AN EXACT PARAMETRIC EQUIVALENT OF THE PERIODOGRAM

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ABSTRACT
A method is presented that provides the exact equivalent representation of the periodogram by means of an MA(\(n-1\)) model. This method will be compared to inferring an MA(\(q\)) model directly from the data by estimating the parameters and selecting the optimal model order. Representing the periodogram by means of an MA(\(n-1\)) model, enables the use of The Prediction Error to make an exact quantitative comparison of the periodogram to parametric spectral estimation methods. It follows that parametric methods provide better spectral estimates than the periodogram. This is explained by the fact that the periodogram is a non-selective transformation of all data, while the parametric methods discriminate between statistically significant information and noise by means of order selection.

1 INTRODUCTION
In spectral analysis of signals generated by stationary stochastic processes, non parametric techniques, like the periodogram, or parametric techniques, like ARMA models, can be used to estimate the spectrum. The periodogram is a direct transformation of the data. Its behaviour can be enhanced by using different kind of tapers and smoothing windows. These attribute their own characteristics to the results of the analyses and a statistically reliable choice is not possible without prior knowledge. Only when the characteristics of the signal to be analysed are known in advance, a reasonable good choice for a taper can be made. In many applications, however, this is not the case.

Parametric representations of the spectrum of a time series are generally chosen among the representatives of the class of ARMA models [1]. These models are inferred from the data by estimating the parameters and selecting the order. Thus a difference is made concerning the statistical relevance of the information; only parameters that really contribute to the quality of the model are incorporated.

In order to compare the performance of the periodogram to parametric spectral estimates, a parametric representation of the periodogram is needed. Since the autocovariance function of the periodogram equals zero beyond lag \(n-1\), where \(n\) is the number of observations, it can be represented by a moving average (MA) model of order \(n-1\).

For the calculation of the equivalent MA representation of a periodogram, only methods which yield an approximation are available [8]; no closed-form solution is known from literature [7]. Wilson [8] describes a method which cannot deal with complex valued data and uses an iterative algorithm including matrix inversions. The method of Wensink and Bazen [7], can deal with complex valued data, but since it uses Durbin’s estimation technique with intermediate AR models [4], this method [7] provides an approximation of the MA model.

In this paper, a method is presented to calculate the MA(\(n-1\)) model that matches the periodogram exactly. Thus a quantitative exact comparison between the periodogram and parametric methods is enabled, that shows the importance of model order selection. This is done by comparing the periodogram, represented as an MA(\(n-1\)) model to an MA(\(q\)) model that is inferred from the data; this means that its parameters are estimated and the optimal order, \(q\), is selected.

Maximum likelihood estimation of MA models from a given series of data is a non linear estimation problem. As a result the roots of the characteristic polynomial may lie outside the unit circle, meaning that the estimated model is not invertible. A popular solution is to mirror the estimated roots in the unit circle, thus creating an invertible solution. The spectra of the invertible and its non-invertible counterpart are the same. It may be used when the model order is known in advance, like with the MA(\(n-1\)) model.

For order selection, however, mirroring in the unit circle is not possible. The basis of order selection is lain in the decrease of the residual variance caused by adding extra parameters to the model. Since mirroring affects the residual variance, order selection demands an estimation method that estimates invertible model without exception. Therefore the method of Durbin will be used [4],[2]. This method estimates an MA model on the basis of an intermediate long AR model that has been estimated from the data.
2 DESCRIPTION OF THE METHOD

Since it uses the Fast Fourier Transform, FFT, the periodogram is the most widely used numerical spectral estimator. For a time series \( x \) of length \( n \), tapered by a window \( w \), it is calculated by:

\[
\hat{P}_{xx}(\omega) = \frac{1}{n} (\text{FFT}(w \cdot x))(\text{FFT}(w \cdot x)),
\]

(1)

where the bar denotes the complex conjugate. An MA\((q)\) model given by,

\[
x_i = b_0 \varepsilon_i + b_1 \varepsilon_{i-1} + \ldots + b_q \varepsilon_{i-q}
\]

(2)
describes time series \( x_i \) as the response of a FIR filter of order \( q \) to white noise \( \varepsilon_i \). In general the model is normalized by taking \( b_0 = 1 \). Formula (2) shows that the impulse response \( h_i \) of the MA\((q)\) model equals the parameters \( b_i \) and the autocovariance function equals zero for lags beyond \( q \). The power spectral density function of an MA\((q)\) model is given by,

\[
\hat{S}(\omega) = \sigma^2 \left( 1 + b_1 e^{-j\omega} + \ldots + b_q e^{-jq\omega} \right) \left( 1 + b_1 e^{j\omega} + \ldots + b_q e^{jq\omega} \right)^{-1}
\]

(3)

Representing the periodogram by an MA\((n-1)\) model now comes down to finding the impulse response that fits to the given autocovariance function of the periodogram. Furthermore, the parametric measures require that the impulse response is minimum phase, which means that all zeros have to be inside the unit circle.

