GENERALIZED CONVOLUTION CONCEPT BASED ON DCT

Przemysław Korohoda[1] *and Adam Dąbrowski*[2]

[1]University of Science and Technology, Institute of Electronics, Kraków, Poland,
e-mail: korohoda@uci.agh.edu.pl

[2]Poznań University of Technology, Institute of Control and System Engineering, Poznań, Poland,
e-mail: dabrow@et.put.poznan.pl

ABSTRACT

A generalized approach to the so-called *product filtering* of digital signals valid for a wide class of linear invertible transformations is presented in this paper. Product type of digital filtering consists in multiplication of the transformed signal with some selectivity function in the transform domain and is in this paper interpreted as a generalized convolution process in the primary domain.

Our considerations are based on the observation that the block-wise product filtering of digital signals can be performed by means of multiplication of a block of samples of the transformed signal with some function in a domain of any invertible transformation just in the same way as it is usually done in the frequency domain after the Fourier transformation. The only (sufficient) condition for a suitable forward transformation is the existence of the inverse transformation.

The presented idea of the generalized product filtering and the generalized convolution has been confronted with a family of the DCT transformations and the Karhunen-Loeve transformation. For the DCT-III the convolution formula has been derived.

1. INTRODUCTION

Multiplication of Fourier-transformed signals with some function of frequency is a common realization of the frequency selective filtering (referred in this paper to as the *product filtering*) [6, 7]. An equivalent operation in the primary domain (e.g., in time or space) is the *convolution*. Strictly speaking, in the area of digital signal processing, the discrete Fourier transform (DFT) is typically computed before the block-wise product filtering, thus in such a case the corresponding primary domain operation is the circular convolution.

There exist numerous block transformations other than DFT that are widely used in a large area of digital signal processing. Authors hope that the concept presented in [4] and developed in this paper, based on the idea of the generalized convolution, which may, in fact, be defined for any block reversible transformation, can be useful for investigation of the transform domain product filtering.

One of the well known applications of the procedure called in this paper "product filtering" is in transform coding of images [1]. Signal multiplication in the transform domain – such as in the zone coding of images – may just be considered as a generalized product filtering operation in the sense described above. In such a case it is useful to have a mathematical tool not only for the analysis but also for the realization of such filtering directly in the primary domain. This tool introduced by the authors in [4] is referred to as the *generalized convolution*.

It is used in this paper for comparison of four similar transformations of the DCT family and afterwards for the derivation a convolution formula for the DCT-III, which is similar to the circular convolution (related to the DFT).

Having this generalized convolution concept we are able not only to study the filtering properties of signal processing transformations but we can also realize the corresponding product filtering straightforwardly in the primary domain.

The comparative study has been supplemented with an example for the discrete Karhunen-Loeve transformation, often mentioned in association with the DCT-II.

It should be stressed that the term *generalized convolution* has already been used in several contexts. For example in [5] the well known convolution concept has been extended for primary domain representation of the nonstationary linear filtering in the Fourier domain. In [2] the *generalized convolution theorem* has been used for the needs of cryptography for extension of the technique originally applied to two *n*-Boolean functions and their Walsh transforms toward multiple functions. In [3] the *generalized convolution* has served for efficient distance computation between the representatives of a set separated into parts, where the kernels of all parts were convolved. The convolution generalization proposed in this paper consists, however, in introducing specially selected weights into the original convolution formula and this concept is exploited in our studies.

