IMPLEMENTATIONS OF NON-SEPARABLE GABOR SCHEMES

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ABSTRACT

In this paper we consider non-separable Gabor schemes for discrete-time signals. We show that three different interpretations of a non-separable lattice lead to three different types of implementation. First, a non-separable lattice can be seen as a union of rectangular lattices, which leads to a filter bank implementation. Secondly, a rectangular lattice can be obtained by shearing a non-separable lattice. As a direct consequence, conventional algorithms developed for rectangular lattices can be re-used for the non-separable case. And finally, interpreting the non-separable lattice as a sub-lattice of a denser rectangular lattice allows the use of the Zak transformation.

1. INTRODUCTION

In 1946, Gabor suggested to represent a time signal in a combined time-frequency domain; he proposed to expand a signal as a series of properly scaled shifted and modulated versions of a Gaussian window on a rectangular time-frequency lattice (see [1] for Gabor's original paper, and [2] for a modern treatment of Gabor analysis). Since the time-frequency representation of a Gaussian has circular contour lines, a better packing of the time-frequency plane is achieved by using a (non-separable) hexagonal lattice instead of a rectangular lattice. This results in a better behaviour of the Gabor scheme in terms of mathematical properties (to be more precise: it leads to tighter frames [3]). Other types of windows, like the one- and two-sided exponentials, yield different shapes in the time-frequency plane. Conceivably, also in these situations other (non-separable) lattices are more suitable than the rectangular lattice. In this paper, we give an overview of implementations of these non-separable Gabor schemes for discrete-time signals.

2. GABOR'S SIGNAL EXPANSION ON A NON-SEPARABLE LATTICE

We start with the usual Gabor expansion [1, 2, 4, 5] on a rectangular time-frequency lattice for a discrete-time signal φ,

$$\varphi[n] = \sum_{k=0}^{K-1} \sum_{m=-\infty}^{\infty} a_{mk} g_{mk}[n], \tag{1}$$

with

$$g_{mk} = \mathcal{M}_{1/K}^{k} \mathcal{T}_{N}^{m} g \tag{2}$$

the shifted and modulated versions of the window g. For convenience, we have used the modulation and translation operators $\mathcal{M}_\omega$ and $\mathcal{T}_\tau$ in this expression, defined as

$$(\mathcal{M}_\omega f)[n] = e^{j2\pi n\omega} f[n] \quad \text{and} \quad (\mathcal{T}_\tau f)[n] = f[n-\tau],$$

with $\omega \in \mathbb{R}$ and $\tau \in \mathbb{Z}$, respectively. The array of Gabor coefficients $\{a_{mk}\}$ is periodic in k with period K, and can be found via the Gabor transform

$$a_{mk} = \sum_{\ell=-\infty}^{\infty} \varphi[\ell] \gamma_{mk}^{*}[\ell] = \langle \varphi, \gamma_{mk} \rangle, \tag{3}$$

with γ_{mk} the shifted and modulated versions of the analysis window (or dual window) γ [cf. Eq. (2)].

The rectangular (or separable) lattice can be obtained by integer combinations of two orthogonal vectors $\underline{v}_0 = [N,\ 0]^T$ and $\underline{v}_1 = [0,\ 1/K]^T$. We thus express the lattice Λ in the form

$$\Lambda = \{ n_0 \underline{v}_0 + n_1 \underline{v}_1 | n_0, n_1 \in \mathbb{Z} \}.$$

We now consider Gabor's signal expansion on a time-frequency lattice that is no longer separable. We call a time-frequency lattice non-separable, if the time-shifts and the modulations in the shifted and modulated windows $g_{\Lambda;mk}$ and $\gamma_{\Lambda;mk}$ are no independent operations anymore. Such a lattice is obtained by integer combinations of two linearly independent, but no longer orthogonal vectors, which we express in the forms $\underline{v}_0 = [aN,\ c/DK]^T$ and $\underline{v}_1 = [bN,\ d/DK]^T$ with a, b, c and d integers, N and K integers, and $D = ad - bc$. The first component in the vectors $\underline{v}_0$ and $\underline{v}_1$ corresponds to a time-shift aN and bN, respectively, while the second component corresponds to a modulation by a frequency c/DK and d/DK, respectively.

