Polynomial Extension Method for Size-Limited Paraunitary Filter Banks *

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ABSTRACT

In this paper we investigate the polynomial extension technique, which has been used as an alternative to symmetric extension when dealing with orthogonal (non linear-phase) filter banks, since it does not introduce artificial discontinuities either; the drawback of symmetric extension is that it has been traditionally implemented as an expansionist transform. Considering a tree-structured paraunitary FIR filter bank with a minimum number of vanishing moments, we show that polynomial extension leads to a non expansionist invertible subband transform for finite signals. Hence, there is no need of extra samples of the subband signals to achieve perfect reconstruction. Additional advantages of the proposed extension method are also illustrated in our experimental results.

1 INTRODUCTION

During the last years, maximally decimated FIR filter banks have become a popular tool to perform subband signal decompositions. In this work we concentrate on tree-structured FIR paraunitary filter banks, which are commonly used to construct orthogonal wavelet packet transforms. The issue of concern here is the investigation of the properties of some extension methods for the treatment of finite signal boundaries in order to maintain perfect reconstruction (PR).

Among the classical extension techniques for size-limited filter banks [1, 2, 7, 8], the only ones that guarantee perfect reconstruction are the periodic extension and, for linear phase filters, the symmetric extension. Thus, when using FIR paraunitary filter banks, only periodic extension assures PR after the synthesis stage. The disadvantage is that it creates discontinuities in the extended signal, which can be seen as spurious high frequencies in the transform domain.

In this work, we study an alternative technique for subband processing of finite signals: the polynomial extension. First, we prove that it guarantees PR at the synthesis stage, whenever the paraunitary filter bank has a minimum number of vanishing moments. In this way, it is a non expansionist transform, since the original signal can be perfectly recovered with no need of extra coefficients in the transform domain. We also construct the family of biorthogonal boundary filters associated to the polynomial extension method, both for the analysis and for the synthesis stage. Finally, we show that the proposed technique does not introduce artificial discontinuities for polynomial signals, clearly overcoming the periodic extension.

2 PRELIMINARIES AND NOTATION

In this section we summarize the notation and recall a few results from our previous publications which are necessary to follow the development of the present work. We will consider only real valued signals and filters. Boldface lowercase letters will denote vectors and boldface uppercase ones will denote matrices. We use $\mathbf{H}_{m\times n}$ to represent an m rows n columns matrix; the Nth-order null and identity matrices are respectively denoted by $\mathbf{0}_N$ and $\mathbf{I}_N$; $r(\mathbf{A})$ is the rank of the matrix $\mathbf{A}$.

Throughout this paper we will consider a two channel paraunitary filter bank, given by the low pass filter $\mathbf{h} = [h(0), h(1), \cdots, h(L-1)]$ and its associated high pass filter $\mathbf{g} = [h(L-1), -h(L-2), \cdots, -h(0)]$, assuming that $L = 2K+2$, with K even.

Our investigation is based on the study of the transformation of the linearly extended vector $\mathbf{x}_e$, i.e., $\mathbf{y}_e = \mathbf{H}_{N\times(N+2K)}\mathbf{x}_e$, where

i) $\mathbf{x}_e = [\mathbf{x}_l^T, \mathbf{x}^T, \mathbf{x}_r^T]^T$, being $\mathbf{x}$ the original finite signal of even length $N \geq 2K$, with $\mathbf{x} = [\mathbf{x}_a^T, \cdots, \mathbf{x}_b^T]^T$; $\mathbf{x}_a$ and $\mathbf{x}_b$ contain, respectively, the K first and K last components of $\mathbf{x}$.

ii) $\mathbf{x}_l = \mathbf{C}^{l,a}\mathbf{x}_a + \mathbf{C}^{l,b}\mathbf{x}_b$ and $\mathbf{x}_r = \mathbf{C}^{r,a}\mathbf{x}_a + \mathbf{C}^{r,b}\mathbf{x}_b$, where $\mathbf{C}^{l,a}, \mathbf{C}^{l,b}$ and $\mathbf{C}^{r,a}, \mathbf{C}^{r,b}$ are, respectively, the left and right linear extension matrices.

