SIMPLIFIED BOUNDS FOR THE COMPLEMENTARY ERROR FUNCTION; APPLICATION TO THE PERFORMANCE EVALUATION OF SIGNAL-PROCESSING SYSTEMS

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ABSTRACT

Any signal–processing scheme requires evaluation of the performance. Very often in such applications the complementary error function (the Gaussian Q-function) occurs. In this paper, we present new upper and lower bounds for this function in the form of one exponential function only. We give examples of applying the derived bounds for the performance evaluation relating to a few recently proposed signal processing schemes. We show that using the suggested estimates results in simple closed-form expressions for the error probability in the considered schemes and provides rather accurate approximations of the exact decisions obtained numerically. Application areas for the proposed approximations are also discussed.

1. INTRODUCTION

The complementary error function *erfc(x)* (the Gaussian Q-function) plays a very important role in many signal-processing applications relating to the error probability evaluation under different scenarios [1-6]. The *erfc*-function occurs also in some other applications, e.g. [7].

The complementary error function *erfc(x)* and the Gaussian Q-function are defined as:

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-t^2)dt \quad . \tag{1}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp(-\frac{t^2}{2})dt = 1/2 erfc(\frac{x}{\sqrt{2}}) \cdot \tag{2}$$

A large number of approximations have been derived for the above functions, e.g. [1–4]. In [1] an alternative (integral-type) representation of (2) is given:

$$Q(x) = \frac{1}{\pi} \int_0^{\pi/2} \exp(-\frac{x^2}{2\sin^2\theta})d\theta \quad \cdot \tag{3}$$

Formula (3) is valid for $x > 0$.

The argument of the Q-function in (3) is contained in the integrand rather than in the integration limit as in (2). This fact was pointed out in [2] and it was shown that representation (3) leads to a significant simplification of the error probability evaluation in a series of applications.

One of widely used approximations of the *erfc*-function is the Chernoff bound [3, 4, 6]:

$$erfc(x) \le 2\exp(-x^2) \, . \tag{4}$$

The bound (4) can be made tighter [3, 4]:

$$erfc(x) \le \exp(-x^2) \, . \tag{5}$$

In [3, 4] another exponential-type upper bound for (1) was derived on the basis of (3) in the form of the sum of two exponential functions:

$$erfc(x) \le \frac{1}{6}\exp(-x^2) + \frac{1}{2}\exp(-\frac{4}{3}x^2) \, . \tag{6}$$

The right side of (6) provides a rather tight bound for

$$x > 0.5 \, . \tag{7}$$

The applicability of (6) was discussed in [3, 4] where examples of the applications of (6) were presented. The use of (6) remarkably facilitates the performance evaluation of signal-processing systems under various scenarios and due to the use of (6) solutions of a series of complex problems can be derived in a closed form.

In this paper, we present exponential-type upper and lower bounds for functions (1) and (2). Each estimate is derived in the form of one exponential function. The obtained estimates are compared with some other known approximations. The applicability of the derived bounds is confirmed by a series of examples. We discuss areas of the applicability of the obtained bounds. The presented approach allows us to derive other approximations of the *erfc*-function if its argument is beyond the range considered in the paper.

The paper is organised as follows. In Section 2 we present our approach and, derived on this basis, new exponential-type bounds for the *erfc*-function. In Section 3 we give examples of the applicability of the obtained bounds for a few recently proposed signal-processing schemes and show that the suggested estimates not only provide rather accurate approximations of the exact decisions obtained numerically but also lead to simple closed-form expressions for the error probability evaluation in the considered schemes. Finally, Section 4 is devoted to conclusions and discussion.

