SVD-BASED THEOREM FOR DESIGNING VARIABLE DIGITAL FILTERS

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ABSTRACT

Arbitrary desired variable frequency response can be uniformly sampled to construct a multi-dimensional (M-D) complex array. In this paper, we propose a new method called vector-array decomposition (VAD) for decomposing M-D complex array into the products of complex vectors and real arrays. Based on the VAD, the difficult problem of designing variable digital filters can be reduced to some easier sub-problems that require one-dimensional (1-D) constant filter designs and M-D polynomial approximations. Since the VAD-based method can design variable 1-D filters with arbitrary variable magnitude and specified phase responses, the direct design using the weighted least-squares (WLS) method is very general, but requires high computational complexity [3]. This paper proposes a straightforward method for designing variable 1-D filters with arbitrary variable magnitude and specified phase responses, which includes the following steps:

1. Construct an M-D complex array by using the samples of desired variable frequency response;
2. Decompose the M-D complex array into the products of complex vectors and M-D real arrays;
3. Design constant 1-D digital filters to approximate the complex vectors;
4. Find M-D polynomials to approximate real arrays;
5. Interconnect the obtained constant 1-D filters and M-D polynomials to form a variable 1-D filter.

Since the VAD-based method can design variable 1-D filters with arbitrary variable magnitude responses and linear or non-linear phase responses by decomposing the difficult problem (variable filter design) into easier sub-problems involving constant filter designs and polynomial approximations, the design approach is very straightforward.

1. INTRODUCTION

Variable one-dimensional (1-D) digital filters are applicable to the fields where the frequency responses of digital filters are required to be tunable. Most existing methods for designing variable 1-D filters are based on frequency-transformations [1, 2], which can be utilized to design low-pass, bandpass, and highpass filters with variable cut-off frequencies. But when more complicated variable design specifications are given, the frequency-transformation-based methods are not applicable. To design variable filters with arbitrary frequency responses, the direct design using the weighted least-squares (WLS) method is very general, but requires high computational complexity [3]. This paper proposes a straightforward method for designing variable 1-D filters with arbitrary variable magnitude and specified phase responses, which includes the following steps:

- Construct an M-D complex array by using the samples of desired variable frequency response;
- Decompose the M-D complex array into the products of complex vectors and M-D real arrays;
- Design constant 1-D digital filters to approximate the complex vectors;
- Find M-D polynomials to approximate real arrays;
- Interconnect the obtained constant 1-D filters and M-D polynomials to form a variable 1-D filter.

Since the VAD-based method can design variable 1-D filters with arbitrary variable magnitude responses and linear or non-linear phase responses by decomposing the difficult problem (variable filter design) into easier sub-problems involving constant filter designs and polynomial approximations, the design approach is very straightforward.

2. VAD-BASED VARIABLE FILTER DESIGN

Let

\[ H_j(\omega, \Psi_1, \Psi_2, \ldots, \Psi_K) = M_j(\omega, \Psi_1, \Psi_2, \ldots, \Psi_K)e^{j\Phi_j(\omega)} \]

be the ideal variable frequency response (complex-valued), where \( \omega \in [-\pi, \pi] \) is the normalized angular frequency, and

\[ \Psi_k = [\Psi_{kmin}, \Psi_{kmax}], \quad k = 1, 2, \ldots, K \]

are the spectral parameters for tuning the magnitude response \( \Phi_j(\omega) \) is arbitrary but not tunable. By sampling the parameters \( \Psi_1, \Psi_2, \ldots, \Psi_K \) uniformly, we obtain the samples

\[ \omega(l) = -\pi + \frac{2\pi(l-1)}{L-1}, \quad \Psi_k(m_k) = \Psi_{kmin} + \frac{(\Psi_{kmax} - \Psi_{kmin})(m_k - 1)}{M_k - 1} \]

where

- \( l = 1, 2, \ldots, L \)
- \( k = 1, 2, \ldots, K, \quad m_k = 1, 2, \ldots, M_k \).

By using the samples of \( H_j(\omega, \Psi_1, \Psi_2, \ldots, \Psi_K) \), we can construct a \((K+1)\)-D complex array \( \tilde{A} \) as

\[ \tilde{A} = [\tilde{a}(l, m_1, m_2, \ldots, m_K)] \]