When comparing the periodogram (1) to MA spectral estimates (3) and noticing that \( h_i = b_i \), the equivalence can be seen:

\[
\hat{P}_{xx}(\omega) \sim \left( \text{FT}(x) \right) \left( \text{FT}(h) \right)
\]

\[
\hat{S}_{xx}(\omega) \sim \left( \text{FT}(h) \right) \left( \text{FT}(h) \right)
\]

(4)

Therefore, by taking

\[
h_i = x_i + 1
\]

(5)
an impulse response \( h_i \) that matches the periodogram is found. However, this impulse response is not minimum phase in general. Therefore the poles have to be mirrored inside the unit circle. Furthermore the parameters have to be normalized by setting \( b_0 = 1 \), which can be done without loss of generality.

In the \( z \)-domain, the autocovariance function \( R_{x\kappa}(z) \) of a transfer function model is given by:

\[
R_{x\kappa}(z) = \sigma^2 \tilde{h}(z) \tilde{h}^*(z)
\]

(6)

where \( \tilde{h}^* \) denotes the complex conjugated reversed sequence \( h \). The zeros of \( R_{x\kappa}(z) \) are determined by the zeros of \( h(z) \) and their counterparts of \( h^*(z) \), which are mirrored in the unit circle. In order to find the minimum phase solution, the zeros of \( R_{x\kappa}(z) \) are redistributed over \( h(z) \) and \( h^*(z) \), in such a way that \( h(z) \) is the polynomial, built up from the zeros of the autocovariance function that are inside the unit circle. This way, the impulse response \( h_i \) is the minimum phase solution we are looking for.

Remarks:

1. The effect of determining the minimum phase solution is mirroring all zeros that are outside, inside the unit circle.

2. When a given autocovariance function has to be represented by an MA model, first the zeros of \( R_{x\kappa}(z) \) are determined and then the remainder of the procedure followed.

Let \( x_i \) be an MA\((n-1)\) process with impulse response \( h_{x\kappa}(i+1) \) for \( i = 0, \ldots, n-1 \) and \( \varepsilon_i \) a white noise process with \( \sigma^2 = 1/n \), then follows for the autocovariance function:

\[
R_{x\kappa}(\tau) = \sigma^2 \sum_{i=0}^{n-1} \tilde{h}_i h_{x\kappa}(i+\tau) + |\tau|
\]

(7)

The autocovariance is estimated from the data as:

\[
\hat{R}_{x\kappa}(\tau) = \frac{1}{n} \sum_{i=1}^{n} x_i x_{i+\tau}, \quad \text{with } x_i = 0, \quad i \in \{1, \ldots, n\}
\]

(8)

The equivalence of the spectrum calculated on the basis of the impulse response or on the basis of the covariances can now be shown.

Expression (3) can be rewritten as:

\[
\hat{S}_{xx}(\omega) = \sigma^2 \left| \hat{h}(\omega) \hat{h}^*(\omega) \right|
\]

(9)

The spectrum, \( S_{xx}(\omega) \), is the Fourier transform of the biased autocovariance function. Taking into account that \( h_0 = 1 \) formula (9) is equivalent to:

\[
\sigma^2 \left| \hat{h}(\omega) \right|^2 = \sigma^2 \left| \sum_{i=0}^{n-1} h_i \exp(-j\omega i) \right|^2 = \frac{1}{n} \left| \sum_{i=1}^{n} x_i \exp(-j\omega i) \right|^2
\]

(10)
\[ S_{xx}(\omega) = \frac{1}{n} |\text{FT}(x)|^2 = \frac{1}{n} \left| \sum_{i=1}^{n} x_i \exp(-i\omega t) \right|^2, \quad (11) \]

showing that the MA(n-1) model can be used as a representation of the periodogram.

This method provides the exact MA(n-1) representation of the periodogram. It can deal with complex valued data and will always provide an invertible model. This representation can be used to make an exact comparison of the periodogram to parametric spectral estimates.

3 ESTIMATION OF MA(q)

Although several methods exist to solve the non linear problem of MA parameter estimation, MA models are not used as much as AR models in Signal Analysis. Many algorithms have difficulty with convergence or converge to non invertible models. In a simulation example with an (invertible) MA(1) process, it has been shown that the sum of squared errors as function of the parameter can easily have its minimum outside the unit circle, whereas no local minimum was found at the mirrored position. Simulation experiments show that only Durbin’s method yields useful models in all simulation runs. Therefore MA parameters are estimated using Durbin’s MA method [4].