iii) $\mathbf{H}_{N\times(N+2K)}$ is the matrix whose rows contain the even shifts of $\mathbf{h}$ and $\mathbf{g}$, and can be written as a block Toeplitz form as shown in the previous literature [6]. In the other way, $\mathbf{H}_{K\times 3K}$ can be split

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into three block-Toeplitz submatrices of order K: $\mathbf{H}_{K\times 3K} = [\mathbf{D}\ \mathbf{E}\ \mathbf{F}]$. $\mathbf{D}$ and $\mathbf{F}$ are, respectively, upper and lower block triangular matrices. Moreover, we can write [5] $\mathbf{D} = \mathbf{Q_1}\mathbf{K_D}\mathbf{P_1}$, $\mathbf{F} = \mathbf{Q_0}\mathbf{K_F}\mathbf{P_0}$ and $\mathbf{E} = \mathbf{Q_1}\mathbf{K_D}\mathbf{C}\mathbf{P_0} - \mathbf{Q_0}\mathbf{K_F}\mathbf{C}^T\mathbf{P_1}$, where $[\mathbf{Q_0}\ \ \mathbf{Q_1}]$ and $[\mathbf{P_0}^T\mathbf{P_1}^T]$ are unitary, and $\mathbf{K_F}$, $\mathbf{K_D}$ and $\mathbf{C}$ are square matrices of order $K/2$.

This transformation of $\mathbf{x}$ amounts to processing the extended signal $\mathbf{x}_e$ by means of the analysis filter bank given by $\mathbf{h}$ and $\mathbf{g}$, only retaining the N central output samples. The whole transformation of the original signal $\mathbf{x}$ can be expressed as $\mathbf{y}_e = \mathbf{G}\mathbf{x}$, where [4, 5]

$$\mathbf{G} = \begin{bmatrix} \mathbf{D}\mathbf{C}^{l,a}+\mathbf{E} & \mathbf{F}\ \mathbf{0}_{K\times(N-3K)} & \mathbf{D}\mathbf{C}^{l,b} \\ & \mathbf{H}_{(N-2K)\times N} & \\ \mathbf{F}\mathbf{C}^{r,a} & \mathbf{0}_{K\times(N-3K)}\ \mathbf{D} & \mathbf{E}+\mathbf{F}\mathbf{C}^{r,b} \end{bmatrix}.$$

The results presented in this work are based on the study of matrix $\mathbf{G}$ when the extension technique is polynomial.

3 POLYNOMIAL EXTENSION TRANSFORMATION MATRIX

In [4], we proposed the polynomial extension technique of any finite signal $\mathbf{x}$: it consists of extrapolating its first K samples ($\mathbf{x}_a$) and its last K samples ($\mathbf{x}_b$) by using polynomials of degree $< K$, so as to obtain the respective extended vectors at the left and right border $\mathbf{x}_l$ and $\mathbf{x}_r$; the extended signal $\mathbf{x}_e$ will be processed later. Figure 1 displays a finite signal $\mathbf{x}$ and its polynomial extension $\mathbf{x}_e$.

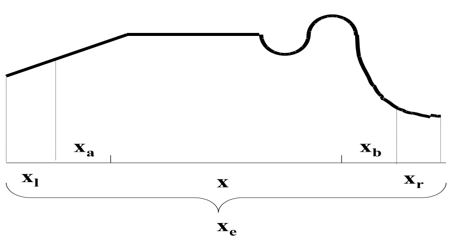

Figure 1: Polynomial extension of the finite signal $\mathbf{x}$.

