DESIGN OF VARIABLE FRACTIONAL-DELAY DIGITAL FILTERS USING WEIGHTED-LEAST-SQUARES SINGULAR-VALUE-DECOMPOSITION

Tian-Bo Deng

Department of Information Science
Faculty of Science, Toho University
Miyama 2-2-1, Funabashi, Chiba 274-8510, Japan
E-mail: deng@is.sci.toho-u.ac.jp

ABSTRACT

This paper proposes a weighted-least-squares singular-value-decomposition (WLS-SVD) method and shows that the problem of designing one-dimensional (1-D) variable fractional-delay (VFD) digital filter can be elegantly reduced to the easier sub-problems that involve 1-D constant filter designs and 1-D polynomial approximations. By utilizing the WLS-SVD of the variable design specification, we prove that both 1-D constant filters and 1-D polynomials possess either symmetry or anti-symmetry simultaneously. Therefore, a VFD filter can be efficiently obtained by designing 1-D constant filters with symmetrical or anti-symmetrical coefficients and performing 1-D symmetrical or anti-symmetrical polynomial approximations. Our computer simulations have shown that the WLS-SVD design can achieve much higher design accuracy with significantly reduced filter complexity than the existing WLS design method.

1. INTRODUCTION

The variable digital filters with variable fractional-delay (VFD) responses have been found useful in the applications such as timing adjustment in digital receivers, speech coding and synthesis, and other applications whenever new sample values at arbitrary time instants between the existing discrete-time samples need to be interpolated [1]-[6]. Among the developed design methods, the weighted-least-squares (WLS) methods can yield more accurate designs with reduced filter complexity than other methods such as the Lagrange interpolator [3]-[6].

This paper proposes a very straightforward method for designing VFD filters based on the weighted-least-squares singular-value-decomposition (WLS-SVD) of the desired variable frequency response. We theoretically prove that the proposed WLS-SVD method generates very interesting results that are complex vectors and real vectors with special symmetries. The key point is that the resulting complex vectors and real vectors from the WLS-SVD are “meaningful” in the filter design sense such that

- the complex vectors can be regarded as the desired frequency responses of one-dimensional (1-D) constant FIR filters (sub-filters) with either symmetrical coefficients or anti-symmetrical coefficients; and
- the real vectors can be regarded as the desired values of either symmetrical (even-degree) or anti-symmetrical (odd-degree) 1-D polynomials, and the aforementioned 1-D filters and 1-D polynomials possess identically the same symmetry (mirror-image symmetry or anti-symmetry) simultaneously.

Based on the WLS-SVD results, one with minimum knowledge of digital filters can design a VFD filter easily since the whole design process just includes the least-squares design of constant 1-D sub-filters and the least-squares curve fitting for 1-D polynomial approximations. Our computer simulations have shown that the WLS-SVD method can achieve higher design accuracy with significantly reduced filter complexity than the normal WLS design technique.

2. SVD-BASED THEOREM AND DESIGN

The objective of designing a VFD filter is to find a variable transfer function $H(z, p)$ such that it can approximate the desired variable frequency response

$$H_d(\omega, p) = e^{-j\alpha p}$$

as accurately as possible in the region

$$\omega \in [-\alpha\pi, \alpha\pi], \quad 0 < \alpha < 1$$
$$p \in [-0.5, 0.5]$$

where $\omega$ is the normalized angular frequency, $\alpha$ is a fixed number for specifying the interested passpand, and $p$ is the desired variable fractional-delay (VFD). By uniformly sampling the parameters $\omega$ and $p$ as

$$\omega_l = -\alpha\pi + \frac{2\alpha\pi(l - 1)}{L - 1}, \quad l = 1, 2, \ldots, L$$
$$p_m = -0.5 + \frac{m - 1}{M - 1}, \quad m = 1, 2, \ldots, M$$

we can obtain the corresponding samples

$$H_d(\omega_l, p_m) = e^{-j\alpha p_m}$$

which can be used to construct a complex-valued matrix

$$\tilde{H} = [H_d(\omega_l, p_m)]$$

whose size is $L$-by-$M$. The singular value decomposition (SVD) of the complex-valued matrix $\tilde{H}$ with rank $r$ can be written as

