THE CHARACTERISTIC FUNCTION OF THE *K*-DISTRIBUTED INTERFERENCE

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ABSTRACT

Several types of electromagnetic interference as well as ultrasound backscatter signals can be accurately modelled by K-distribution. The probability density and cumulative distribution functions and expressions for moments of a K-distributed random variable are known. However, the form of the characteristic function has remained unsolved. Here, a closed form expression of the characteristic function of K-distribution is derived and investigated. Also, the form of the kernel for empirical characteristic function is examined in a view of designing a test for K-distribution.

1. INTRODUCTION

K-distribution has been used as a model of interference in several applications. It can accurately model the statistics of backscatter sea echo [1, 2], the acoustic reverberations in sonar [3], and ultrasound RF backscatter signal [4, 5].

The statistics of a K-distributed random variable X can be described by the probability density function

$$f_X(x) = \frac{2}{a\Gamma(\nu+1)}\left(\frac{x}{2a}\right)^{\nu+1} K_\nu\left(\frac{x}{a}\right) \quad x > 0. \qquad (1)$$

where $a > 0$ and $\nu > -1$ denote the scale and shape parameters, respectively, $K_\nu(\cdot)$ is the modified Bessel function of the second kind of order ν, and $\Gamma(\cdot)$ is the Euler's gamma function. Figure 1 shows the plots of $f_X(x)$ for several values of the shape parameter ν and a unit scale, $a = 1$.

The two parameters of K-distribution, a and ν, can be estimated using an iterative maximum likelihood (ML) estimation method [6], with suboptimal methods [7, 8, 9] or by applying a neural network based technique [10].

Modelling the statistics of interference by a K-distributed random variable is often justified physically. However, in some applications, K-distribution is just one of the models available along with Weibull, Inverse-Gaussian, and the Log-Normal distributions, to mention a few. Thus, the question often arises whether a set of random data is K-distributed or whether it has a different statistical law.

Several statistical tests exist that are based on the probability density function such as the Kolmogorov-Smirnov test. However, their power is usually moderate when compared to a specialized test designed entirely for a particular type of distribution. To the best of our knowledge there are no dedicated tests for K-distribution.

Characteristic function based tests have been advocated as an alternative to the ones based on probability density functions [13]. The characteristic function of a random variable X with probability density function $f_X(x)$ is given by

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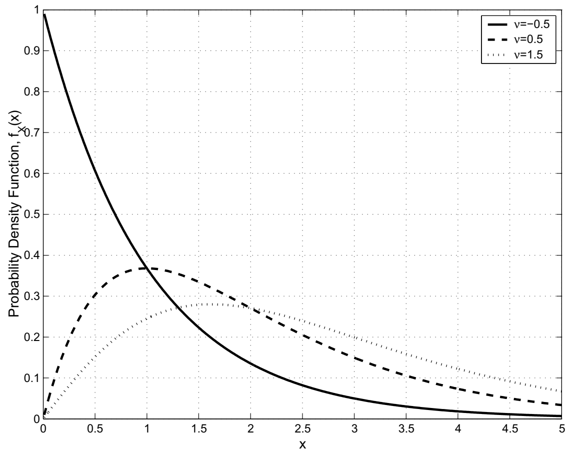

Figure 1: Probability density function of a K-distributed random variable for several values of the shape parameter ν.

$$\phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x)e^{j\omega x}dx.$$

Given N independent realizations of the random variable $X = [X_1, \ldots, X_N]$, the characteristic function can be estimated using the so-called empirical characteristic function (ECF)

$$\hat{\phi}_X^e(\omega) = \frac{1}{N}\sum_{n=1}^{N} \exp(j\omega X_n). \qquad (2)$$

A test for a distribution can be constructed by taking a distance measure between the ECF and $\phi_X(\omega)$ with its parameters replaced by the ML estimates. It has been demonstrated that tests based on smoothed versions of (2) are superior to their probability density function counter-representations [14, 15]. However, in order to take the advantage of such a representation in our case, a closed form of the characteristic function of K-distribution needs to be derived.

