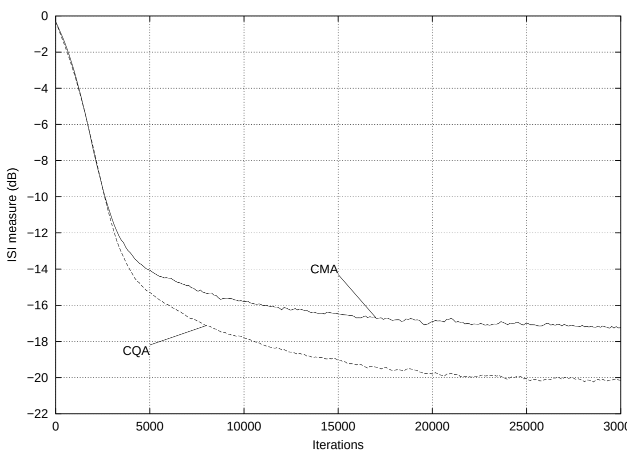

Figure 3: Proakis 1 channel.

As can be seen from fig. 3 and 4, for the first two channels, the CQA out performs the CMA in terms of ISI at steady state, for almost the same convergence rate, which proves the efficiency of the CQA against the

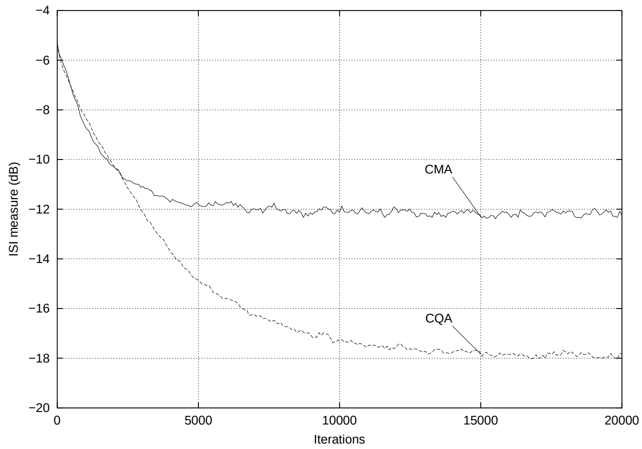

Figure 4: Lee channel.

excess MSE. The improvement is bigger for the Lee channel; this could be explained by the intrinsic phase-recovery of the CQA.

3.2 Comparison CMA/CQA/CDNA

The comparison of the CDNA with the CMA and CQA was done by setting the CDNA's parameters as follows: λ is initialized to 0 (so the CDNA starts off like a CMA); the coefficient α is set to 1.18; and the step size μ_λ is fixed at $2{\cdot}10^{-2}$. Contrary to the other simulations, this simulation was performed on the Altekar and Beaulieu channel (fig. 5) in order to compare the speed of convergence. Therefore, the step size of the algorithms was chosen to have the same steady state performance. We observe there that the CQA converges faster than the CMA. The CDNA is also as good as the CQA because it tends to choose the best algorithm between the CMA and CQA, as expected.

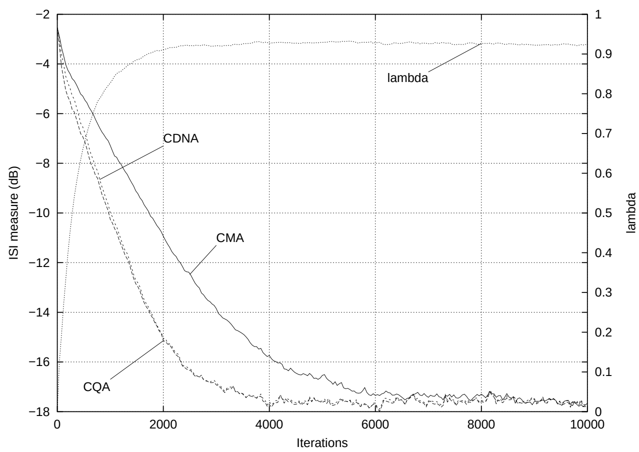

Figure 5: Altekar and Beaulieu channel.

Moreover, in order to check that the parameter λ is dynamic enough with regards to the output variation, we have added noise between the samples 10000 and 12000. In this range, the SNR goes from 20 dB to 9.6 dB.