2. NEW EXPONENTIAL - TYPE BOUNDS FOR THE COMPLEMENTARY ERROR FUNCTION

Unlike the approach in [3, 4], we aim at searching approximations of (1) and (2) in the form of one exponential function. We note that being applied to solving the problems of the error probability evaluation, the Chernoff bound and its improved version (5) provide rather rough estimates of the real error probability [3, 4, 6]. At the same time, we note that in the above applications the argument of the Gaussian Q-function contains the actual signal-to-noise ratio, i.e. under a realistic scenario the condition (7) usually holds. Therefore the bound (5) can be made more accurate at the expense of a slightly smaller range of the validity. From the mean-value theorem applied to (3) it follows that there exists a more accurate upper bound than that described by (5), which is also expressed in the form of one exponential function.

Taking into account that the integrands in (1) and (2) are positive, we search suitable bounds for the integrands. Moreover, we search these bounds in such a form that the integrals could be calculated analytically. If we are able to find a solution for the integrand in the form

$$f(x) = C \times 2ax \exp(-ax^2), \tag{8}$$

where C and a are constants, then the bounds for (1) and (2) can be derived in the form of one exponential function. We note that (8) should not be larger (for an upper bound) or smaller (for a lower bound) than the integrand in (1) elsewhere. It is only important that the corresponding relations would be valid for the integrals. In doing so, we find that the function

$$g1(x) = 0.56 \exp(-1.275 x^2) \tag{9}$$

provides a rather accurate lower bound for the *erfc*-function if the condition (7) holds.

At the same time we find that the function

$$g2(x) = 0.6 \exp(-1.01 x^2) \tag{10}$$

gives an upper bound for the *erfc*-function if

$$x > 0.535 \ . \tag{11}$$

Fig.1 presents the obtained bounds as well as the upper bound derived in [3, 4]. It is clearly seen from Fig.1 that the function at the right-hand side of (6) provides a more accurate approximation for *erfc(x)* than (10) while the latter estimate is represented by one exponential function only. In its turn, the approximation (10) is more accurate than the Chernoff bounds (4) and (5) that are widely used as upper bounds for the *erfc*-function.

Obviously, that the derived bounds (9) and (10) can be used if the argument of the *erfc*-function satisfies (7) and (11) respectively. As it is seen from the comparison of (7) and (11), the areas of the validity of (6) and (10) almost coincide. The right-hand side of (6) gives a more accurate approximation of the *erfc*-function than the formula (10) at the expense of a slightly more complex analytical description.

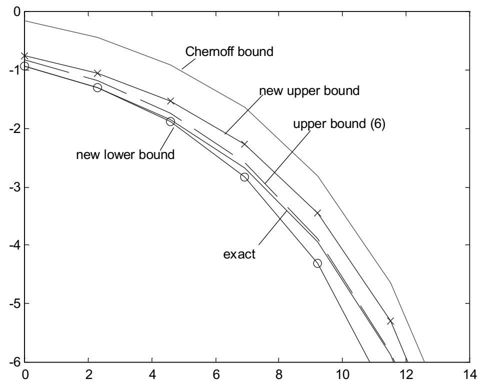
If the average error probability is evaluated in a multipath environment, averaging through statistics of the channel parameters is required. Thus, estimates (9), (10), as well as the bound (6) can be employed for the evaluation of the average error probability in fading channels if the probability of deep fading is rather low.

In Section below we present a few examples of the application of the derived formulas.

3. APPLICATION FOR THE EVALUATION OF THE PERFORMANCE OF SIGNAL-PROCESSING SYSTEMS

3.1 Error probability in a system with a concatenated space-time coding

In [5], the Alamouti transmit diversity scheme [8], with $L_t = 2$ transmit (TX) antennas and L_r receive (RX) antennas, was combined with a linear outer error-correcting code. In this scheme, the encoded bits are processed by a combined multiplexer-interleaver and next each pair (or



x, dB

Figure 1: Comparison of different types of approximation of $erfc(x)$.

quadruple) of bits is mapped on a sequence $\{s(i)\}_{i=1}^{N}$ of two-dimensional vectors of BPSK (or QPSK) symbols with Gray mapping. The bits are transmitted over two TX antennas according to the Alamouti scheme. Such a combination of the code diversity with that of the Alamouti scheme increases the total degree of the diversity. The maximum diversity degree of the suggested scheme may be as large as dL, where d is the diversity degree of the code and $L = L_t L_r$ is that of the antenna.