Let us explain how to build the extended vector. It is known that any polynomial p of degree $< K$ verifies

$$p(n+K) = \sum_{j=1}^{K} (-1)^{(j+1)} \begin{pmatrix} K \\ j \end{pmatrix} p(n+K-j). \qquad (1)$$

for any integer n. This means that any new sample $p(n+K)$ obtained via polynomial extension at the right edge can be computed as a fixed linear combination of the K previous ones $(p(n+K-1), \ldots, p(n))$, whose coefficients are:

$$c_j = (-1)^{(j+1)} \begin{pmatrix} K \\ j \end{pmatrix} \qquad \forall j = 1, ..., K.$$

Thus, equation (1) provides a simple algorithm for obtaining the samples of the extended vector at the right edge, $\mathbf{x}_r$: from $m = 1$ to $m = K$,

$$x_r(m) = \sum_{j=1}^{m-1} c_j x_r(m-j) + \sum_{j=0}^{K-m} c_{j+m} x_b(K-j).$$

This procedure can be expressed matricially:

$$\begin{bmatrix} x_r(1) \\ \vdots \\ x_r(K) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(K-1)\times 1} & \mathbf{I}_{K-1} \\ c_K & \ldots \quad c_1 \end{bmatrix} \begin{bmatrix} x_b(K) \\ x_r(1) \\ \vdots \\ x_r(K-1) \end{bmatrix}.$$

Let $\mathbf{M}$ denote this Frobenius matrix; by iteration over the vector at the right hand side, we obtain $\mathbf{x}_r = \mathbf{M}^K\mathbf{x}_b$, so $\mathbf{V} = \mathbf{M}^K$ is the right extension matrix $\mathbf{C}^{r,b}$ associated to the proposed polynomial extension, and $\mathbf{C}^{r,a}$ is null. Analogously, we build the extended vector $\mathbf{x}_l$ by polynomial extension on the left edge: from the expression (1), we also obtain $p(n) = \sum_{m=1}^{K} c_m p(n+m)$; this means that the samples at the left hand edge can also be computed iteratively from the subsequent samples:

$$x_l(m) = \sum_{j=1}^{K-m} c_j x_l(m+j) + \sum_{j=1}^{m} c_{j+K-m} x_a(j),$$

starting from $m = K$ and decreasing up to $m = 1$. Notice that the outermost components of the appended segments $\mathbf{x}_l$, $\mathbf{x}_r$ are always computed by means of the innermost ones. Matricially,

$$\begin{bmatrix} x_l(1) \\ \vdots \\ x_l(K) \end{bmatrix} = \begin{bmatrix} c_1 \ldots & c_K \\ \mathbf{I}_{K-1} & \mathbf{0}_{(K-1)\times 1} \end{bmatrix} \begin{bmatrix} x_l(2) \\ \vdots \\ x_l(K) \\ x_a(1) \end{bmatrix}.$$

If $\mathbf{M}'$ denote this Frobenius matrix, by iteration over the vector at the right hand side we obtain $\mathbf{x}_l = \mathbf{M}'^K\mathbf{x}_a$, so $\mathbf{W} = \mathbf{M}'^K$ is the left extension matrix $\mathbf{C}^{l,a}$ associated to polynomial extension, and $\mathbf{C}^{l,b}$ is null.
We conclude that the subband transform matrix associated to the polynomial extension is

$$\mathbf{G}_{pol} = \begin{bmatrix} \mathbf{E}+\mathbf{D}\mathbf{W} \quad \mathbf{F} & & \mathbf{0}_{K\times(N-2K)} \\ & \mathbf{H}_{(N-2K)\times N} & \\ \mathbf{0}_{K\times(N-2K)} & & \mathbf{D} \quad \mathbf{E}+\mathbf{F}\mathbf{V} \end{bmatrix}.$$

4 INVERTIBILITY OF THE POLYNOMIAL EXTENSION SUBBAND TRANSFORM

In this section we prove that the polynomial extension method is non expansionist; in other words, the associate matrix $\mathbf{G}_{pol}$ is nonsingular. In our previous work [4] we provided some invertibility criteria for the family of matrices $\mathbf{G}$; in order to show that $\mathbf{G}_{pol}$ is invertible, we first present a new characterization:

Proposition 1: *If $\mathbf{C}^{l,b}$ and $\mathbf{C}^{r,a}$ are null, then $\mathbf{G}$ is invertible if and only if*

$$r((\mathbf{E}+\mathbf{D}\mathbf{C}^{l,a})\mathbf{P_0}^T) = r((\mathbf{E}+\mathbf{F}\mathbf{C}^{r,b})\mathbf{P_1}^T) = K/2.$$