2. CONVOLUTION AS A WEIGHTED SUM

The well known formula for the circular convolution of two blocks of samples, say x_n and h_n for $n = 0,1,\dots,N-1$, may be rewritten in a general form (proposed in this paper for a definition of the generalized convolution) as the following weighted sum of products of both blocks of samples

$$y_n = \sum_{\substack{\text{for}\\ \text{all}\\ m}} \sum_{\substack{\text{for}\\ \text{all}\\ k}} w_{m,k}^n \cdot x_m \cdot h_k \ , \tag{1}$$

where $w_{m,k}^n$ are weights with three indices (two lower regarding two input blocks of samples and one upper regarding the output block of samples). For each value of n the set of weights forms a weighting matrix of the form

$$W_n = \begin{bmatrix} w_{0,0}^n & \cdots & w_{0,N-1}^n \\ \vdots & \ddots & \vdots \\ w_{N-1,0}^n & \cdots & w_{N-1,N-1}^n \end{bmatrix} . \tag{2}$$

For the circular convolution given by (1) a set of N weigthing matrices is necessary. If $N = 4$ these matrices are

$$W_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

For generalized convolutions (1), i.e., for cases other than the circular convolution and for larger N's it is rather difficult to comprehend the whole set of matrices W. Therefore, to enable recognition of interesting properties hidden in them, the following graphical representation has been used. The grey-level scale is adapted in such a way that the *average grey* represents 0, while *white* indicates the maximum absolute value of all weight coefficients in the set, and *black* represents the negative of this value. For (3) the adequate representation is depicted in Fig. 1.

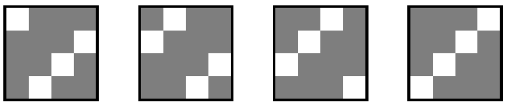

Fig. 1. The set of W matrices (3) depicted in the graphical form

3. COMPUTATION OF THE WEIGHT MATRICES

Assume that $a_{i,j}$ are the elements of the transformation matrix, while $b_{i,j}$ are elements of its inverse. It was proved in [4] that weight coefficients in (1) may be computed in the following way

$$w_{m,k}^n = \sum_{p=0}^{N-1} b_{n,p} \cdot a_{p,m} \cdot a_{p,k} \ .$$

Note, that the weighting matrices W_n must be symmetric. This property immediately results from the fact that changing the order of multiplication in the transform domain does not affect the result. If the transformation is orthogonal, additional property for the weighting matrices is valid, namely

$$\frac{1}{N} \sum_m \sum_k \sum_n \left(w_{m,k}^n\right)^2 = 1 \ .$$

4. COMPARATIVE STUDY

The sets of weights in expression (1) may be studied for a given transformation in many ways leading to the detection of various properties. For a block of samples of length N the set of N square matrices W_n defines the whole set of weights. In general, distribution of their values may be quite complicated, but it is useful to find out, which of them form products with input samples, which affect the result much more than the others. In many cases we can accept approximate but efficiently computed results by zeroing some of the smallest (thus irrelevant) weights. Furthermore, there exist transformations, for which precise convolution formulae can be derived (examples are presented below).

4.1 Introductory example – DAT (Discrete Accumulative Transformation)

As the first example let us consider a very simple non-orthogonal transform, namely the discrete accumulative transformation (DAT), defined for the block of samples as

$$X_n = \sum_{k=0}^{n} x_k \quad for\ n = 0,1,\ldots,N-1 \ . \tag{4}$$

The result may be interpreted as a discrete approximation of integration. The inverse transformation matrix contains the basis vectors that represent the "moving difference". The set of matrices (2) for N=8 is depicted in Fig. 2.

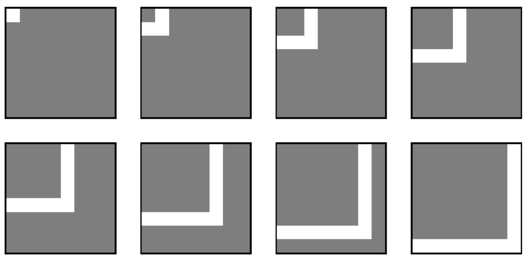

Fig. 2. A set of matrices W_n for the DAT, $N = 8$, depicted with the adapted grey-level scale

Analysing Fig. 2 by taking into account that all non-zero values are ones (white pixels) it is easy to derive the following convolution formula for this transformation

$$y_n = x_n \cdot h_n + \sum_{k=0}^{n-1} \left[x_n \cdot h_k + x_k \cdot h_n \right]. \qquad (5)$$