2. DERIVATION OF THE CHARACTERISTIC FUNCTION

The probability density function of a K-distributed random variable X from (1) can be written as

$$f_X(x) = A_1 x^{\nu+1} K_\nu\left(\frac{x}{a}\right)$$

where

$$A_1 = \frac{1}{a^2\Gamma(\nu+1)}\left(\frac{1}{2a}\right)^{\nu}.$$

Thus, the characteristic function of K-distribution is given by

$$\phi_X(\omega) = A_1 \int_0^{\infty} x^{\nu+1} K_\nu\left(\frac{x}{a}\right) e^{j\omega x} dx \tag{3}$$

In order to to solve equation (3) the following integral formula is used [11, p. 712, Eq. 6.621-3]

$$\begin{aligned}\int_0^{\infty} x^{\mu-1} e^{-\alpha x} K_\nu(\beta x) dx &= \frac{\sqrt{\pi}(2\beta)^{\nu}}{(\alpha+\beta)^{\mu+\nu}} \\ &\times \frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma\left(\mu+\frac{1}{2}\right)} \\ &\times {}_2F_1\left(\mu+\nu, \nu+\frac{1}{2}; \mu+\frac{1}{2}; \frac{\alpha-\beta}{\alpha+\beta}\right)\end{aligned}$$

where $\Re\mu > |\Re\nu|$, $\Re(\alpha+\beta) > 0$, and ${}_2F_1(A,B;C;z)$ is the Gauss hypergeometric function.

Letting $\mu = \nu+2$, $\alpha = -j\omega$, and $\beta = \frac{1}{a}$ one obtains

$$\begin{aligned}&\int_0^{\infty} x^{\nu+1} e^{j\omega x} K_\nu\left(\frac{x}{a}\right) dx = \\ &\frac{\sqrt{\pi}\left(\frac{2}{a}\right)^{\nu}}{\left(\frac{1}{a}-j\omega\right)^{2\nu+2}} \frac{\Gamma(2\nu+2)}{\Gamma\left(\nu+\frac{5}{2}\right)} \\ &\times {}_2F_1\left(2\nu+2, \nu+\frac{1}{2}; \nu+\frac{5}{2}; \frac{-j\omega-\frac{1}{a}}{-j\omega+\frac{1}{a}}\right).\end{aligned} \tag{4}$$

Equation (4) can be simplified noting that [12, p. 256, Eq. 6.1.18]

$$\Gamma[2(\nu+1)] = \frac{1}{\sqrt{\pi}}\Gamma(\nu+1)\Gamma\left(\nu+\frac{3}{2}\right)2^{2\nu+1}$$

which yields (after multiplying by A_1) the final closed form expression of the characteristic function of K-distribution

$$\begin{aligned}\phi_X(\omega) &= \frac{1}{2\nu+3}\left(\frac{2}{1-j\omega a}\right)^{2\nu+2} \\ &\times {}_2F_1\left(2\nu+2, \nu+\frac{1}{2}; \nu+\frac{5}{2}; \frac{-j\omega-\frac{1}{a}}{-j\omega+\frac{1}{a}}\right).\end{aligned} \tag{5}$$

Figure 2 shows the plots of the magnitude characteristic function, $\Phi_X(\omega) = |\phi_X(\omega)|$, for several values of shape parameter ν and a unit scale, $a = 1$. Note that in general the characteristic function of K-distribution has heavier tails to that of a Gaussian random variable. These tails are becoming lighter when $\nu \to \infty$ and the distribution approaches that of a Rayleigh distribution. However, shape parameter $-1 < \nu < 0$ is mostly of interest in non-Rayleigh modelling of interference, as it represents the measure of "spikyness" of interference [7].

