UNIFIED FORMULATION OF CLOSED-FORM ESTIMATORS FOR BLIND SOURCE SEPARATION IN COMPLEX INSTANTANEOUS LINEAR MIXTURES

Vicente Zarzoso and Asoke K. Nandi

Signal Processing and Communications Group, Department of Electrical Engineering and Electronics, The University of Liverpool, Brownlow Hill, Liverpool L69 3GJ, England, UK
Tel/Fax: +44 151 794 4525/4540, `e-mail: {vicente, aknandi}@liv.ac.uk`
`web: http://www.liv.ac.uk/~{vicente, aknandi}`

ABSTRACT

The blind separation of unknown independent source signals from sensor observations is addressed in environments of instantaneous linear mixtures. After prewhitening the sensor data, the source extraction reduces to the identification of certain parameters defining a unitary transformation. This contribution develops the algebraic devices which allow us: 1) to provide a unified formulation of the closed-form estimators of the separation parameters in the complex-mixture scenario — typical of multi-user digital communications —, and 2) to disclose the remarkable parallelism existing between the real and the complex problem in the context of their analytic solutions.

1 INTRODUCTION

Owing to the vast number of application areas it encompasses, blind source separation (BSS) has aroused an increasing research interest over the last decade. The BSS problem consists of recovering a set of unknown source signals from sensor observations generated as mixtures of the sources. Although several types of mixture exist, we address the case where each sensor output can be regarded as an unknown linear combination of the source signals. Mathematically, vector $\mathbf{x} \in \mathbb{C}^q$ representing the q sources and vector $\mathbf{y} \in \mathbb{C}^p$ containing the p sensor observations are linearly related through an unknown transformation $M \in \mathbb{C}^{p \times q}$:

$$\mathbf{y} = M\mathbf{x}. \tag{1}$$

The goal of BSS is to estimate the source vector $\mathbf{x}$ and the mixing matrix M from the exclusive knowledge of the observation vector $\mathbf{y}$. It is only assumed that the sources are statistically independent and the mixing matrix is full column rank.

The convenience of a two-step approach, comprising an initial stage based on second-order statistics (SOS) followed by another based on higher-order statistics (HOS), has been endorsed in a great number of works [2–6, 9]. The SOS-step (so-called pre-whitening), yields a set of normalized uncorrelated components $\mathbf{z} \in \mathbb{C}^q$, which are related to the sources through a unitary transformation $Q \in \mathbb{C}^{q \times q}$:

$$\mathbf{z} = Q\mathbf{x}. \tag{2}$$

In the two-signal case, matrix Q becomes an elementary complex Givens rotation, exhibiting the general form:

$$Q = \mathcal{Q}(\theta, \alpha), \tag{3}$$

with $\mathcal{Q}(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}^{2 \times 2}$ such that

$$\mathcal{Q}(\theta, \alpha) \triangleq \begin{bmatrix} \cos\theta & -\mathrm{e}^{-j\alpha}\sin\theta \\ \mathrm{e}^{j\alpha}\sin\theta & \cos\theta \end{bmatrix}. \tag{4}$$

The mixture identification is hence reduced to the estimation of parameters θ and α, which requires the use of HOS in the i.i.d. case [8]. A wide variety of techniques have been suggested for this second stage of the separation, but we concentrate on the methods whereby these parameters are estimated in analytic or closed-form, as opposed to procedures requiring iterative optimization or other computationally demanding mechanisms.

In the real case, $\alpha = 0$, and only parameter θ is relevant to the source extraction. It was first proven in [3] that the analytic estimation of θ is indeed feasible, and an estimator was proposed (Comon's formula, CF). The maximum-likelihood (ML) approach together with some simplifications produced the approximate ML (AML) estimator [5], which — via the extended ML (EML) estimator — was later generalized to practically any source distribution [7, 9]. Another closed-form expression was unveiled in [4] departing from the optimization of a certain contrast function. A unifying common framework for all these closed-form methods in the real case was accomplished in [10, 12].

