Nonlinear processing of nongaussian and noncoherent signals in the presence of colored noise

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ABSTRACT

The problem of nonlinear processing of nongaussian and noncoherent signals is common in many applications such as remote sensing, hydroacoustics, radioastronomy and communication technologies. Optimal solution of this problem requires taking into account the space-time statistical properties of signals, medium and noise, which are usually known a priori. One of the best methods is suggested by the theory of conditional Markov processes [1-3]. Using this technique, we may describe the signal fluctuations in the form of stochastic differential equations. According to the standard procedure, to obtain the filtering algorithms, we must write the evolution equation for the conditional density functional. The disadvantage of this procedure is that we may write closed equations for the conditional density functional only for the case of signals received in the presence of white noise.

In this paper on the base of the functional approach [4,5], we consider the case of signals, which are observed as a random field in the presence of noise. Using the technique, we may describe the signal fluctuations in the form of stochastic differential equations. According to the standard procedure, to obtain the filtering algorithms, we must write the evolution equation for the conditional density functional. The disadvantage of this procedure is that we may write closed equations for the conditional density functional only for the case of signals received in the presence of white noise.

In this paper on the base of the functional approach [4,5], we consider the case of signals, which are observed as a random field in the presence of noise. We obtain optimal algorithms for a set of modulated signals and colored noises, including the noise described by Ornstein-Uhlenbeck process.

1. Functional representation for the conditional density functional

Let us suppose that received signal $y(t,r)$ is observed as a random field in the presence of noise $n(t,r)$

$$y(t,r) = s(t,r,\lambda(t,r)) + n(t,r),$$

Let us also assume that the Markov process describes the useful message, which we have to estimate

$$\frac{d\lambda(t,r)}{dt} = a(\lambda, t, r) + b(\lambda, t, r) f(t, r),$$

where $f(t, r)$ is a Gaussian white process with

$$f(t, r) f(t', r') = F(r-r')\delta(t-t'), \quad \int f(t, r) = 0 $$

We can represent the conditional (posterior) density functional in the following functional-integral form

$$P[\lambda, y_0^T]/f = \int Df D\lambda D\lambda' / f p[y_0^T]/f \int D\lambda / f p[\lambda],$$

where $y_0^T$ is a realization of $y(t,r)$ over interval $[0, t]$, $\lambda$ is a realization of useful process $\lambda$, given a realization of white random process $f$ and known initial condition, $\lfloor Df \rfloor$ denotes a functional integral over $f$, $\lfloor D\lambda \rfloor$ denotes a functional integral over space realization of $\lambda(t,r)$ at moment $t$. This is the basic formula of the study. To prove this expression (1), we may use the following equalities

$$P[\lambda, f] = p[\lambda, f, y_0] = \delta[\lambda - \lambda_0(t, r, \lambda(0, r))],$$

where functionals $K$ and $\theta$ are defined by the following rules

$$K[y_0] = y(t, r), \quad \theta[\lambda, \lambda_0] = \begin{cases} 1, & \text{if } \lambda(r) = \lambda_0(r), \quad \forall r \\ 0, & \text{if } \lambda(r) \neq \lambda_0(r), \quad \exists r. \end{cases}$$

Using these expressions and the known formulas of the theory of probabilities, we obtain (1).

2. Case of white noise. Evolution equation for the conditional density functional (the Stratonovich equation)

To analyze (1), we must know the functional $P[y_0^T]/f$. The most simple case, for which we can write $P[y_0^T]/f$, is the case of white noise $n(t,r)$.

$$n(t,r) = N(r-r')\delta(t-t'), \quad n(t,r) = 0 $$

Then

$$P[y_0^T]/f = N_x \times \exp \left(-\frac{1}{2} \int_0^T \int \int \int \int \int dt dt' \int \int \int \int \int s(t,r,\lambda,0,r,\lambda) s(t',r',\lambda,0,r,\lambda) N \right)$$

where $D$ is the aperture area.

According to the traditional approach, to perform synthesis and analysis of suboptimal processing algorithms we should write the evolution equation for the conditional density functional (the Stratonovich equation) and assume that the posteriori probability density functional may be approximated by Gaussian functional. In this Gaussian approximation we may write equations for mean aposteriori estimates and covariance functions which describe the desired processing algorithms.