2.1 A special case

For $\nu = -0.5$,

$${}_2F_1\left(1, 0; 2; \frac{-j\omega-\frac{1}{a}}{-j\omega+\frac{1}{a}}\right) = 1.$$

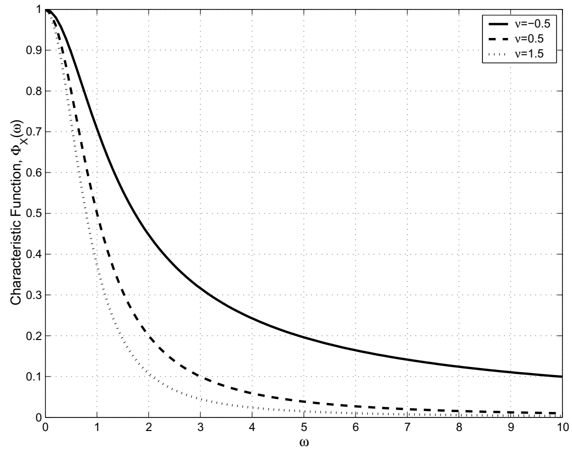

Figure 2: The magnitude characteristic function of a K-distributed random variable for several values of shape parameter ν.

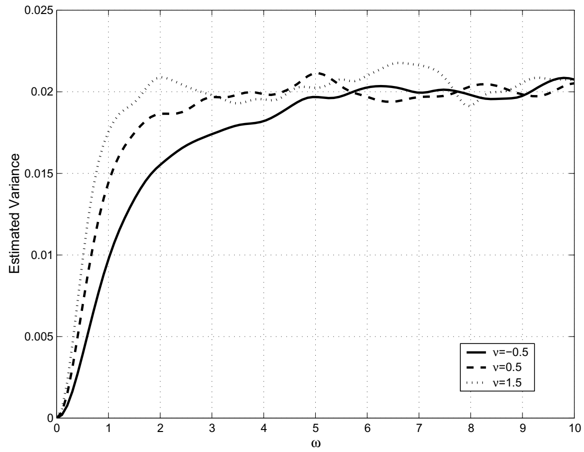

Figure 3: The estimated variance of the ECF, $\hat{\phi}_X^e(\omega)$, for a K-distributed random variable with $N = 50$ and 1000 independent realizations of X.

and the characteristic function of the K-distribution reduces to

$$\phi_X(\omega) = \frac{1}{1-j\omega a},$$

the characteristic function of the Gamma distribution with the shape parameter $\beta = 1$.

3. CONSTRUCTION OF A KERNEL EMPIRICAL CHARACTERISTIC FUNCTION

The ECF given in (2) is not an optimal choice as its behaviour strongly depends on ω. In Figure 3, the estimated variance of the ECF is shown for several K-distributed random variables and $N = 50$. 1000 independent Monte Carlo realizations of $X = [X_1, \ldots, X_{50}]$ were used to estimate each of the variances. Theoretically, it approaches $1/N$ when $\omega \to \infty$. Note, however, that the convergence to $1/N$ (dependent on ν) is rather quick. This non-optimality presents a major problem for designing tests based on the ECF.

A better choice is to use the concept of kernel empirical characteristic function (KECF) in which the ECF is appropriately smoothed [14], i.e.

$$\hat{\phi}_X(\omega) = K(\omega)\hat{\phi}_X^e(\omega) \tag{6}$$

where $K(\omega)$ is an appropriate characteristic function domain kernel function.

An optimal kernel function in the sense of the minimum integrated mean square error (MSE) could be derived in the following manner. Let us first separate the real and imaginary components of the KECF, i.e.

$$\begin{aligned}\Re\hat{\phi}_X(\omega) &= \Re K(\omega)\Re\hat{\phi}_X^e(\omega) - \Im K(\omega)\Im\hat{\phi}_X^e(\omega)\\ \Im\hat{\phi}_X(\omega) &= \Re K(\omega)\Im\hat{\phi}_X^e(\omega) + \Im K(\omega)\Re\hat{\phi}_X^e(\omega).\end{aligned}$$