Proof: It is known [4] that $\mathbf{G}$ is invertible if and only if the submatrix containing its K first and K last columns has maximum rank $2K$. When $\mathbf{C}^{l,b}$ and $\mathbf{C}^{r,a}$ are null matrices, it is equivalent to

$$r\left(\begin{bmatrix} \mathbf{E}+\mathbf{D}\mathbf{C}^{l,a} \\ \mathbf{D} \end{bmatrix}\right) = K = r\left(\begin{bmatrix} \mathbf{F} \\ \mathbf{E}+\mathbf{F}\mathbf{C}^{r,b} \end{bmatrix}\right).$$

This means that the null spaces of both matrices are zero. The null space of the first matrix is the set of vectors $\mathbf{v}$ such that $(\mathbf{E}+\mathbf{D}\mathbf{C}^{l,a})\mathbf{v} = \mathbf{D}\mathbf{v} = \mathbf{0}_{K\times 1}$; as $\mathbf{D}\mathbf{P_0}^T$ is null [5], they can be written as $\mathbf{v} = \mathbf{P_0}^T\mathbf{x}$, with $(\mathbf{E}+\mathbf{D}\mathbf{C}^{l,a})\mathbf{P_0}^T\mathbf{x} = \mathbf{0}_{K\times 1}$. This subspace is zero if and only if the $K/2$ columns of $(\mathbf{E}+\mathbf{D}\mathbf{C}^{l,a})\mathbf{P_0}^T$ are linearly independent, or, equivalently, $r((\mathbf{E}+\mathbf{D}\mathbf{C}^{l,a})\mathbf{P_0}^T) = K/2$. The second identity is derived analogously.

We will use the previous proposition in order to prove our main result, whose demonstration can be found in the Appendix:

Theorem 1: *If $\mathbf{g}$ is a high pass filter of length L of at least $K = L/2-1$ vanishing moments, then the polynomial extension yields an invertible transform $\mathbf{G}_{pol}$.*

Remark: As a consequence, polynomial extension yields a family of $2K$ border filters: in fact, the K first and K last rows of $\mathbf{G}_{pol}$ contain the respective left and right analysis biorthogonal boundary filters. In [3], a similar set of $4K$ biorthogonal border filters was given, but we have just proved that, by using only $2K$ filters, PR is also achieved.

5 EXPERIMENTAL RESULTS

Length L Daubechies filters have exactly $L/2 = K+1 > K$ vanishing moments, so Theorem 1 assures that polynomial extension always satisfies PR property when using such prototype filters. The associate transform matrix $\mathbf{G}_{pol}$ is invertible; the corresponding left and right analysis boundary filters are alternately low pass and high pass, with lengths $K+2, K+4, \ldots, L-2$. When using Daubechies filters of length $L = 10$ and polynomial extension, we obtain $K = 4$ boundary filters per border. Table 1 and Table 2 show the coefficients of the four left and right border filters, respectively.

For the synthesis stage, it is clear that the first K and last K rows of the inverse of $\mathbf{G}_{pol}$ provide the set of biorthogonal synthesis boundary filters. Tables 3 and 4 display the family of the four left and right synthesis boundary filters corresponding to the ones given in Tables 1 and 2.

We have also studied the behaviour of the transform vector. Considering an original signal which corresponds to a polynomial of degree 7, and Daubechies filters of length 10, in Figure 2 we compare its transform vectors obtained by using the classical periodic extension and using polynomial extension. We observe that periodization introduces discontinuities at the edges, which produce artificial high frequencies in the transform domain, while the performance of the polynomial extension is clearly better. Symmetric extension would not introduce such discontinuities either, but it is an expansionist method when using paraunitary (non linear-phase) filters; for this reason, we only compare the (invertible) polynomial extension to the (invertible) periodic extension. Finally, notice that, although Daubechies filters of length 10 have only 5 vanishing moments, polynomial extension overcomes the periodization technique, even for polynomial test signals of greater degree.