Durbin’s method is based on the theoretical equivalence of a finite order MA process with an infinite order AR process. In practice this equivalence doesn’t create a workable situation, since the number of available observations is finite. First, an intermediate high-order AR model is fitted to the data. Then, the MA parameters are estimated as AR parameters, using the parameters of the intermediate AR model as observations. This method consists of two consecutive linear estimation problems; thus ensuring the estimated MA model to be invertible.

Great care has to be taken when determining the size of the intermediate AR order. It turns out that good results are obtained when the intermediate AR order is determined by the sliding window method [2]. This consists in first selecting an AR order, say \( p \), with a suitable finite sample order selection criterion. In addition the intermediate AR order for a MA model of order \( q \) will be equal to \( p + q \). In this way the intermediate AR order will slide with the MA order that is estimated, till some maximum order is reached.

For MA models, order selection is carried out using the GIC, or Generalized Information Criterion [6],

\[ GIC(q, \alpha) = \log \left( S^2(q) \right) + \alpha \cdot \frac{q}{N}, \quad (12) \]

The estimation of the residual variance of an estimated MA model may e.g. be carried out by the back forecasting technique [1].

4 PREDICTION ERROR

In model order estimation two important quantities have to be clearly distinguished. These quantities are the residual variance and the prediction error.

The residual variance \( S^2(p) \) describes the fit of a \( p \)-th order model to the series, from which the model is estimated. Estimation of an extra parameter will always cause the residual variance to decrease, since the given realisation of the data is described better.

The prediction error PE describes the fit of an estimated model to the process generating the data. Therefore, the prediction error decreases when relevant parameters are estimated and added to the model, while it increases when extra parameters are estimated that are not significant anymore. It can be shown that models that have smallest PE in the time domain, have at the same time the best fit to the spectral envelope.

If the process is known, the prediction error can be calculated. It equals the variance \( \sigma^2_{E} \) of the non-white innovations \( \eta_{t} \), that the estimated model needs, in order to have the same output characteristics as the generating process.

In z-domain ARMA process can be written as:

\[ A(z)X(z) = B(z)E(z) \quad (13) \]

When \( B(z)/A(z) \) represents the process then \( B(z)/A(z) \) represents the model. Now, the prediction error can be calculated as the variance of and ARMA(\( p + q', p' + q \)) model, in which, \( p \) is the real AR order, \( p' \) is the estimated AR order, \( q \) is the real MA order and \( q' \) is the estimated MA order. For the split process and model holds:

\[ C(z) = \hat{A}(z)B(z) \]
\[ D(z) = \hat{A}(z)B(z) \quad (14) \]

The prediction error can now be calculated as

\[ PE(p', q') = \sigma^2_{E} = \begin{bmatrix} 1 & d^1 & \ldots & d^p & p' + q \end{bmatrix} \begin{bmatrix} R_{vv}(0) & R_{vv}(-1) & \ldots & R_{vv}(-p' - q) \\ R_{vv}(1) & R_{vv}(0) & \ldots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ R_{vv}(p' + q) & \ldots & R_{vv}(0) & d \\ \end{bmatrix} \begin{bmatrix} 1 \\ d^1 \\ \vdots \\ d^p \\ p' + q \end{bmatrix}, \quad (15) \]

where \( R_{vv}(s) \) is calculated from the AR part \( C(z) \), using the Yule-Walker equations.

The time domain prediction error is related to the model fit in the frequency domain, expressed by NATFE, [5], via the Parseval relations.

For this comparison of the quality of the representations the
Prediction Error (PE) is used, since it provides an objective, quantitative, measure for assessing the quality of the different representations [7].

5 SIMULATIONS

In parametric spectral estimation, selection of the model order is used to distinguish statistically relevant information from noise. To compare the performance of the periodogram and MA(q) estimation, simulation experiments have been carried out. In the example given here, both periodogram and MA(q) models have been estimated from series, generated by a complex valued AR(2) process. The prediction error is used to evaluate the simulation experiments. An example of the estimated spectra and the average prediction error are depicted in Figure 1.

From these results can be seen that distinguishing between statistically relevant information and noise by means of order selection will lead to a considerably lower prediction error and so to a more accurate spectral estimate than when the periodogram is used.

6 CONCLUSIONS

In order to compare the performance of the periodogram to parametric spectral estimates, a parametric representation of the (tapered) periodogram has been developed. Since the autocovariance function of the periodogram equals zero beyond lag n-1, where n is the number of observations, it can be represented by a MA model of order n-1.

When comparing the periodogram to MA(q) models, it follows that parametric methods provide better spectral estimates than the periodogram. This can be explained by the fact that the periodogram is a non-selective transformation of all data, while the parametric methods discriminate between statistically significant information and noise by means of model order selection [3].

References