Each point $\underline{\lambda} \in \Lambda$ in the time-frequency plane can be obtained by a matrix-vector product

$$\forall_{\underline{\lambda} \in \Lambda} \exists_{\underline{n} \in \mathbb{Z}^2} \ \ \underline{\lambda} = \boldsymbol{UL}\underline{n},$$

with

$$\boldsymbol{U} = \frac{1}{DK} \begin{bmatrix} NDK & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The column vectors $\underline{v}_0$ and $\underline{v}_1$ are the columns of the lattice generator matrix $\boldsymbol{UL}$. Note, moreover, that D is equal to the determinant of the matrix $\boldsymbol{L}$. We assume that the integers a and b have no common divisors, and that the same holds for the integers c and d; hence $\gcd(a,b) = 1$ and $\gcd(c,d) = 1$. A possible common divisor can be unified with N and K. The area of a cell (a parallelogram) in the time-frequency plane is equal to the determinant of the lattice generator matrix $\boldsymbol{UL}$, which is equal to N/K. It is well known that the set of shifted and modulated versions of the window is not complete in the case that $N/K > 1$. The equality $N/K = 1$ corresponds to critical sampling, and $N/K < 1$ corresponds to oversampling. For convenience, we introduce the three integers p, q, and J for which the relationships $J = \gcd(K,N)$, $K = pJ$ and $N = qJ$ hold. Note that the integers p and q are relatively prime, and that the oversampling equals $K/N = p/q$.

For a given matrix $\boldsymbol{U}$, there are many matrices $\boldsymbol{L}$ that generate the same lattice Λ. One form, the Hermite normal form [6], is very interesting; it reads

$$\boldsymbol{L}' = \begin{bmatrix} 1 & 0 \\ -r & D \end{bmatrix},$$

where the integer $-r$ equals $h_0 c + h_1 d + C_0 D$, with the integers h_0 and h_1 such that $h_0 a + h_1 b = 1$ and C_0 an arbitrary integer. Note that these integers h_0 and h_1 exist, since $\gcd(a,b) = 1$, and that they can be obtained by the Euclidean algorithm (see, for instance, [7]). The columns of the matrix $\boldsymbol{UL}'$ are equal to $[N,\ -r/DK]^T$ and $[0,\ 1/K]^T$, respectively. Consequently, the shifted and modulated versions $g_{\Lambda;mk}$ of the synthesis window g_Λ on the lattice Λ take the form

$$g_{\Lambda;mk} = \mathcal{M}_{-r/DK}^{m} \mathcal{M}_{1/K}^{k} \mathcal{T}_{N}^{m} g_\Lambda. \tag{4}$$

The shifted and modulated versions $\gamma_{\Lambda;mk}$ of the analysis window γ_Λ are defined similarly. With this modified definition (4) of the set of shifted and modulated window functions (2), the original expressions for Gabor's signal expansion (1) and the Gabor transform (3) remain valid in the non-separable case. Note that in the case of a rectangular lattice, i.e., $D = 1$ and $r = 0$, Eq. (4) indeed reduces to Eq. (2).

Alternatively, the non-separable lattice Λ can be interpreted as a sub-lattice of the lattice Λ_s,

$$\Lambda_s = \{\mathbf{\Lambda}_s \underline{n} | \underline{n} \in \mathbb{Z}^2\} \quad \text{with} \quad \mathbf{\Lambda}_s = \boldsymbol{U} = \frac{1}{DK}\begin{bmatrix} NDK & 0 \\ 0 & 1 \end{bmatrix},$$

which refines the non-separable lattice Λ. The non-separable lattice Λ is obtained by assigning the value zero to those shifted and modulated windows $g_{\Lambda_s;mk}$ on the separable lattice, which do not belong to the non-separable lattice Λ. This is achieved by using the Poisson summation formula

$$P_{\mathbf{\Lambda}}(m,k) = \frac{1}{D}\sum_{i=0}^{D-1} e^{j2\pi i(mr+k)/D} = \begin{cases} 1 & \text{if } \mathbf{\Lambda}_s \left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right] \in \Lambda, \\ 0 & \text{if } \mathbf{\Lambda}_s \left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right] \notin \Lambda. \end{cases}$$

The shifted and modulated versions of the window g_Λ on the lattice Λ now take the form

$$\tilde{g}_{\Lambda_s;mk} = P_{\mathbf{\Lambda}}(m,k) g_{\Lambda_s;mk}. \tag{5}$$