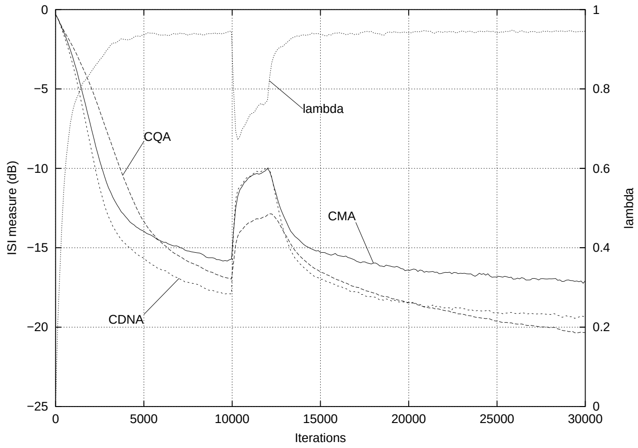

Figure 6: Proakis 1 channel.

We notice in fig. 6 that the CDNA starts like the CMA, then converges faster and finally reaches a performance equivalent to the CQA. The CDNA's steady state behavior is slightly higher than that of the CQA because of the contribution of the CMA part of the algorithm. Therefore, the CDNA follows well to the original intuitive idea. Furthermore, parameter λ that guides the CDNA dynamically reacts to the equalizer output. We notice that when the noise increases, it has a tendency to return to the CMA mode. The CDNA is overall the best performing algorithm, in the sense that it converges faster than the CMA. It is also worth noting that the CQA appears much less disturbed by jumps in the noise level than the CMA.

Another set of simulations analyzes the behavior of the different algorithms on a channel that changes the phase of the constellation. For this, we use the same parameters as before applied on the Lee channel (fig. 7).

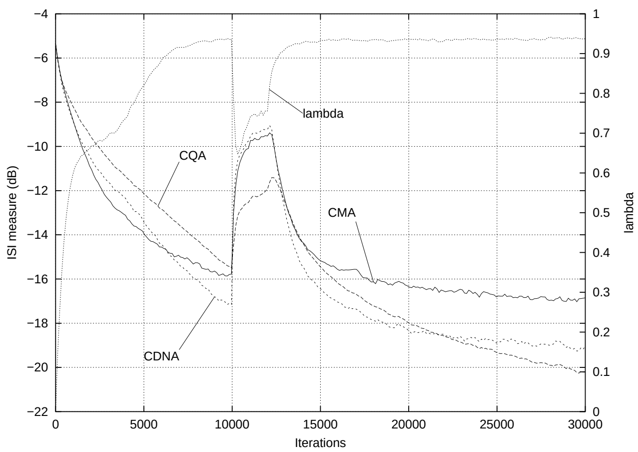

Figure 7: Lee channel.

We can see that since it is necessary to recover the phase, the CDNA doesn't immediately converge to the CQA. Once the phase is sufficiently recovered, the convergence takes place.

4 Conclusion

In this article, we have presented two new algorithms (CQA and CDNA) belonging to the same general class (CNA) as the CMA.

Simulation results clearly show that CQA generates much less algorithm noise than CMA. It is also possible to show this result analytically, by analyzing the residual MSE. The fact that the MSE excess is small allows us to increase the step size, and thus the speed of convergence.

The same holds for the CDNA, allowing us to continue the positive aspects of CMA (*i.e.*, its insensitivity to carrier residue) and CQA. This algorithm is a weighting of the other two, but remains in the CNA class. The interesting point in this weighting is that it can be dynamically carried out.

Results on CDNA and CQA suggest that for all the modulation schemes, there exists an optimized norm, which result in a better steady state performance like the CQA for the QAM, and the CMA for the PSK.

The analogy between the CMA and CQA is so strong that we could extend the CMA's derivations as the Multiple Modulus of [6] to the CQA.

References

[1] Dominique N. Godard. Self-recovering equalization and carrier tracking in two-dimensional data communication systems. *IEEE Trans. Commun.*, COM-28(11):1867–1875, November 1980.

[2] Jian Yang, Jean-Jacques Werner, and Guy A. Durmont. The multimodulus blind equalization algorithm. In *Proceedings of 13th Int. Conf. Digital Sig. Processing*, pages 127–130, Santorini, Greece, July 1997. European Space Agency.

[3] John G. Proakis. *Digital communications*. McGraw-Hill, 1989.

[4] Wonho Lee and Kyungwhoon Cheun. Convergence analysis of the stop-and-go equalization algorithm. *IEEE Trans. Commun.*, 47(2):177–180, February 1999.

[5] Shirish A. Altekar and Norman C. Beaulieu. Upper bounds to the error probability of decision feedback equalization. *IEEE Trans. Inform. Theory*, 39(1):145–156, January 1993.

[6] W. A. Sethares, G. A. Rey, and Jr C. R. Johnson. Approaches to blind equalization of signals with multiple modulus. In *Proceedings of ICASSP'89*, pages 972–975, Glasgow, Scotland, UK, May 1989. IEEE Signal Processing Society.