The expectations and variances of these components are

$$\begin{aligned}\mathsf{E}[\Re\hat{\phi}_X(\omega)] &= \Re K(\omega)\Re\hat{\phi}_X(\omega) - \Im K(\omega)\Im\hat{\phi}_X(\omega)\\ \mathsf{E}[\Im\hat{\phi}_X(\omega)] &= \Re K(\omega)\Im\hat{\phi}_X(\omega) + \Im K(\omega)\Re\hat{\phi}_X(\omega).\end{aligned}$$

and

$$\begin{aligned}\mathsf{Var}[\Re\hat{\phi}_X(\omega)] &= \frac{1}{2N}\{\Re^2 K(\omega)(1+\Re\phi_X(2\omega))\\ &- 2\mathsf{E}^2[\Re\hat{\phi}_X(\omega)] + \Im^2 K(\omega)(1-\Re\phi_X(2\omega))\\ &- 2\Re K(\omega)\Im K(\omega)\Im\phi_X(2\omega)\}\\ \mathsf{Var}[\Im\hat{\phi}_X(\omega)] &= \frac{1}{2N}\{\Re^2 K(\omega)(1-\Re\phi_X(2\omega))\\ &- 2\mathsf{E}^2[\Im\hat{\phi}_X(\omega)] + \Im^2 K(\omega)(1+\Re\phi_X(2\omega))\\ &+ 2\Re K(\omega)\Im K(\omega)\Im\phi_X(2\omega)\}\end{aligned}$$

respectively. Then, the integrated MSE for the real and imaginary parts of the KECF are given by

$$\begin{aligned}I_{MSE}[\Re\hat{\phi}_X(\omega)] &= \int_{-\infty}^{\infty} \mathsf{Var}[\Re\hat{\phi}_X(\omega)]\\ &+ \left(\mathsf{E}[\Re\hat{\phi}_X(\omega)] - \Re K(\omega)\right)^2 d\omega \end{aligned}\tag{7}$$

and

$$\begin{aligned}I_{MSE}[\Im\hat{\phi}_X(\omega)] &= \int_{-\infty}^{\infty} \mathsf{Var}[\Im\hat{\phi}_X(\omega)]\\ &+ \left(\mathsf{E}[\Im\hat{\phi}_X(\omega)] - \Im K(\omega)\right)^2 d\omega. \end{aligned}\tag{8}$$

A criterion based on the minima of (7) and (8) has been used to find the optimal kernel for a Gaussian characteristic function [14]. However, in the case of the K-distribution an analytical form of a kernel is difficult to realize. Given certain form of the $K(\omega)$, parameterised with a and ν, a numerical solution can be considered. Below, we will investigate possible shapes of the kernel function of the KECF that are suitable for K-distributed data.

Let us first investigate the behaviour of the Gaussian KECF with

$$K(\omega) = \exp\left(-j\frac{s^2\omega^2}{2}\right)$$

where

$$s^2 = \mathsf{Var}[X]\left(\frac{2}{3N}\right)^{1/5}$$

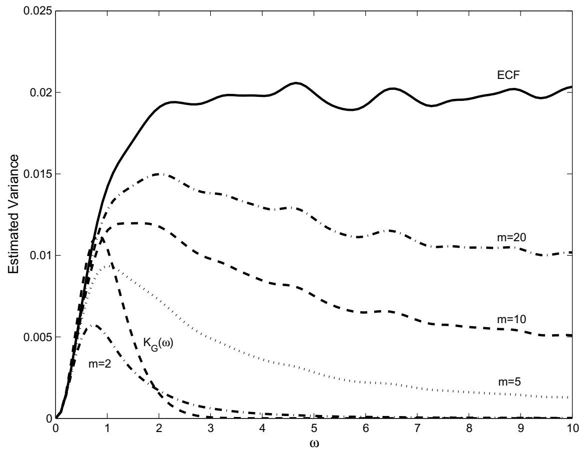

Figure 4: Estimated variance of the ECF, the KECF with a Gaussian kernel $K_G(\omega)$, and the KECF with kernel $K(\omega; a, \nu; m)$ for K-distributed data with $\nu = 0.5$, $a = 1$, and 1000 independent realizations of X..