Herein, we are interested in solving the complex problem, characterized by eqns. (2)–(4), which arises in as important applications as digital communications [4] and seismic exploration [6]. Our primary objective is to extend the common framework of [10, 12] to the complex-mixture scenario. With this aim in mind, we

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define a new number class — so-called *bicomplex numbers* — in compliance with the structure and properties of the unitary matrix to be identified. This allows straightforward generalizations of results previously encountered in the real case, evidencing a beautiful connection in the formulations of the real- and the complex-mixture problems.

2 PRELIMINARIES

2.1 Complex-Variate Statistics

The real-mixture methods cited in the introduction rely on the higher-order cumulants of the data, and so do the complex versions considered in the sequel. Among the possible cumulant definitions for complex variables [1], we find more convenient to choose:

$$\mathrm{Cum}^z_{i_1 i_2 i_3 \dots} \triangleq \mathrm{Cum}[z^*_{i_1},\, z_{i_2},\, z^*_{i_3},\, \dots]. \tag{5}$$

The same pairwise conventions as in the real case are adopted, namely:

$$\kappa^z_{n-r,\,r} \triangleq \mathrm{Cum}^z_{\underbrace{1\dots1}_{n-r}\underbrace{2\dots2}_{r}}. \tag{6}$$

The cumulants of the other separation-system elements (sources $\mathbf{x}$, observations $\mathbf{y}$, and separator output $\mathbf{s} = \hat{\mathbf{x}}$) are defined in a totally analogous manner.

2.2 Recollections of the Real-Mixture Case

Before getting deeper into our discussion of the complex problem, we recall some results on the real case. The cornerstone of the real-mixture unified formulation developed in [7,9,10] is the complex scatter diagram of the signals involved:

$$\left.\begin{array}{l}(x_1,\, x_2) = x_1 + jx_2 = r\mathrm{e}^{j\phi'} = r\angle\phi' \\ (z_1,\, z_2) = z_1 + jz_2 = r\mathrm{e}^{j\phi} = r\angle\phi\end{array}\right\} \quad j^2 \triangleq -1, \tag{7}$$

where, from (2) and (3)–(4) (with $\alpha = 0$), $\phi = \phi' + \theta$, yielding the fundamental relationship:

$$z_1 + jz_2 = \mathrm{e}^{j\theta}(x_1 + jx_2). \tag{8}$$

Among other benefits, the scatter diagram provided insightful geometrical interpretations of some of the methods studied [9].

The central result of [10] was the derivation of an nth-order (with n arbitrary) estimation family for the parameter of interest. Such family was characterized by the following theorem:

Theorem 1. *Define* $\xi_n(\mathbf{z})$ *as the following complex weighted sum of pairwise* n*th-order cumulants of the components of* $\mathbf{z}$*, for* $n \in \mathbb{N}^+$*:*

$$\xi_n(\mathbf{z}) \triangleq \sum_{r=0}^{n} \binom{n}{r} j^r \kappa^z_{n-r,\,r}. \tag{9}$$

If $\mathbf{z} = \mathcal{Q}(\theta,\, 0)\,\mathbf{x}$*, with* $\mathbf{x}$ *made up of independent components, then*

$$\xi_n(\mathbf{z}) = \mathrm{e}^{jn\theta}\xi_n(\mathbf{x}), \tag{10}$$

where, according to (9), $\xi_n(\mathbf{x}) = \kappa^x_{n0} + j^n\kappa^x_{0n}$.

The terms $\xi_n(\mathbf{x})$ and $\xi_n(\mathbf{z})$ are called *nth-order complex centroids* of the sources and the whitened observations, respectively. Eqn. (10) leads at once to an analytic estimator of θ based on nth-order cumulants.

3 BICOMPLEX NUMBERS

As reviewed in the preceding section, the use of complex numbers facilitates the closed-form identification of the real orthogonal matrix. Since we now want to deal with complex signals, it seems natural to seek certain type of extension of the complex numbers. We carry out such extension as follows.