If the presented system operates in a correlated Rayleigh fading channel, a channel vector is introduced:

$$\mathbf{h}_l = \begin{pmatrix} h_l(i_1) \\ h_l(i_2) \\ \dots \\ h_l(i_d) \end{pmatrix}, l = \overline{1, L}, \tag{12}$$

where the index i stands for the time/frequency variance.

For the usual Alamouti diversity scheme, the pair error probability (PEP) (the probability of deciding $\hat{s}$ while indeed the codeword s was transmitted) for a fixed channel $\mathbf{h}$ is defined as [5]:

$$P\left(s \to \hat{s} \middle| \mathbf{h}\right) = Q\left(\sqrt{\|\mathbf{h}\| \gamma^2 / 2\sigma^2}\right), \tag{13}$$

where γ is a constant that depends on the signal constellation and the code used , σ^2 is the variance of additive noise of the channel.

In order to derive the average PEP, it is necessary to average (13) over the channel statistics. In [5], a technique for the PEP evaluation for the general case of a correlated Rayleigh fading channel was presented. The final expression for the PEP is:

$$P_{av.} = \int_0^\infty \int_0^\infty \dots \int_0^\infty Q\left(\sqrt{\sum_{i=1}^{d} \lambda_i \|\tilde{\mathbf{h}}\| \gamma^2 / 2\sigma^2}\right) p\left(\left|\tilde{h}_1\right|, \dots \left|\tilde{h}_L\right|\right) d\left|\tilde{h}_1\right| \dots d\left|\tilde{h}_L\right|, \tag{14}$$

where λ_i ($i = \overline{1, d}$) are the eigenvalues of the autocorrelation matrix of the channel $\mathbf{R}_{\mathbf{hh}} = E\left\{\mathbf{h}_l \mathbf{h}_l^H\right\}$, the

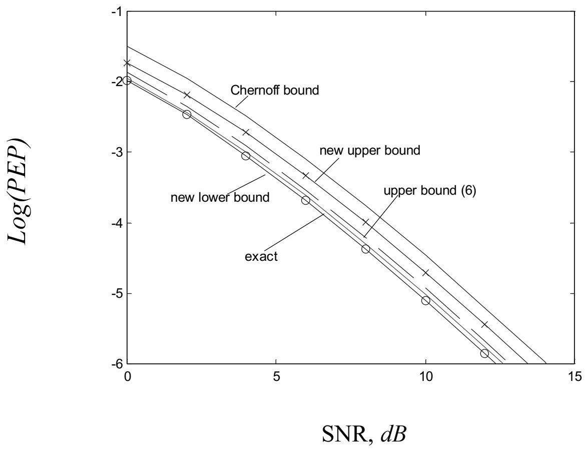


Figure 2: PEP in the Alamouti diversity system with two RX antennas for independently fading Rayleigh channel.

index H denotes the Hermitian conjugate. The matrix $\mathbf{R_{hh}}$ does not depend on the path number l (see [5] for details). The vector $\widetilde{\mathbf{h}}_l = \mathbf{G}\mathbf{h}_l$ is the transformed channel vector, $\mathbf{G}$ is the matrix that has as its columns the set $\{\mathbf{g}_i\}$ of the eigenvectors of $\mathbf{R_{hh}}$ and $p\left|(\widetilde{h}\right|_1,..\left|\widetilde{h}_L\right|)$ is the L-fold Rayleigh probability density function of independent variables $\{\widetilde{h}_i\}_{i=1}^{L}$.