TABLE 1: Left analysis boundary filters

$l1$	25.234	-55.969	44.44	-12.282	-.012	.003		
$l2$	.036	.013	-.42	.814	-.603	.16		
$l3$	4.74	-6.686	4.574	-1.276	.077	-.006	-.012	.003
$l4$	-.077	.219	-.068	-.216	-.138	.724	-.603	.16

TABLE 2: Right analysis boundary filters

$r1$	.16	.603	.724	.138	-.243	-.032	.086	-.023
$r2$	-.003	-.012	.006	.077	-.004	-.256	.282	-.09
$r3$	0	0	.006	.077	-.004	-.256	.282	.08
$r4$	0	0	-.003	-.012	.083	-.142	.1	-.025

TABLE 3: Left synthesis boundary filters

$\tilde{l}1$	-.057	-.088	.51	.038	.281	-.009	-.027	.0006
$\tilde{l}2$	-.023	-.266	.109	.084	.628	-.014	-.007	.0001
$\tilde{l}3$	.027	-.089	-.166	.018	.674	.009	.172	-.003
$\tilde{l}4$	.007	.711	-.052	-.238	.151	.076	.6	-.012

TABLE 4: Right synthesis boundary filters

$\tilde{r}1$	-.2	-8.3	.7	-18.5	-1.2	-4.1	1.4	-94.8
$\tilde{r}2$	-.6	-28.8	2.4	-68.7	-3.7	-15.9	2.6	-357.7
$\tilde{r}3$	-1.4	-65.5	5.3	-165.2	-7.5	-53.8	4.3	-812.7
$\tilde{r}4$	-2.6	-122.6	9.6	-322.4	-12.9	-134.7	6.6	-1528.8

6 CONCLUSIONS

In this paper we have proposed a non expansionist polynomial extension method as a technique for processing finite length signals via paraunitary FIR filter banks. We have first presented a new and general criterium for the invertibility of linear extension methods. It has been used to prove that polynomial extension transformation guarantees perfect reconstruction of the original finite signal at the synthesis stage, when using filter banks with a minimum number of vanishing moments. As a consequence, polynomial extension yields a new set of biorthogonal boundary filters. Experimental results show that this technique outperforms the classical extension methods, as it avoids artificial discontinuities and achieves PR with paraunitary filter banks.

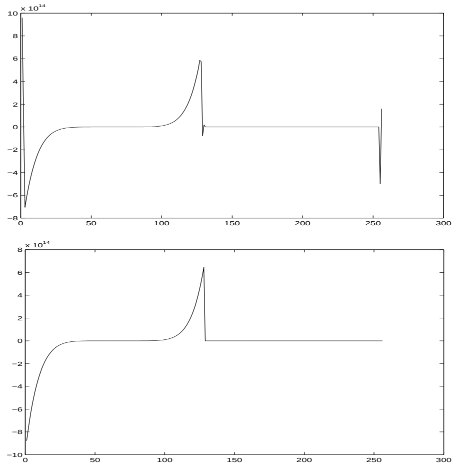

Figure 2: Transform vector of a polynomial test signal of degree 7 via Daubechies filters of length 10, by using: periodization (top) and polynomial extension (bottom).

APPENDIX: PROOF OF THEOREM 1.

Proposition 1 guarantees that it is necessary and sufficient to check that the null space of the matrices $(\mathbf{E}+\mathbf{DW})\mathbf{P_0}^T$ and $(\mathbf{E}+\mathbf{FV})\mathbf{P_1}^T$ is zero. Let us consider any vector $\mathbf{x} \in \mathbb{R}^{K/2}$ such that $(\mathbf{E}+\mathbf{DW})\mathbf{P_0}^T\mathbf{x} = \mathbf{0}_{K\times 1}$; our aim is to show that $\mathbf{x}$ is zero.