We put a tilde on top of $g_{\Lambda_s;mk}$ to indicate that the multiplication operator $P_{\mathbf{\Lambda}}(m,k)$ is involved. The shifted and modulated versions $\tilde{\gamma}_{\Lambda;mk}$ of the analysis window $\tilde{\gamma}_\Lambda$ are defined similarly. Note that the shifted and modulated versions $\tilde{g}_{\Lambda_s;mk}$ are periodic in the frequency variable k with period DK. With the modified expression (5) for the shifted and modulated versions $\tilde{g}_{\Lambda_s;mk}$ of the window g_Λ, Gabor's signal expansion takes the form

$$\varphi = \sum_{k=0}^{DK-1} \sum_{m=-\infty}^{\infty} \tilde{a}_{mk} \tilde{g}_{\Lambda_s;mk}, \tag{6}$$

where

$$\tilde{a}_{mk} = \langle \varphi, \tilde{\gamma}_{\Lambda_s;mk} \rangle \tag{7}$$

is the array of Gabor expansion coefficients. Note that the array $\{\tilde{a}_{mk}\}$ is periodic in the variable k with period DK. Moreover, due to the operator $P_{\mathbf{\Lambda}}(m,k)$, the array $\{\tilde{a}_{mk}\}$ of Gabor expansion coefficients contains many zeros.

In order to use the Fourier transformation and the Zak transformation for periodic signals, the non-separable Gabor scheme has to be periodized. Therefore, we restrict the class of signals φ and dual windows γ_Λ to signals that have a finite support of length not more than N_φ and N_γ, respectively. The condition of finite support implies that the arrays $\{a_{mk}\}$ and $\{\tilde{a}_{mk}\}$ have a finite support in the m-variable for all signals φ in the class. In this paper we only periodize Gabor's signal expansion (6) and the Gabor transform (7) (in [3] we consider more approaches). We shall denote the support of the array $\{\tilde{a}_{mk}\}$ in the m-variable by M, where M satisfies the condition

$$MN \geq N_\varphi + N_\gamma - 1. \tag{8}$$

Note that this condition is necessary to prevent time-aliasing, if we want to use overlap-add techniques. From a group theoretical point of view (see [8] for related work within the scope of group theory), this condition is not necessary. Then the condition $MN \geq \max(N_\varphi, N_\gamma)$ is sufficient, but for this choice of M it is not possible to use overlap-add techniques, due to time-aliasing. As mentioned above, we want to use overlap-add techniques and we therefore assume that M satisfies condition (8). We shall write capital letters Φ, G_Λ, Γ_Λ, and A_{mk}, to indicate that we deal with the periodized version of φ, g_Λ, γ_Λ, and a_{mk}, respectively. So, for example, the periodized version Φ of the signal φ is defined by

$$\Phi[n] = \sum_{i=-\infty}^{\infty} \varphi[n + MNi].$$

Without going into detail, one can show that the integer M should satisfy $M = pLD$, with L an integer such that the condition (8) is fulfilled. Then we have the periodized Gabor transform (7)

$$\tilde{A}_{mk} = \langle \Phi, \tilde{\Gamma}_{\Lambda_s;mk} \rangle, \tag{9}$$

which yields the bi-periodic array $\{\tilde{A}_{mk}\}$; this array is periodic in the m-variable with period M and periodic in the k-variable with period DK. The signal Φ can be reconstructed with the periodized Gabor expansion (6)

$$\Phi = \sum_{m=0}^{M-1} \sum_{k=0}^{DK-1} \tilde{A}_{mk} \tilde{G}_{\Lambda_s;mk}. \tag{10}$$

We will use the periodized Gabor transform (9) and Gabor's expansion (10) to show the connection between the periodized Gabor scheme and the Zak transformation (see Section 3.3).

The relationship between the periodized window G_Λ and the periodized dual window Γ_Λ follows from substituting the Gabor transform (9) into Gabor's signal expansion (10):

$$\begin{aligned} &K \sum_{m=0}^{M-1} e^{-j2\pi mkr/D} \Gamma_\Lambda^*[n - kK - mN] G_\Lambda[n - mN] \\ &= \sum_{\ell=-\infty}^{\infty} \delta[k - \ell qLD], \end{aligned} \tag{11}$$

where δ is the Kronecker delta, with $\delta[0] = 1$ and $\delta[k] = 0$ for $k \neq 0$. The condition (11) should hold for $k = 0 \ldots qLD - 1$ and $n = 0 \ldots MN - 1$.