Definition 2. A *bicomplex number*, $\bar{x} \in \mathcal{B}$, is an expression of the form:

$$\bar{x} = a + \jmath b, \qquad a,\, b \in \mathbb{C}, \quad \jmath^2 \triangleq -1. \tag{11}$$

Although symbol $\jmath$ has the same numeric value as the usual imaginary unit j, they actually represent distinct algebraic elements.

Definition 3. Given $\bar{x} = a + \jmath b \in \mathcal{B}$, $\mathbb{R}\mathrm{e}(\bar{x}) = a \in \mathbb{C}$ is the *bicomplex-real (breal)* part of $\bar{x}$, and $\mathbb{I}\mathrm{m}(\bar{x}) = b \in \mathbb{C}$ its *bicomplex-imaginary (bimaginary)* part.

This terminology prevents confusion with the commonplace real [$\mathrm{I\!Re}(\cdot)$] and imaginary [$\mathrm{I\!Im}(\cdot)$] parts of complex numbers, which are always real valued. Accordingly, $\jmath$ is named *bimaginary unit.* Due to the nature to the matrix to be identified, our attention is restricted to the subset:

Definition 4. $\mathcal{B}_u = \{\bar{x} \in \mathcal{B} \,:\, |a|^2 + |b|^2 = 1\} \subset \mathcal{B}$ is the set of *unitary* bicomplex numbers.

It is straightforward to establish an isomorphism between the set of unitary bicomplex numbers and the set $\mathcal{U}$ of (2×2) unitary matrices. Any matrix $U \in \mathcal{U}$ can be written in general form as:

$$U = \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}, \qquad a,\, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1. \tag{12}$$

The product of $U_1,\, U_2 \in \mathcal{U}$ is then given by:

$$U_1U_2 = \begin{bmatrix} a_1a_2 - b_1^*b_2 & -a_1b_2^* - b_1^*a_2^* \\ b_1a_2 + a_1^*b_2 & -b_1b_2^* + a_1^*a_2^* \end{bmatrix}. \tag{13}$$

Now, we can define the *product* of two bicomplex numbers $\bar{x}_1$ and $\bar{x}_2$ in compliance with the product of $\mathcal{U}$-matrices by considering the first column of the previous expression, so that:

$$\begin{aligned}\bar{x}_1\bar{x}_2 \triangleq{}& \left[\mathbb{R}\mathrm{e}(\bar{x}_1)\mathbb{R}\mathrm{e}(\bar{x}_2) - \mathbb{I}\mathrm{m}^*(\bar{x}_1)\mathbb{I}\mathrm{m}(\bar{x}_2)\right] \\ &+ \jmath\left[\mathbb{I}\mathrm{m}(\bar{x}_1)\mathbb{R}\mathrm{e}(\bar{x}_2) + \mathbb{R}\mathrm{e}^*(\bar{x}_1)\mathbb{I}\mathrm{m}(\bar{x}_2)\right].\end{aligned} \tag{14}$$

With this product operation: $\jmath^2 = -1$, in agreement with Dfn. 2. Also, if both the breal and bimaginary parts of the bicomplex operands are real, such product reduces (changing j for $\jmath$) to the usual complex-number multiplication. Hence, eqn. (14) is a natural generalization of the complex product. By virtue of these conventions, the mapping

$$\aleph:\ \mathcal{U} \mapsto \mathcal{B}_u, \tag{15}$$

with

$$\aleph(U) = u_{11} + \jmath u_{21} = a + \jmath b, \qquad U \in \mathcal{U}, \tag{16}$$

represents an isomorphism between $\mathcal{U}$ under normal matrix multiplication and $\mathcal{B}_u$ under the bicomplex-number product as defined in (14). Having established this connection, the definitions of bicomplex conjugation, modulus and inverse naturally follow [11]. As interesting particular consequences, we cite $\jmath^{-1} = -\jmath$, and $\jmath^n = (-1)^{\frac{n}{2}}$, for n even, $(-1)^{\frac{n-1}{2}}\jmath$, for n odd, identities which highlight the similarity between the bimaginary and the imaginary units.