Considering the K samples of the signal $\mathbf{P_0}^T\mathbf{x}$, we extrapolate this vector by appending $2K$ samples per border in a polynomial way, so as to form $\mathbf{x}_{pol}$. This way, $\mathbf{x}_{pol}$ is a discrete polynomial of degree $< K$; we process it by means of the 2-channel cell, and obtain:

$$\begin{aligned}
\mathbf{z} &= \mathbf{H}_{3K\times 5K}\mathbf{x}_{pol} = \\
&= \begin{bmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} & \mathbf{0}_K & \mathbf{0}_K \\ \mathbf{0}_K & \mathbf{D} & \mathbf{E} & \mathbf{F} & \mathbf{0}_K \\ \mathbf{0}_K & \mathbf{0}_K & \mathbf{D} & \mathbf{E} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{W^2P_0}^T\mathbf{x} \\ \mathbf{WP_0}^T\mathbf{x} \\ \mathbf{P_0}^T\mathbf{x} \\ \mathbf{VP_0}^T\mathbf{x} \\ \mathbf{V^2P_0}^T\mathbf{x} \end{bmatrix} = \\
&= \begin{bmatrix} (\mathbf{DW}^2+\mathbf{EW}+\mathbf{F})\mathbf{P_0}^T\mathbf{x} \\ \mathbf{FVP_0}^T\mathbf{x} \\ (\mathbf{E}+\mathbf{FV})\mathbf{VP_0}^T\mathbf{x} \end{bmatrix}.
\end{aligned}$$

We have applied the fact that $\mathbf{DP_0}^T = \mathbf{0}_{K\times K/2}$ and the assumption $(\mathbf{DW}+\mathbf{E})\mathbf{P_0}^T\mathbf{x} = \mathbf{0}_{K\times 1}$. As $\mathbf{x}_{pol}$ is a polynomial sequence of degree $< K$, the hypothesis about the K vanishing moments guarantees that the even components of $\mathbf{z}$ (say, the high pass coefficients of $\mathbf{x}_{pol}$) are zero. In particular, the $K/2$ central even components of $\mathbf{z}$ are the even elements of $\mathbf{FVP_0}^T\mathbf{x}$, which must be zero. But the even rows of $\mathbf{F}$ are linearly independent and generate the rest of the rows, so the whole vector $\mathbf{FVP_0}^T\mathbf{x}$ must be zero; moreover, this means that we can write $\mathbf{VP_0}^T\mathbf{x} = \mathbf{P_1}^T\mathbf{y}$ for some vector $\mathbf{y}$. Then, the last K components of $\mathbf{z}$ are $(\mathbf{FV}+\mathbf{E})\mathbf{P_1}^T\mathbf{y} = \mathbf{F}(\mathbf{VP_1}^T - \mathbf{P_0}^T\mathbf{C}^T)\mathbf{y}$, where we have used that $\mathbf{EP_1}^T = -\mathbf{Q_0K_FC}^T = \mathbf{FP_0}^T\mathbf{C}^T$, which is a consequence of the preliminaries given in Section 2, iii). Thus, we have another vector such that $\mathbf{F}(\cdot)$ has null even components, so it must be identically zero for the same reason. We summarize that

$$\mathbf{z}^T = [z_1, 0, z_3, 0, \ldots, z_{K/2-1}, 0, \mathbf{0}_{1\times 2K}].$$

In other words, not only the $3K/2$ high pass coefficients of the polynomial $\mathbf{x}_{pol}$ are zero, but also its last K low pass coefficients. Recall that the $3K/2$ low pass components form another polynomial of degree $< K$; but it has more roots than its degree, so it must be the null polynomial. We conclude that $\mathbf{z} = \mathbf{0}_{3K\times 1}$. Finally, by computing $\mathbf{P_0}^T\mathbf{x} = [\mathbf{F}^T\ \mathbf{E}^T\ \mathbf{D}^T\ ]\mathbf{z} = \mathbf{0}_{K\times 1}$, we derive that $\mathbf{x}$ must be null. Analogously, we would show that $r((\mathbf{E}+\mathbf{FV})\mathbf{P_1}^T) = K/2$, finishing the proof.

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