3. IMPLEMENTATIONS OF NON-SEPARABLE GABOR SCHEMES

In this section we show that the non-separable Gabor scheme can be implemented efficiently. First, we observe that a non-separable lattice consists of a finite amount of separable lattices, which results in a filter bank implementation (see Section 3.1). Secondly, a non-separable lattice can be transformed to a separable lattice by shearing the frequency variable. As a direct result, the same techniques that hold for separable Gabor schemes can be re-used for the non-separable case (see Section 3.2). And finally, we observe that a non-separable lattice is a sub-lattice of a separable lattice, which results in an implementation that uses the Zak transformation (see Section 3.3).

3.1 Filter banks

Without proof, we state that a non-separable lattice Λ consists of D separable lattices. As an example, we have plotted in Fig. 1 a lattice which is generated by the lattice generator matrix

$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 1/12 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}, \tag{12}$$

i.e., $r = 1$, $D = 3$, $N = 1$ and $K = 4$. In this figure, we see that this non-separable lattice consists of $D = 3$ separable lattices. As a consequence, a non-separable Gabor scheme can be implemented by combining D uniform DFT filter banks (see [9]) corresponding to the implementation of D separable Gabor schemes. In Fig. 2

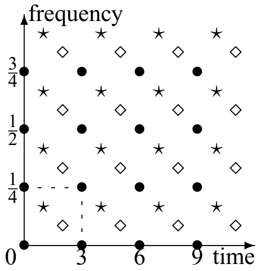

Figure 1: The lattice that is generated by the lattice generator matrix $\mathbf{\Lambda}$ [see Eq. (12)], which consists of $D = 3$ separable lattices indicated by the symbols $\bullet$, $\star$, and $\diamond$. One tile of the lattice is indicated by the dashed lines.

we have depicted this filter bank schematically. One can show that (see [3] for more details)

$$h_{iu}^{(a)} = \mathcal{M}_{1/K}^{u} \mathcal{M}_{r/DK}^{-i} \mathcal{T}_N^{-i} h_{(a)} = \mathcal{M}_{1/K}^{u} h_{(a);i}$$

with $i = 0 \dots D-1$ and $u = 0 \dots K-1$, are the DK impulse responses of the analysis filters, spread uniformly over D analysis banks with the D prototypes $h_{(a);i} = \mathcal{M}_{r/DK}^{-i} \mathcal{T}_N^{-i} h_{(a)}$ and $h_{(a)}[n] = \gamma_\Lambda^*[-n]$. Moreover,

$$h_{iu}^{(s)} = \mathcal{M}_{1/K}^{u} \mathcal{M}_{r/DK}^{-i} \mathcal{T}_N^{i} h_{(s)} = \mathcal{M}_{1/K}^{u} h_{(s);i}$$

are the DK impulse responses of the synthesis filters, spread uniformly over D synthesis banks with the D prototypes $h_{(s);i} = \mathcal{M}_{r/DK}^{-i} \mathcal{T}_N^{i} h_{(s)}$ and $h_{(s)} = g_\Lambda$.

It is well-known that uniform DFT filter banks can be implemented efficiently by using the polyphase representation of the prototype filter (see [9] for more details). Since the non-separable Gabor scheme can be implemented as D uniform DFT filter banks in parallel, this technique can also be used here.

3.2 Shearing

A non-separable lattice can be transformed to a separable lattice by shearing the frequency variable. We define the discrete shear operator as

$$(\mathcal{Q}_{\omega_a} \varphi)[n] = e^{j 2\pi \omega_a n^2} \varphi[n],$$

which is unitary on $\ell_2(\mathbb{Z})$ with corresponding adjoint operator $\mathcal{Q}_{\omega_a}^* = \mathcal{Q}_{-\omega_a}$. The modulation slope of the continuous line through the origin induced by the shear operator $\mathcal{Q}_{\omega_a}$ is $2\omega_a$. To re-shear the non-separable lattice Λ into a separable one, we have to choose $\omega_{a,0} = r/2NDK$. By applying the shear operator $\mathcal{Q}_{\omega_a}$ with $\omega_a = \omega_{a,0}$ to the shifted and modulated windows $g_{\Lambda;mk}$, we obtain

$$\mathcal{Q}_{\omega_{a,0}} g_{\Lambda;mk} = e^{-j\pi m^2 rN/DK} \mathcal{M}_{1/K}^{k} \mathcal{T}_N^{m} \mathcal{Q}_{\omega_{a,0}} g_\Lambda .$$