4.2 Results for the DCT

Four DCT's [8] have been investigated to identify their suitability for the introduced generalized convolution concept. Regularities in the sets of weights were carefully studied. Three types of comparisons were made. Graphical comparison, in which we have studied general location regularities and also indicated possibilities for reduction of the number of weights (cf., Fig. 3). Histogram comparison with logarithmic scales (cf., Fig. 4) for visualization of the statistical distribution and the discretization of weights. Additional quantitative comparison of numbers of the most significant weight values, obtained using an approach of the relative threshold. For each DCT the threshold has been defined with the formula

$$T = 0.1 \cdot \max_{\substack{for\ all \\ k,m,n}} \left(\left| w_{m,k}^{n} \right| \right). \qquad (6)$$

For each set of matrices (2) numbers of weights were computed, whose absolute values exceeded the threshold, indicating, which is the most important portion of weights. The results of the thresholding are presented in Table 1. All three comparison techniques are complementary and should be used simultaneously.

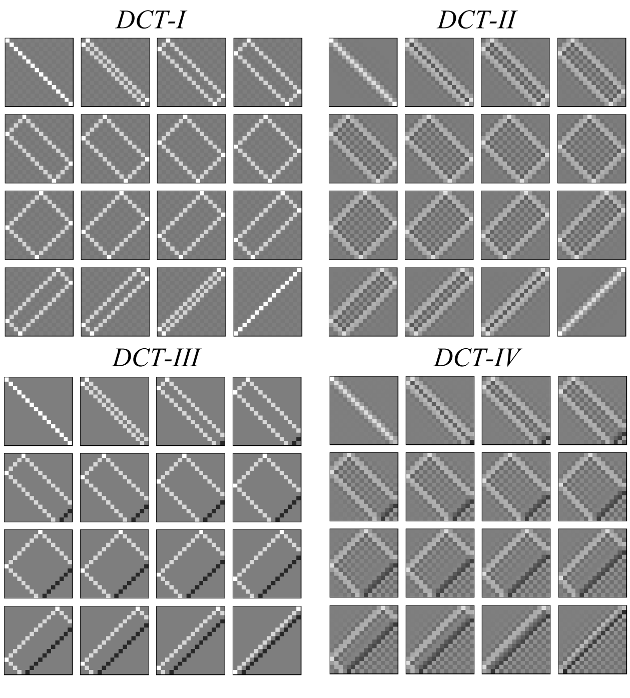

Fig. 3. The sets of W_n matrices for the DCT-family, N=16, depicted with the adapted grey-level scale

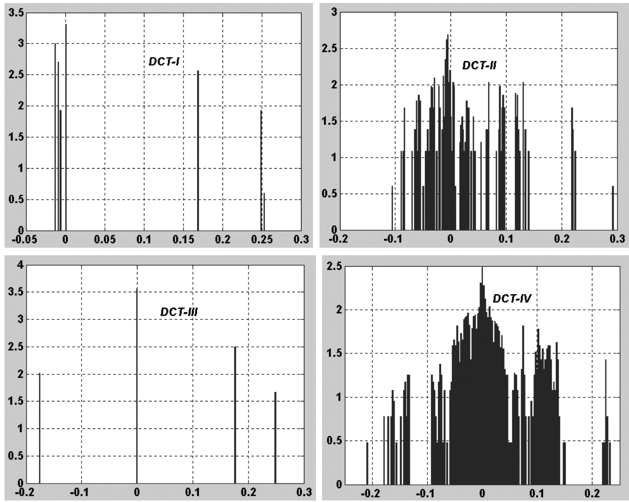

Fig. 4. Log_{10} function of the weight values histograms for the DCT-family, N=16

Table 1. Number of the absolute weight values exceeding the threshold T (6), expressed in % of N^2, for the DCT family and the DFT

N	8	16	32	64
DFT	12.5	6.25	3.12	1.56
DCT-I	28.91	11.04	5.87	3.03
DCT-II	60.94	46.63	23.63	12.03
DCT-III	20.70	11.38	5.96	3.05
DCT-IV	61.72	43.75	23.00	11.91