In [5], an integral-type representation of the Gaussian Q-function (3) is used in order to simplify the calculation of (14). As a result, an integral-type representation of (14) is obtained. By application of (9) and (10) closed-form expressions for an accurate estimate of $P_{av.}$ can be derived:

$$0.28\prod_{i=1}^{d}\frac{1}{(1.275\frac{R_c}{L}\frac{E_b}{N_0}\lambda_i+1)^L} < P_{av.} < 0.3\prod_{i=1}^{d}\frac{1}{(1.01\frac{R_c}{L}\frac{E_b}{N_0}\lambda_i+1)^L}, \tag{15}$$

where R_c is the code rate.

The application of (6) gives a slightly more complex but at the same time a more accurate expression for an upper bound:

$$P_{av.} < \frac{1}{12}\prod_{i=1}^{d}\frac{1}{(\frac{R_c}{L}\frac{E_b}{N_0}\lambda_i+1)^L} + \frac{1}{4}\prod_{i=1}^{d}\frac{1}{(\frac{4R_c}{3L}\frac{E_b}{N_0}\lambda_i+1)^L}. \tag{16}$$

In Fig. 2, we present for the comparison estimates of the $P_{av.}$ obtained by using the approximations (15) and (16) for an encoded system with $L_r = 2$ RX antennas and for a case of independent fading. The exact expression obtained on the basis of an integral-type representation of the Gaussian *Q*-function is also shown in Fig. 2. The curves in Fig. 2 are given for the case of BPSK and QPSK modulation. The parameter γ in (13) in this case is equal to 1. We assume also that $E\{|h_l|^2\}=1$, $(l=\overline{1,L})$.

In Fig. 3, the estimates of the PEP in the coded system with the Walsh-Hadamard (WH) (16,4,8) code operating in a correlated Rayleigh fading channel are shown. The parameters of the channel and the signal are those used in [5]. The channel is assumed to have an exponential delay power spectrum with the mean delay $\tau_m = 1\mu s$. The signal bandwidth *B=2GHz*. The curves in Fig. 3 are given for a case of one RX antenna.

It is worth noting that bounds for the bit-error probability (BEP) in the system can be easily evaluated in a closed form by using either (10) or (6).

3.2 Error probability in a space-time block coding system with antenna selection and power allocation

As another example, we consider the application of the proposed approximations for the estimation of the error probability in the system with a new space-time block coding method suggested in [6]. This is an algorithm with selection for the transmission two antennas from three possible ones ('best', 'intermediate' and 'worst'). The transmitter selects two 'best' antennas on the basis of feedback information from the receiver that estimates the channels in terms of the absolute magnitude of the channel gains. The fraction 'a^2' of the total power is transmitted via the 'best' antenna and the rest ($1-a^2$) is transmitted via the 'intermediate' antenna. The Alamouti block code is used for the transmission.

The technique for estimation of the system performance in a Rayleigh fading channel is given in [6]. In this case, in order to evaluate the average error probability it is necessary to average (13) not only over the statistics of the channel vector (as in the previous example) but also over the statistics of the feedback information.

By using (9) and (10) we obtain simple analytical expressions for the estimates of the error probability (BEP). Both of the bounds are described by the same analytical expression with different parameters. For brevity, we present here the bounds for the conditional bit-error probability (if the 'best' and 'intermediate' antennas are used):

$$P_{upp,low} = C_1\frac{1}{C_2a^2\gamma^2\sigma_h^2/\sigma^2+1}\times\frac{1}{C_2\gamma^2\sigma_h^2/\sigma^2+2}\times\frac{1}{C_2\gamma^2\sigma_h^2/\sigma^2+3}, \tag{17}$$

where $C_1 = 0.3, C_2 = 0.5005$ for the upper bound and $C_1 = 0.28, C_2 = 0.6385$ for the lower bound. The σ_h^2 is the variance of the channel coefficients.

The graphs of the bit-error rate in the considered system are shown in Fig. 4. More precisely, in Fig. 4 we present the numerical results given in [6], the estimates obtained on the basis of the Chernoff bound used by the authors, upper bounds derived analytically on the basis of (6) and (10) as well as a lower bound obtained by using (9) . The graphs in Fig. 4 are given for the case of perfect feedback information.