Applying the unitary shear operator $\mathcal{Q}_{\omega_{a,0}}$ to the Gabor expansion (1) yields a sheared Gabor expansion on a rectangular lattice as well:

$$\begin{aligned} \varphi &= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{K-1} \langle \varphi, \mathcal{M}_{1/K}^{k} \mathcal{T}_N^{m} \mathcal{Q}_{\omega_{a,0}} \gamma_\Lambda \rangle \mathcal{M}_{1/K}^{k} \mathcal{T}_N^{m} \mathcal{Q}_{\omega_{a,0}} g_\Lambda \\ &= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{K-1} \check{a}_{mk} \mathcal{M}_{1/K}^{k} \mathcal{T}_N^{m} \mathcal{Q}_{\omega_{a,0}} g_\Lambda , \end{aligned} \tag{13}$$

where

$$\check{a}_{mk} = a_{mk} e^{-j\pi m^2 rN/DK} = \langle \varphi, \mathcal{M}_{1/K}^{k} \mathcal{T}_N^{m} \mathcal{Q}_{\omega_{a,0}} \gamma_\Lambda \rangle$$

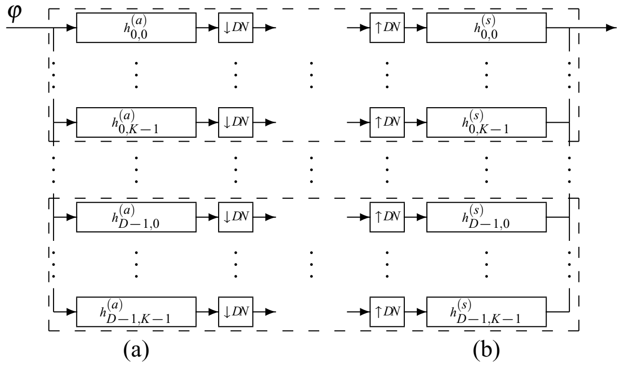

Figure 2: The D uniform DFT filter banks, which are indicated by the dashed lines, consist of D analysis and synthesis banks. (a) The analysis banks. (b) The synthesis banks.

is the array of (modified) Gabor expansion coefficients.

Since Eq. (13) corresponds to a (sheared) Gabor expansion on a rectangular lattice (with windows $\mathcal{Q}_{\omega_{a,0}} g_\Lambda$ and $\mathcal{Q}_{\omega_{a,0}} \gamma_\Lambda$), it is possible to exploit methods of the separable case for the non-separable case, to calculate the dual window γ_Λ and the Gabor expansion coefficients, and to reconstruct the signal φ.

3.3 Discrete Zak transform

The discrete Zak transformation can be very useful and efficient to calculate – for a given dual window Γ_Λ – the (periodized) window G_Λ and the array of Gabor coefficients, and to reconstruct the signal. For the use of the discrete Zak transformation in the case of rectangular lattices in this context, we refer to [10, 11, 12]. In this section we will show how the Zak transformation can be used for the non-separable case, as well.

Condition (11) can be used directly to find the window G_Λ. However this is not very practical. The Zak transformation leads to a more efficient method to find the window G_Λ for a given dual window Γ_Λ. By using the discrete Zak transform

$$(\mathcal{Z}\Phi)[n, \ell; N, M] = \sum_{m=0}^{M-1} \Phi[n + mN] e^{-j2\pi m\ell/M}, \tag{14}$$

condition (11) can be transformed into the sum-of-products form (see [3] for more details)

$$\sum_{k=0}^{fp-1} g_{ik}[n,\ell] \gamma_{sk}^*[n,\ell] = \frac{fp}{K} \delta[i-s], \tag{15a}$$