Let us find the evolution equation, using the functional approach. We differentiate (1) with respect to time
\[ \partial_t P[y/f] = \partial_{y_0} \int I_1 - P[y/f] \partial_{y_0} \int I_2, \]

(2)

where

\[ I_1 = \int \mathcal{D} \delta [\dot{x} - \lambda, t] P[y/f] \int f, \quad I_2 = \int \mathcal{D} \lambda, I_1, \]

and find \( \partial_{I_1} \) and \( \partial_{I_2} \) successively, using properties of \( \delta \)-function

\[ \partial_{I_1} = -\int \mathcal{D} \delta (\dot{x}(\lambda, r, t)) \partial_{x} \quad - \int \mathcal{D} \delta (\dot{x}(\lambda, r, t)) \int f \partial_{x} \]

Rearranging on the base of the known formulas of functional integral theory ([16,7] for example) and substituting these expressions into (2), we obtain the desired evolution equation (the Stratonovich equations)

\[ \partial_t P[y/f] = P[y/f] \int \Lambda(\lambda, t) - \int \mathcal{D} \delta \partial_{x} \Lambda(\lambda, t) P[y/f] + \int \mathcal{D} \delta \partial_{x} \Lambda(\lambda, t) P[y/f] \]

where

\[ \Lambda(\lambda, t) = -\frac{1}{2} \int \mathcal{D} \delta N(x-r) N(t-t) \Lambda(\lambda, t) \int f \partial_{x} \]

and \( N \) satisfies the following equation

\[ \int \mathcal{D} \delta N(x-r) N(t-t) = \delta (x-r) \).

Stratonovich was the first to write the evolution equation for the case of random processes. For the case of random fields, this equation was generalized by Dr. Shmelev, who used another method.

3. The case of colored noise.

We may use the Stratonovich method only in the case when we received the signal in the presence of white noise. In this section we show how we may use functional approach to treat of wider class of colored noises. To point the most important features of the method and to avoid complicated formulas, we consider the case of time fluctuations only. As an example we suppose that \( n(t) \) is colored noise which we may describe by the stochastic equations

\[ \frac{dn(t)}{dt} = -b(t)n(t) + g(t), \]

where \( g(t) \) is a Gaussian white process with

\[ \overline{g(t)} = 0, \quad \overline{[g(t)g(t)]} = \alpha \delta (t-t) = b \delta (t-t) \]

To find \( P[y/f] \), we transform the received signal \( y(t) \)

\[ Y(t) = \frac{d}{dt} n(t) + \epsilon n(t) = R(t) + Q(t)f(t) + g(t), \]

where

\[ R(t) = \frac{\partial (\lambda, t)}{dt} + a(\lambda, t) \frac{\partial (\lambda, t)}{dt} + \epsilon n(t), \quad Q(t) = \frac{\partial (\lambda, t)}{dt} b(t, \lambda). \]

Then

\[ P[y/f] = N_{y} \exp\left\{ -\frac{1}{2\hat{F}} \int \mathcal{D} \delta \int f Y(t) - R(t) - Q(t)f(t) \right\}, \]

\[ P[y/f] = N_{f} \exp\left\{ -\frac{1}{2\hat{F}} \int \mathcal{D} \delta \int f Y(t) - R(t) - Q(t)f(t) \right\} \]

where \( N_{y} \) and \( N_{f} \) are the normalization constants. As in the case of white noise, to obtain the evolution equation we differentiate the conditional density functional with respect to time and try to find \( \partial_{I_1} \)

\[ \partial_{I_1} = \frac{\partial}{\partial t} \left\{ \int \mathcal{D} \delta \partial_{x} \Lambda(\lambda, t) \right\} = \Delta_{1} \partial_{x} \Lambda(\lambda, t) \int f \partial_{x} \]

We have

\[ \Delta_{1} \partial_{x} \Lambda(\lambda, t) \int f \partial_{x} = \int \mathcal{D} \delta \partial_{x} \Lambda(\lambda, t) \int f \partial_{x} \]

and

\[ \int \mathcal{D} \delta \partial_{x} \Lambda(\lambda, t) \int f \partial_{x} \]

Sings.

Here \( Sing \) means singularity like \( \delta \)-function. That is, generally the conditional density functional \( P[y/f] \) is nondifferentiable and we cannot obtain the evolution equations. But for any particular case, we can try to find the corresponding approximations of the required evolution equation. For an example, if \( Q(t)/H = const(t, \lambda) \), then the singularity is reduced when we substitute \( \partial_{I_1} \) and \( \partial_{I_2} \) into (2) and we may write

\[ \frac{\partial}{\partial t} \left\{ b(t, \lambda) I_1 \right\} = \frac{1}{2} \int \mathcal{D} \delta \partial_{x} \Lambda(\lambda, t) \int f \partial_{x} \]

\[ - \int \mathcal{D} \delta \partial_{x} \Lambda(\lambda, t) \int f \partial_{x} \]