The given examples show that the proposed approximations not only significantly simplify the calculation of the error probability in signal-processing systems. Their use results in estimates which well agree with numerical results. The above examples also confirm that the proposed bounds can be successfully used in applications where averaging over the statistics of the *erfc*- function argument is required.

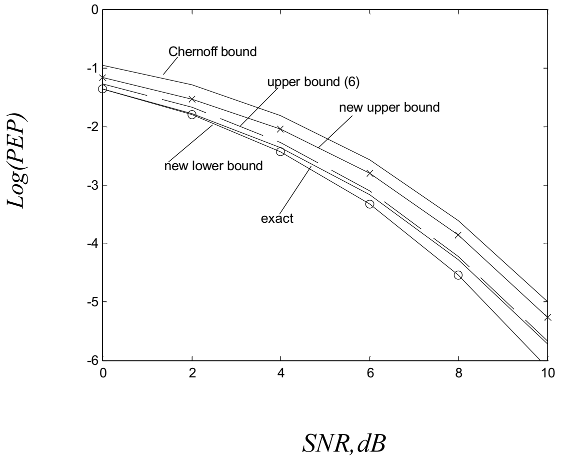


Figure 3: PEP for the WH (16,4,8) system with one RX antenna in Rayleigh fading channel with $B\tau_m = 2$.

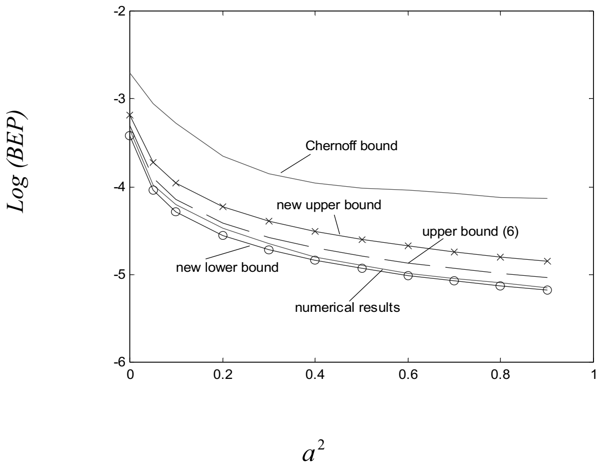


Figure 4: Estimates of BEP in the system with antenna selection obtained in different ways, $\sigma_h^2 / \sigma^2 = 20dB$.

4. CONCLUSIONS

When the *erfc*-function occurs in scientific or engineering calculations, it may be very convenient to use its estimates that have simple analytical descriptions. So far (to the best of our knowledge), the simplest upper bound of the *erfc*-function has been the Chernoff bound that is expressed in the form of one exponential function. But it provides rather rough estimates. In this paper, we have presented a novel upper bound for the *erfc*-function that is also described by one exponential function but gives a more accurate estimation compared with the Chernoff bound. The upper bound given in [3, 4] provides a more accurate approximation of the *erfc*-function than the proposed one at the expense of a slightly more complex analytical description.

The obtained lower bound gives a rather accurate approximation of the *erfc*-function and, as a result, when used for the evaluation of the performance of signal-processing systems provides an accurate estimation of the error probability.

For a good approximation of error probabilities in signal-processing systems, it may be useful to combine the upper bound obtained in [3, 4] with the proposed lower bound.

The derived bounds can be used when solving any practical problem where the erfc-function occurs and the presented estimates are valid. The application of the presented estimates facilitates the derivation of the expressions that contain the *erfc*- function and, in a series of cases, their use leads to obtaining these expressions in a closed form. The proposed estimates can be also used in the applications where averaging over the statistics of the argument is required.

On the basis of the presented approach, depending on the problem solved, simple bounds for the *erfc*-function in the form of one exponential function can be found also if conditions (7) and (11) do not hold.

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