The presented study indicates that both DCT-II and DCT-IV are not suitable for a straightforward utilization of the generalized convolution concept. However, both exhibit some regularity noticeable in Fig. 3. DCT-I and DCT-III have similar values in Table 1 and the regularities in their graphical representations look very much alike. However, the histograms indicate that the DCT-I has numerous small values, while in the case of the DCT-III there are only four value levels. Note also that both DCT-I and DCT-III have only about two times more important weight coefficients than the DFT, being the optimum in the discussed sense.
The detailed study has shown that for the DCT-III there are N nonzero weight values in W_0 and $(2 \cdot N - 2)$ for the rest of matrices W_n. Exploitation of all these observations leads to the formulation of the convolution expression suitable for the DCT-III, i.e., to the formula

$$y_0 = \frac{1}{\sqrt{N}} \cdot \sum_{k=0}^{N-1} x_k \cdot h_k ,$$

while for n>0: (7)

$$y_n = \frac{1}{\sqrt{N}} \cdot (x_0 \cdot h_n + x_n \cdot h_0)$$
$$+ \quad \frac{1}{\sqrt{2 \cdot N}} \cdot \sum_{k=0}^{n-2} [x_{k+1} \cdot h_{n-k-1} - x_{N+k-n+1} \cdot h_{N-k-1}]$$
$$+ \quad \frac{1}{\sqrt{2 \cdot N}} \cdot \sum_{k=1}^{N-1-n} [x_k \cdot h_{k+n-1} + x_{k+n-1} \cdot h_k]$$

The above formula may be also suitable for the DCT-II as the DCT-III is its inverse.

4.3 Results for the Karhunen-Loeve transformation

The DCT-II is commonly used in the area of data (image) compression as it often quite well approximates the optimum Karhunen-Loeve transformation (KLT). One of its applications is in the zonal coding of images [1], which may be interpreted as a kind of the generalized filtering (in the sense proposed in this paper). Therefore the study has been extended to the example of the KLT computed for the “cameraman” grey-level image. Fig. 5 presents the obtained results that indicates that in this case the utilization of the generalized convolution will be rather difficult and inefficient.

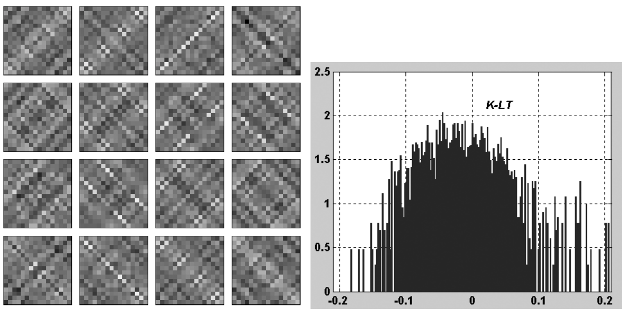

Fig. 5. The sets of W_n matrices for the Karhunen-Loeve transformation, N=16 (blocks 4x4), computed for the image “cameraman” and the $\log_{10}$ function of the weight values histogram

5. CONCLUDING REMARKS

All considered types of the DCT have very similar basis sets and formal definitions. However, they are distinguishingly different if they are compared in the sense of the generalized convolution concept, i.e., according to the general formula (1).

The presented concept of the general convolution is valid for any invertible block transformation and the process we refer in this paper to as the product filtering. It is, in fact, a direct generalization of the classical circular convolution being a primary domain analysis tool for the classical digital product filtering in the frequency domain. The generalized convolution plays the same role for any block invertible transformation, which can be real or complex (even orthogonality is not required).

The properties resulting from our study based on the analysis of the complexity of weighting matrices describing the generalized convolution form an additional indication about the efficiency of the given transformation and its potential to applications in the generalized product filtering.

Further study of the presented idea is in progress and the authors hope that it will lead to suggestions for efficient signal processing procedures and to further generalization of notions typically used exclusively in the case of filtering based on the DFT.

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