Our objective here is to decompose the array \( \tilde{A} \) as

\[ \tilde{A} \approx \sum_{i=1}^{r} C_i \oplus R_i \]  \hspace{1cm} (1)

under the strict constraints

- **Constraint-I**: \( C_i \) being complex conjugate-symmetrical vectors, and \( R_i \) being K-D real-valued arrays
- **Constraint-II**: mean-squared decomposition error

\[ E_r = \left\| \tilde{A} - \sum_{i=1}^{r} C_i \oplus R_i \right\|^2 \] \hspace{1cm} (2)

being minimum, where the notation \( \oplus \) denotes a special product between the vector \( C_i \) and K-D array \( R_i \) as

\[ \tilde{a}(l, m_1, m_2, \ldots, m_K) \approx \sum_{i=1}^{r} C_i(l) \circ R_i(m_1, m_2, \ldots, m_K) \]
Constraint-1 is imposed for the reason that the complex vectors \( C \) can be regarded as the desired frequency responses of constant 1-D filters, and that the K-D real arrays \( R_i \) can be regarded as the desired values of K-D polynomials of the spectral parameters \( \Psi_1, \Psi_2, \ldots, \Psi_K \), respectively. Moreover, the Constraint-2 is for minimizing the decomposition error \( E_r \) for a given number \( r \). Since the complex M-D array \( A \) is approximately decomposed into the sum of the special products of complex vectors and real arrays, thus the decomposition is called vector-array decomposition (VAD). Once the VAD (1) is obtained, the complex vectors \( C \) can be approximated by designing constant 1-D filters \( H_i(z) \), and the K-D real arrays \( R_i \) can be approximated by using K-D polynomials \( P_i(\Psi_1, \Psi_2, \ldots, \Psi_K) \), respectively, then cascading \( H_i(z) \) with \( P_i(\Psi_1, \Psi_2, \ldots, \Psi_K) \) as Fig. 1 can yield a variable filter. Since the 1-D filters \( H_1(z), H_2(z), \ldots, H_r(z) \) are fixed, and K-D polynomials \( P_i(\Psi_1, \Psi_2, \ldots, \Psi_K) \) are variable, thus Fig. 1 represents a variable filter consisting of constant part and variable part. In signal processing applications, only the variable part needs to be tuned, and the constant part is always fixed. Therefore, the VAD-based approach can indirectly obtain a variable filter through designing 1-D constant filters and approximating K-D polynomials, which are easier to solve.

3. SVD-BASED VAD ALGORITHM

Since the VAD (1) is very special and difficult to perform due to the two strict constraints, such a decomposition does not exist so far. In this section, we theoretically prove that the singular-value decomposition (SVD) can be successfully applied to achieve the goal.

First, we convert the complex array \( \hat{A} \) to matrix \( A \) as

\[
\hat{A} = [\hat{a}(l, m_1, m_2, \ldots, m_K)] \Longrightarrow A = [a(l, m)]
\]

based on the one-to-one index mappings

\[
(m_1, m_2, \ldots, m_K) \rightarrow m
\]

such that

\[
a(l, m) = \hat{a}(l, m_1, m_2, \ldots, m_K)
\]

Then, the SVD is directly applied to decompose \( A \) as

\[
A = U \Sigma V^T = \sum_{i=1}^{l} \sigma_i u_i v_i^T = \sum_{i=1}^{l} \hat{u}_i \hat{v}_i^T
\]

where \( I \) is the rank of \( A \).

\[
U = [u_1 \ u_2 \ \ldots \ u_l] \quad V = [v_1 \ v_2 \ \ldots \ v_r]
\]

are unitary matrices, and

\[
\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_l)
\]

is a diagonal matrix with its singular values \( \sigma_i \) \((\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_l \geq 0)\) as its diagonal entries, and

\[
\hat{u}_i = \sqrt{\sigma_i} u_i, \quad \hat{v}_i = \sqrt{\sigma_i} v_i.
\]

In this paper, we theoretically prove the following new theorem that leads to an optimal SVD-based VAD algorithm.

**Theorem:** The SVD in (4) can always be manipulated to the form

\[
A = \sum_{i=1}^{R} \hat{u}_i \hat{v}_i^T
\]

such that the vectors \( \hat{u}_i \) are complex conjugate-symmetric about their mid-points, and \( \hat{v}_i \) are real-valued, where \([\cdot]^T\) denotes the transpose of \([\cdot]\).

Based on this new theorem, if we let

\[
C_i = \hat{u}_i
\]

and convert the real-valued vector \( \hat{v}_i \) to K-D array \( R_i \) as

\[
\hat{v}_i = [\hat{v}_i(m)] \Rightarrow R_i = [R_i(m_1, m_2, \ldots, m_K)]
\]

in the reverse order of the index mapping (3), then

\[
\hat{A} = \sum_{i=1}^{R} C_i \otimes R_i
\]

satisfies the Constraint-1. Furthermore, if we truncate the last few terms in (6) and approximate \( A \) as

\[
\hat{A} \approx \hat{A} = \sum_{i=1}^{R} C_i \otimes R_i
\]

the SVD can guarantee that the mean-squared error

\[
\|\hat{A} - \hat{A}\|^2
\]

is always minimum for any \( R < L \), thus the decomposition (7) also satisfies the Constraint-2. Therefore, the decomposition (7) is the optimal solution for the VAD.