where

$$g_{ik}[n,\ell] = (\mathcal{Z}G_\Lambda) \left[n + iK, \ell - \frac{M}{fp} k - \frac{rM}{D} i; N, M \right] \tag{15b}$$

and

$$\gamma_{ik}[n,\ell] = (\mathcal{Z}\Gamma_\Lambda) \left[n + iK, \ell - \frac{M}{fp} k - \frac{rM}{D} i; N, M \right], \tag{15c}$$

with $f = D/\gcd(D,q)$, $i = 0 \dots fq-1$, $k = 0 \dots fp-1$, and n and ℓ extending over an interval of length DK and M/fp, respectively. Note that M/fp and M/D are integers (recall $M = pLD$). Now we combine these functions g_{ik} and γ_{ik} into the $(fq \times fp)$ matrices of functions

$$\boldsymbol{G}[n,\ell] = \begin{bmatrix} g_{00}[n,\ell] & g_{01}[n,\ell] & \dots & g_{0,fp-1}[n,\ell] \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ g_{fq-1,0}[n,\ell] & g_{fq-1,1}[n,\ell] & \dots & g_{fq-1,fp-1}[n,\ell] \end{bmatrix}$$

and

$$\boldsymbol{\Gamma}[n,\ell] = \begin{bmatrix} \gamma_{00}[n,\ell] & \gamma_{01}[n,\ell] & \cdots & \gamma_{0,fp-1}[n,\ell] \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \gamma_{fq-1,0}[n,\ell] & \gamma_{fq-1,1}[n,\ell] & \cdots & \gamma_{fq-1,fp-1}[n,\ell] \end{bmatrix},$$

respectively. With the help of these matrices $\boldsymbol{G}$ and $\boldsymbol{\Gamma}$, Eqs. (15a-c) can now be expressed as

$$\boldsymbol{G}\boldsymbol{\Gamma}^* = \frac{fp}{K}\boldsymbol{I}_{fq}, \tag{16}$$

where $\boldsymbol{I}_{fq}$ is the $(fq \times fq)$ identity matrix. Note that in the case of oversampling, $p > q$, the matrix $\boldsymbol{\Gamma}^*$ in Eq. (16) is not a square matrix and does not have an inverse, but in general will have a (non-unique) left inverse. The optimum solution $\boldsymbol{G}_{\text{opt}}$ in the sense of minimum ℓ_2-norm can be found with the help of the generalized (Moore-Penrose) inverse $\boldsymbol{\Gamma}^\dagger = \boldsymbol{\Gamma}^*(\boldsymbol{\Gamma}\boldsymbol{\Gamma}^*)^{-1}$ as

$$\boldsymbol{G}_{\text{opt}} = \frac{fp}{K}(\boldsymbol{\Gamma}^\dagger)^*,$$

which corresponds to the minimum ℓ_2-norm window $G_{\Lambda;\text{opt}}$.

Using the (two-dimensional) discrete Fourier transformation $\mathcal{F}$ defined by

$$(\mathcal{F}\tilde{A})[n,\ell;DK,M] = \sum_{m=0}^{M-1}\sum_{k=0}^{DK-1} \tilde{A}_{mk} e^{-j2\pi(m\ell/M - kn/DK)},$$

the discrete Zak transformation [see Eq. (14)], and the shifted and modulated windows $\tilde{\Gamma}_{\Lambda_S;mk}$ [see Eq. (5)], it can be shown that the periodized Gabor transform (9) can also be transformed into a sum-of-products form (see [3] for more details)

$$(\mathcal{F}\tilde{A})\left[n,\ell - \frac{M}{fp}k;DK,M\right] = K\sum_{i=0}^{fq-1}\gamma_{ik}^*[n,\ell]\varphi_i[n,\ell], \tag{17}$$

where

$$\varphi_i[n,\ell] = (\mathcal{Z}\Phi)\left[n+iK,\ell - \frac{rM}{D}i;fpN,\frac{M}{fp}\right],$$

with $i = 0\ldots fq-1$, $k = 0\ldots fp-1$, and where the variables n and ℓ extend over an interval of length DK and M/fp, respectively. The Fourier transform $(\mathcal{F}\tilde{A})[n,\ell;DK,M]$ is completely determined by the functions

$$a_k[n,\ell] = (\mathcal{F}\tilde{A})\left[n,\ell - \frac{M}{fp}k;DK,M\right], \quad k = 0\ldots fp-1.$$