\[ \left\{ \frac{\partial}{\partial t} \left\{ b(t, \lambda) I_1 \right\} \right\} = \frac{1}{2} \int \mathcal{D} \delta \partial_{x} \Lambda(\lambda, t) \int f \partial_{x} \]

\[ + \int \mathcal{D} \delta \partial_{x} \Lambda(\lambda, t) \int f \partial_{x} \]

\[ \frac{\partial}{\partial t} \left\{ b(t, \lambda) I_1 \right\} = \frac{1}{2} \int \mathcal{D} \delta \partial_{x} \Lambda(\lambda, t) \int f \partial_{x} \]
\[ \frac{\partial J_1}{\partial I_1} = \int d\sigma \frac{\partial J_1}{\partial J_2}, \]

where
\[ \Phi_Y = \frac{Y(t) - R(t)}{\sqrt{\mathcal{H}}} \quad \Phi_Z = \frac{LQ}{2\sqrt{\mathcal{H}}}. \]

\[ \partial p[y_0 / x_0] = \frac{\partial J_1}{\partial I_1} = -p[y_0 / x_0] \frac{\partial J_1}{\partial I_1}. \]

### 3.1. Estimation of phase fluctuations

As an example we find nonlinear filtering algorithm for
\[ s(t, \lambda) = A \cos(\omega t + \lambda_t), \]
\[ a(\lambda, t) = -\gamma \lambda, \quad b(\lambda, t) = 1. \]

For this signal we may write
\[ \mathcal{R}(t) = A \cos(\omega t + \lambda) + A \gamma \lambda \sin(\omega t + \lambda), \]
\[ Q(t) = -A \sin(\omega t + \lambda). \]

Assuming that frequency \( \omega \) is high enough, we may average the fast oscillating terms over a few cycles, so that \( Q(t)^2 = \mathcal{A} \mathcal{F} = \text{const} \) and we obtain the evolution equations for conditional density functional without singular terms. In Gaussian approximations, this equation splits into the equations for required estimation \( \lambda \) and the second order cumulant \( K(t) \)

\[ \frac{d\lambda}{dt} = -\gamma \lambda - K(t) \frac{Y(t) \cos(\omega t + \lambda)}{2\mathcal{A} / (1+\alpha)^2} - \frac{4\alpha \omega}{\mathcal{A} (1+\alpha)^2} - \frac{Y(t) \sin(\omega t + \lambda)}{2\mathcal{A} / (1+\alpha)^2} \left( K(t) \frac{F}{2} + 2 + \alpha \right) - \frac{\alpha \omega (2 + \alpha)}{1+\alpha)^2}, \]

\[ \frac{dK(t)}{dt} = F \left( \frac{2 + \alpha}{2(1+\alpha)^2} - 2y \right) - K(t) \left( \frac{2 \omega \alpha (2 + \alpha)}{1+\alpha)^2} \right) - \frac{\alpha \omega (2 + \alpha)}{1+\alpha)^2} \left( \frac{1}{1 + \alpha^2} \right). \]

where \( \alpha = FA^2/4H \). \( K(t) \) describes the processing error.

In the limit of white noise \( n(t) \rightarrow \infty, H/\mathcal{A} = \mathcal{G} = \text{const} \), these equations are transformed into well known equations for nonlinear processing of phase fluctuations in the presence of white noise with spectral density \( G \)

\[ \frac{d\lambda}{dt} = -\gamma \lambda - \frac{A K(t)}{G} \frac{Y(t) \sin(\omega t + \lambda)}{G}, \]
\[ \frac{dK(t)}{dt} = F - 2y K(t) - \frac{A^2}{2G} K(t). \]

### 3.2 Numerical simulation

Figures 1-3 presents the results of simulating nonlinear algorithms for case of colored noise. Fig.1 show relationship between the square of normalized stationary errors of processing of phase from the prior variance of phase fluctuating. The curve \( a \) is a theoretical results. The curve \( b \) represent of digital simulating. This figure demonstrates of validity of suggested algorithms. Fig.2 show realizations of phase, its optimal estimation (good correspondence) and non-optimal estimation, which consider noise as white one.

![Fig.1](image1.png)

![Fig.2](image2.png)

### 3.3 Another case

I would like to point another important case, for
\[ s(t, \lambda) = A \lambda, \quad b(\lambda, t) = 1 \text{ then } Q = \text{const}. \]

which we can write evolutional equations. This signal due to the restriction of the paper volume, we can not to publish the results in this paper.

### 4. Conclusions

In conclusion it worth to emphasize that the functional approach provide an effective synthesis of processing algorithms, which may take into account the known statistical properties of wave signal and noise.

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### References