In considering the hardware cost for implementing the resulting variable filter, it is desired to use a small number \( r \) to achieve high design accuracy, but too small \( r \) will lead to poor design accuracy, thus the trade-off between the hardware cost and design accuracy must be taken into account.

Next, let us prove the property stated in (5). Without loss of generality, we assume that \( L \) is an even number, Then the matrix \( A \) can be partitioned as

\[
A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}
\]

By using the reversal permutation matrix

\[
I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

and noting the relation

\[
A_2 = \hat{I} A_1
\]

where \([\cdot]^\ast\) denotes the complex-conjugate of \([\cdot]\), we get

\[
A = \begin{bmatrix} \hat{A}^T & 0 \end{bmatrix}, \quad \hat{A}^T = \begin{bmatrix} \hat{A}^T & \hat{A}^T \end{bmatrix}.
\]
Thus
\[
\begin{align*}
\bar{A}^T &= \begin{bmatrix} A \bar{A}_1^T & A \bar{A}_1^T \hat{i} \\
\hat{i} A \bar{A}_1^T & \hat{i} A \bar{A}_1^T \end{bmatrix} \\
&= \begin{bmatrix} B & C \hat{i} \\
\hat{i} C & \hat{i} B \end{bmatrix} \hat{i}
\end{align*}
\]
where
\[
B = A \bar{A}_1^T \quad C = A \bar{A}_1^T
\]
and \(I\) is an identity matrix. From the SVD (4) we obtain
\[
\bar{A}^T u_i = \sigma_i^2 u_i.
\]
Partitioning the vector \(u_i\) to upper and lower part as
\[
u_i = \begin{bmatrix} u_{i1} \\
u_{i2}
\end{bmatrix}
\]
we have
\[
\begin{bmatrix} B & C \hat{i} \\
\hat{i} C & \hat{i} B \end{bmatrix} \begin{bmatrix} u_{i1} \\
u_{i2}
\end{bmatrix} = \sigma_i^2 \begin{bmatrix} u_{i1} \\
u_{i2}
\end{bmatrix}
\]
which leads to
\[
\begin{align*}
Bu_{i1} + Cu_{i2} &= \sigma_i^2 u_{i1} \\
\hat{i}Bu_{i1} + \hat{i}Cu_{i2} &= \sigma_i^2 u_{i2}
\end{align*}
\]
and
\[
\begin{align*}
Bu_{i1} + Cu_{i2} &= \sigma_i^2 u_{i1} \\
\hat{i}Bu_{i1} + \hat{i}Cu_{i2} &= \sigma_i^2 u_{i2}
\end{align*}
\]
Hence,
\[
\begin{bmatrix} B & C \hat{i} \\
\hat{i} C & \hat{i} B \end{bmatrix} \begin{bmatrix} \mu_{i1} \\
\mu_{i2}
\end{bmatrix} = \sigma_i^2 \begin{bmatrix} \mu_{i1} \\
\mu_{i2}
\end{bmatrix}.
\]
Furthermore, multiplying both sides of (10) by \(I\) obtains
\[
\begin{bmatrix} B & C \hat{i} \\
\hat{i} C & \hat{i} B \end{bmatrix} \begin{bmatrix} \mu_{i1} \\
\mu_{i2}
\end{bmatrix} = \sigma_i^2 \begin{bmatrix} \mu_{i1} \\
\mu_{i2}
\end{bmatrix}.
\]
From (11) and (12) it is known that both
\[
\begin{bmatrix} u_{i1} \\
u_{i2}
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mu_{i1} \\
\mu_{i2}
\end{bmatrix}
\]
are the orthonormal eigen-vectors corresponding to the eigen-value \(\sigma_i^2\), thus
\[
\begin{bmatrix} u_{i1} \\
u_{i2}
\end{bmatrix} = e^{j\theta_i} \begin{bmatrix} \mu_{i1} \\
\mu_{i2}
\end{bmatrix}
\]
where \(\theta_i\) is arbitrary angle. As a result, we have
\[
u_{i1} = e^{j\theta_i} \mu_{i1}
\]
and
\[
u_{i2} = e^{j\theta_i} \mu_{i2}
\]
Obviously, the upper and lower parts of the complex vector
\[
\bar{\mu}_i = \begin{bmatrix} e^{j\theta_i} \mu_{i1} \\
e^{-j\theta_i} \mu_{i2}
\end{bmatrix}
\]
are conjugate-symmetrical. Furthermore, from the SVD (4) we have
\[
V = \bar{A}^T U \Sigma^{-1}
\]
i.e.,
\[
\nu_i = \sigma_i^{-1} \bar{A}^T u_i = \sigma_i^{-1} e^{j\theta_i} \bar{A} \mu_i
\]
\[
= e^{j\theta_i} \sigma_i^{-1} \left[ e^{j\theta_i} \bar{A}^T \mu_{i1} + e^{-j\theta_i} \bar{A}^T \mu_{i2} \right]
\]
\[
= e^{j\theta_i} \nu_i
\]
Obviously, the vector
\[
\nu_i = \sigma_i^{-1} \left[ e^{j\theta_i} \bar{A}^T \mu_{i1} + e^{-j\theta_i} \bar{A}^T \mu_{i2} \right]
\]
is real-valued. By considering
\[
\bar{u}_i \nu_i^T = \bar{\mu}_i \bar{\nu}_i^T
\]
we can replace \(\bar{u}_i \nu_i^T\) in (4) by \(\bar{\mu}_i \bar{\nu}_i^T\) and obtain (5), where
\[
\begin{bmatrix}
\bar{\mu}_i = \sqrt{\sigma_i} \mu_i \\
\bar{\nu}_i = \sqrt{\sigma_i} \nu_i
\end{bmatrix}
\]
4. DESIGN EXAMPLE
This section provides an example to illustrate the effectiveness of the VAD-based design method.
[Design Example]: The ideal frequency response of lowpass filter with variable magnitude response and constant fractional-delay (FD) is given by
\[
H_f(\omega, \Psi_1, \Psi_2) = M_f(\omega, \Psi_1, \Psi_2) e^{j\Phi_f(\omega)}
\]
where