The functions a_k are now combined into the fp-dimensional column vector of functions

$$\underline{a}[n,\ell] = (a_0[n,\ell], a_1[n,\ell], \ldots, a_{fp-1}[n,\ell])^T,$$

and, likewise, the functions φ_i with $i = 0\ldots fq-1$ into the fq-dimensional column vector of functions

$$\underline{\phi}[n,\ell] = (\varphi_0[n,\ell], \varphi_1[n,\ell], \ldots, \varphi_{fq-1}[n,\ell])^T.$$

With the help of the vector functions $\underline{a}$ and $\underline{\phi}$, Eq. (17) can then be expressed in the matrix-vector product

$$\underline{a} = K\boldsymbol{\Gamma}^*\underline{\phi}. \tag{18}$$

The relation (16) applied to an arbitrary vector $\underline{\phi}$ leads to the condition

$$\boldsymbol{G}\boldsymbol{\Gamma}^*\underline{\phi} = \frac{fp}{K}\underline{\phi}.$$

Substitution of Eq. (18) into this expression yields

$$\underline{\phi} = \frac{1}{fp}\boldsymbol{G}\underline{a}. \tag{19}$$

Note that this vector $\underline{\phi}$ is unique, since $\boldsymbol{\Gamma}^*$ is injective, and $\boldsymbol{G}$ is a (proportional) left inverse of $\boldsymbol{\Gamma}^*$. These matrix-vector products (18) and (19) now provide a method to calculate the array $\{\tilde{A}_{mk}\}$ of Gabor expansion coefficients and to reconstruct the signal Φ (and therefore φ) from a given array $\{\tilde{A}_{mk}\}$.

Note that in the separable case, $D = 1$, the sum-of-products forms (15a) and (17) reduce to product forms in the case of critical sampling. In the non-separable case, $D > 1$, the number of elements in the sum-of-product forms is equal to the determinant D and, hence, the sum-of-products forms do not reduce to product forms.

4. CONCLUDING REMARKS

We have considered implementations of Gabor's signal expansion and the Gabor transform for discrete-time signals. Three types of implementations were considered: (i) a filter bank implementation, (ii) shearing of the time-frequency lattice, and (iii) implementation by using the Zak transformation. These three implementations were a direct result of interpreting a non-separable lattice in three different ways. Most likely, other interpretations of a non-separable lattice may lead to other (and different) implementations.

REFERENCES

[1] D. Gabor. Theory of communication. *J. Inst. Elec. Eng.*, 93 (III):429–457, 1946.

[2] H.G. Feichtinger and T. Strohmer, editors. *Gabor analysis and algorithms: theory and applications*. Birkhäuser, Berlin, 1998.

[3] A.J. van Leest. *Non-separable Gabor schemes: their design and implementation*. PhD thesis, Eindhoven University of Technology, Eindhoven, Netherlands, 2001.

[4] M.J. Bastiaans. On the sliding-window representation in digital signal processing. *IEEE Trans. Acoust. Speech Signal Process.*, ASSP-33:868–873, 1985.

[5] M.J. Bastiaans. Gabor's signal expansion and the Zak transform. *Appl. Opt.*, 33:5241–5255, 1994.

[6] C. Hermite. Sur l'introduction des variables continues dans la théorie des nombres. *J. Reine Angew.*, 41:191–216, 1851.

[7] J.R. Durbin. *Modern algebra: an introduction, 3rd edition*. Wiley, New York, 1992.

[8] H.G. Feichtinger, T. Strohmer, and O. Christensen. Group theoretical approach to Gabor analysis. *Optical Engineering*, 34:1697–1704, 1995.

[9] P.P. Vaidyanathan. *Multirate systems and filter banks*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1993.

[10] M.J. Bastiaans and M.C.W. Geilen. On the discrete Gabor transform and the discrete Zak transform. *Signal Process.*, 49:151–166, 1996.

[11] L. Auslander, I.C. Gertner, and R. Tolimieri. The discrete Zak transform application to time-frequency analysis and synthesis of nonstationary signals. *IEEE Trans. Signal Process.*, 39:825–835, 1991.

[12] R.S. Orr. The order of computation for finite discrete Gabor transforms. *IEEE Trans. Signal Process.*, 41:122–130, 1993.