If the unitary transformation is parameterized like in (4), then the associated unitary bicomplex number becomes an important particular case.

Definition 5. Expression

$$\mathrm{e}_{\alpha}^{\jmath\theta} \triangleq \cos\theta + \jmath \mathrm{e}^{j\alpha}\sin\theta, \qquad \theta,\, \alpha \in \mathbb{R}, \tag{17}$$

is called *bicomplex exponential*, by analogy with its familiar complex counterpart.

With these tools, now it is simple to prove that a expression similar to (8) also holds in the complex-mixture case.

Proposition 6. *If* $\mathbf{z} = \mathcal{Q}(\theta,\, \alpha)\,\mathbf{x}$, *then*

$$z_1 + \jmath z_2 = \mathrm{e}_{\alpha}^{\jmath\theta}(x_1 + \jmath x_2). \tag{18}$$

By analogy with the real case, the bicomplex numbers $(z_1 + \jmath z_2)$ *and* $(x_1 + \jmath x_2)$ *are called* bicomplex scatter-diagram points *of the whitened output and the source signals, respectively.*

This scatter-plot relation is algebraically analogous to its real equivalent, but lacks an intuitive geometric interpretation in the complex case.

The beauty of bicomplex numbers is that by constraining the breal and bimaginary parts to be real quantities, i.e., the associated transformation to have real elements, one is at once dealing with the familiar complex numbers and, in the context of BSS, with the real-mixture scenario.

4 COMPLEX ANALYTIC ESTIMATORS

With the aid of the bicomplex-number tools developed in the previous section, Theorem 1 accepts the following extension to complex mixtures.

Theorem 7. *Define* $\xi_n(\mathbf{z})$ *as the following bicomplex weighted sum of pairwise nth-order cumulants of the components of* $\mathbf{z}$, *with* $n \in \mathbb{N}^+$:

$$\xi_n(\mathbf{z}) \triangleq \sum_{r=0}^{n} \binom{n}{r} \jmath^r \kappa_{n-r,\,r}^{z}. \tag{19}$$

If $\mathbf{z} = \mathcal{Q}(\theta,\, \alpha)\,\mathbf{x}$, *with* $\mathbf{x}$ *made up of independent components, then, for* n *even:*

$$\xi_n(\mathbf{z}) = \mathrm{e}_{\alpha}^{\jmath n\theta}\xi_n(\mathbf{x}) \tag{20}$$

where, according to (19), $\xi_n(\mathbf{x}) = \kappa_{n0}^{x} + \jmath^n\kappa_{0n}^{x}$; *and, for* n *odd:*

$$\xi_n(\mathbf{z}) = \mathrm{e}_{-\alpha}^{\jmath n\theta}\kappa_{n0}^{x} + \jmath^n \mathrm{e}_{\alpha}^{\jmath n\theta}\kappa_{0n}^{x}. \tag{21}$$

Sketch of the proof. The proof [11] runs along the lines of that in the real case [10], and is essentially based on the multilinearity property of cumulants, the source statistical independence assumption, the unitary nature of matrix Q, Euler's formula, De Moivre's identity and certain algebraic simplifications. ■

For real mixtures, eqns. (20) and (21) merge into a unique expression, eqn. (10). By analogy with the real case, cumulant combination (19) is termed whitened-observation *nth-order bicomplex centroid.* The above key relationship between source and whitened bicomplex centroids does not seem to lead to a simple estimation of the unknown parameter at odd orders [eqn. (21)]. In the real case, by contrast, a third-order estimator (TO-BSE) was derived from the associated family [10, 12]. This is not a strong shortcoming, since in practice many complex signals of interest (e.g., in areas such as communications) are circularly distributed processes, whose odd-order cumulants all vanish [1]. At even order, $\xi_n(\mathbf{x})$ is always a real number, and then parameters $(\theta,\, \alpha)$ may be readily estimated in closed-form from (20) via:

$$(n\hat{\theta},\, \hat{\alpha}) = \operatorname{barg}\left(\frac{\xi_n(\mathbf{z})}{\xi_n(\mathbf{x})}\right), \qquad n \text{ even}, \tag{22}$$

where $(\psi,\, \varphi) = \operatorname{barg}(\bar{x})$ represents the *bicomplex argument* function (analogous to the complex argument, or phase, function "arg(·)"), that we define as [11]:

$$\begin{cases} \psi = \arg\big(\mathbb{R}\mathrm{e}(\bar{x}) + j|\mathbb{I}\mathrm{m}(\bar{x})|\big) \\ \varphi = \arg\big(\mathbb{I}\mathrm{m}(\bar{x})\big). \end{cases} \tag{23}$$

This definition results in $\mathrm{e}_{\varphi}^{\jmath\psi} = \bar{x}$, $\forall \bar{x} \in \mathcal{B}_u$ of the form $\mathrm{e}_{\alpha}^{\jmath\theta}$.

The indeterminacy issues discussed in [10] for the real estimators derived from Theorem 1 also apply to estimators (22). This indeterminacy lies in the fact that $\exp(jn\theta) = \exp\big(jn(\theta + 2\pi m/n)\big)$, for any $n \in \mathbb{N}^+$, $m \in \mathbb{Z}$, so a non-valid separation solution may be produced. In addition, the value of $\xi_n(\mathbf{x})$ (i.e., the source marginal n th-order cumulants) is not known a priori in a genuine blind problem. However, these two difficulties do not always arise.

4.1 Order 4

At fourth-order, (20) becomes:

$$\xi_4(\mathbf{z}) = (\kappa_{40}^z - 6\kappa_{22}^z + \kappa_{04}^z) + \jmath 4(\kappa_{31}^z - \kappa_{13}^z) = \mathrm{e}_\alpha^{\jmath 4\theta}(\kappa_{40}^x + \kappa_{04}^x) = \mathrm{e}_\alpha^{\jmath 4\theta}\gamma, \quad (24)$$

where $\gamma = \xi_4(\mathbf{x}) = \kappa_{40}^x + \kappa_{04}^x$ is the source kurtosis sum (sks). Hence, the unknown parameters can be obtained from:

$$(4\hat{\theta},\, \hat{\alpha}) = \mathrm{barg}\big(\xi_4(\mathbf{z})/\xi_4(\mathbf{x})\big). \quad (25)$$

Due to the evident connection with its real counterpart (the EML estimator [9]), this is called the *Complex EML (CEML)* estimator. Just like its real twin, the sks can be obtained from the available signals as:

$$\gamma = \kappa_{40}^z + 2\kappa_{22}^z + \kappa_{04}^z = \kappa_{40}^x + \kappa_{04}^x. \quad (26)$$

As an interesting by-product, eqns. (24) and (26) easily lead to the complex version of the whitened-sensor 4th-order cumulant identity found in [3] for the real case.

Simulations on several kinds of complex signals [11] show that the CEML performance is independent of parameter α, but depends on the actual value of θ. Why this dependency occurs is still unclear; a theoretical analysis of the estimator's performance would shed some light on this point. Performance with respect to the sks is similar to that exhibited by its real version. The CEML behaviour in noisy environments is very satisfactory, and surpasses JADE [2] for certain source distributions.

5 CONCLUSIONS

The bicomplex numbers have been defined, forming a set isomorphic to the group of transformations relevant to the BSS problem after pre-whitening. This algebraic formalism has permitted an elegant extension to the complex-mixture case of a number of closed-form estimation ideas in real scenarios, including the concepts of scatter diagram and centroid. The extended centroids appear as specific bicomplex linear combinations of the whitened-vector cumulants which are able to retain the information contained in the unitary mixing transformation, giving rise to compact closed-form expressions for the estimation of the pertinent parameters. Altogether, a unified formulation of the real and the complex problems within the framework of their analytical solutions has been devised. It is envisaged that the full potential of the bicomplex formalism may be employed in further research to obtain new results on BSS, and even in other fields.

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