\[ M_f(\omega, \Psi_1, \Psi_2) = \begin{cases} 
1 & 0 \leq |\omega| \leq \omega_p \\
\frac{\omega_p - \omega}{\omega_p - \omega_p} & \omega_p \leq |\omega| \leq \omega_s \\
1 & \omega_s \leq |\omega| \leq \pi 
\end{cases} \]  (18)

\[ \omega_p = 0.25\pi + \Psi_1, \quad \Psi_1 \in [-0.16\pi, 0.16\pi] \]
\[ \omega_s = 0.75\pi + \Psi_2, \quad \Psi_2 \in [-0.08\pi, 0.08\pi] \]
\[ \Phi_f(\omega) = -t\omega, \quad t = 10.9. \]

The spectral parameters \( \Psi_1 \) and \( \Psi_2 \) control the variable passband width and variable stopband width independently. By taking \((L, M_1, M_2) = (101, 9, 5)\), we first construct a 3-D complex array \( \tilde{\mathbf{A}} \) whose size is \( 101 \times 9 \times 5 \), then the complex array \( \tilde{\mathbf{A}} \) is decomposed by the proposed VAD method. Table 1 lists the VAD errors when increasing the channel number \( r \). To perform the variable filter design, we select \( r = 4 \), thus the decomposition error is \( 1.5970\% \). Furthermore, the orders of the 4 constant filters \( H_f(z) \) are \{23, 23, 22, 24\}, and the degrees of the 4 polynomials \( \Psi_1^{(i)}(\Psi_1, \Psi_2) \) are \{(2,2), (3,2), (3,3), and (4,2)\}. To evaluate the final design accuracy, we take \((L, M_1, M_2) = (601, 41, 5)\). Fig. 2 shows the resulting variable magnitude response for \( \Psi_2 = -0.08\pi \). The average of the variable magnitude response errors for different \( \Psi_1 \) and \( \Psi_2 \) values is \( 1.5526\% \), and that of passband FD error is \( 0.0247\% \). Moreover, the total normalized RMS error of the obtained variable frequency response is \( 1.5915\% \). Therefore, considerably high design accuracy has been achieved.

5. CONCLUSION

In this paper, we have proposed a VAD-based design method for reducing the difficult problem of designing a variable filter to easier sub-problems that require constant filter designs and M-D polynomial approximations. Compared with the existing frequency transformation-based methods, the VAD-based method can obtain variable filters with arbitrary tunable magnitude responses and linear or non-linear phases. The VAD-based design method is particularly efficient for complicated design specifications, where direct designs like the WLS method are extremely difficult [3].

REFERENCES


Table 1: Decomposition Errors

<table>
<thead>
<tr>
<th>Channel ( r )</th>
<th>VAD [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.6902</td>
</tr>
<tr>
<td>2</td>
<td>5.5747</td>
</tr>
<tr>
<td>3</td>
<td>2.6092</td>
</tr>
<tr>
<td>4</td>
<td>1.5970</td>
</tr>
<tr>
<td>5</td>
<td>1.1838</td>
</tr>
<tr>
<td>6</td>
<td>0.7806</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Fig. 2. Variable magnitude response for \( \Psi_2 = -0.08\pi \).