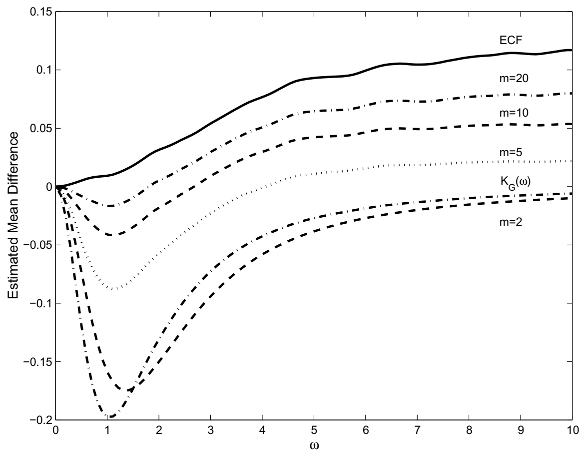

Figure 5: Corresponding to Figure 4 mean difference between the ECF (KECF) and the true characteristic function $\phi_X(\omega)$.

that we will apply to K-distributed data. The variance of a K-distributed random variable X is given by

$$\mathsf{Var}[X] = 4a^2(\nu+1) - \left(\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)}{\Gamma(\nu+1)}2a\right)^2.$$

Figure 4 shows the estimated variance of the Gaussian based KECF while in Figure 5 the estimated mean difference between the KECF and the true characteristic function is presented for K-distributed data with $\nu = 0.5$ and $a = 1$. 1000 independent realizations of the K-distributed random variable were used in the simulation. The sample size was set to $N = 50$ as in the previous simulation.

It could be noted that although the variance of the Gaussian based KECF is much less than that of the ECF, there is a significant change in the bias, especially at lower range of ω. Experiments with different s^2 lead to a conclusion that Gaussian type kernels are not suitable in the estimation of characteristic function for K-distributed data.

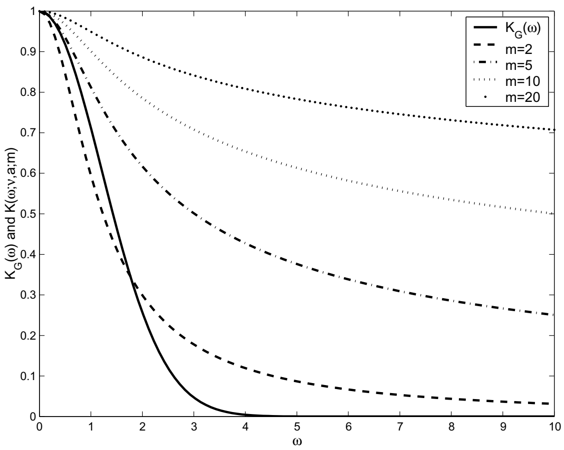

Figure 6: The proposed kernel function $K(\omega;a,\nu;m)$ and the Gaussian kernel $K_G(\omega)$ for K-distributed data with $\nu = 0.5$, $a = 1$, and several values of m.

A new kernel function, dependent on the parameters of K-distribution is proposed,

$$K(\omega;a,\nu;m) = \left(\frac{1}{1+\omega^2 a^2}\right)^{(\nu+1)/m} \tag{9}$$

where $m > 0$ is a "match" parameter that determines the trade-off between minimum variance and minimum mean difference. Note that the proposed kernel corresponds to the modulation component of the Gauss hypergeometric function in the characteristic function of the K-distribution (5). The shape of the kernel function for a range of m is shown in Figure 6. It is evident that the proposed kernel has heavier tails than those of a Gaussian function.

The results of the variance and mean difference estimation of the KECF with the proposed kernel function (9) are shown in Figures 4 and 5, respectively. We note that when parameter m is small, the performance of the proposed KECF is comparable to a Gaussian based KECF while for $m \to \infty$ it approaches that of the ECF.

4. CONCLUSIONS

The main contribution of this paper is the derivation of the closed form expression for the characteristic function of K-distribution. The paper also considers methodology for designing a kernel function for the empirical characteristic function estimator under the assumption that the data is K-distributed. It has been found that traditional Gaussian type kernels are not suitable for this purpose. A new kernel function with heavier tails has been proposed and its performance evaluated using numerical simulations. It was found that the proposed kernel represent a trade-off between minimum variance of the KECF and the minimum mean difference between the KECF and the true characteristic function. Other choices of $K(\omega)$ are possible. In any case, it should be ensured that the kernel used for estimating the characteristic function of K-distribution has much heavier tails than those of Gaussian. The underlying methodology is useful for designing a statistical test for K-distribution in the future.

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