$$\tilde{H} = USV^* = \sum_{i=1}^r \sigma_i u_i v_i^* = \sum_{i=1}^r u_i v_i^*$$

where $U$ and $V$ are unitary matrices

$$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix}, \quad V = \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix}$$
and 
\[ \Sigma = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r) \]
is a diagonal matrix with the distinctive singular values \( \sigma_i \) as its diagonal entries, where the singular values are arranged in the decreasing order as \( \sigma_1 > \sigma_2 > \cdots > \sigma_r > 0 \), and

\[
\begin{align*}
\tilde{u}_i &= \sqrt{\sigma_i} u_i \\
\tilde{v}_i &= \sqrt{\sigma_i} v_i
\end{align*}
\] (5)

Here, we present a very important theorem that is closely related to the SVD-based VFD filter design.

**Theorem:** The SVD (3) can always be manipulated to

\[ \hat{H} = \sum_{i=1}^{K} \tilde{\mu}_i \tilde{v}_i^T \] (6)
such that either (Case-I)

- \( \tilde{\mu}_i \) is a real-valued vector whose entries are symmetrical about its mid-entry and
- \( \tilde{v}_i \) is also a real-valued vector whose entries are symmetrical about its mid-entry;

or (Case-II)

- \( \tilde{\mu}_i \) is a pure imaginary vector whose imaginary parts are anti-symmetrical about its mid-entry and
- \( \tilde{v}_i \) is a real-valued vector whose entries are also anti-symmetrical about its mid-entry.

In Case-I, the resulting vectors \( \tilde{\mu}_i \) and \( \tilde{v}_i \) are meaningful in the filter design sense because

- \( \tilde{\mu}_i \) can be regarded as a vector containing the desired frequency response samples of a 1-D constant FIR filter (sub-filter) with symmetrical coefficients, and thus a 1-D zero-phase sub-filter with symmetrical coefficients can be designed for approximating the vector \( \tilde{\mu}_i \);
- \( \tilde{v}_i \) can be regarded as a vector containing the desired samples of a 1-D symmetrical (even-degree) polynomial of \( p \), and thus an even-degree polynomial can be obtained for approximating the vector \( \tilde{v}_i \).

In Case-II, the resulting vectors \( \tilde{\mu}_i \) and \( \tilde{v}_i \) are also meaningful in the filter design sense because

- \( \tilde{\mu}_i \) can be regarded as a vector containing the desired frequency response samples of a 1-D constant FIR filter (sub-filter) with anti-symmetrical coefficients, and thus a 1-D \( \pi/2 \)-phase sub-filter with anti-symmetrical coefficients can be designed for approximating the vector \( \tilde{\mu}_i \);
- \( \tilde{v}_i \) can be regarded as a vector containing the desired samples of a 1-D anti-symmetrical (odd-degree) polynomial of \( p \), and thus an odd-degree polynomial can be obtained for approximating the vector \( \tilde{v}_i \).

3. WEIGHTED-LEAST-SQUARES SVD

The SVD (6) produces a very important result that the squared decomposition error \( \| \epsilon_H^2 \| \) is minimum, where

\[ \epsilon_H = \hat{H} - \sum_{i=1}^{K} \tilde{\mu}_i \tilde{v}_i^T \] (7)
is the residual error matrix from the SVD, and \( K \leq r \). Generally speaking, the absolute error distribution \( \epsilon_H \) is not uniform (flat), i.e., the absolute errors in some region are small, but those in other region are relatively large. To make the decomposition errors as flat as possible in the whole region shown in (2), a weighted-least-squares (WLS) singular value decomposition (WLS-SVD) method must be developed. Our objective here is to find the optimal vectors \( \tilde{\mu}_i \) and \( \tilde{v}_i \) such that the WLS decomposition error

\[ J(\tilde{\mu}_i, \tilde{v}_i) = \| W \odot \epsilon_H^2 \| \]

is minimized, where \( W = [W(l,m)] \) is a weighting matrix whose elements \( W(l,m) \) are positive, and the notation \( \odot \) denotes the Hadamard product, i.e., the element-wise product between two matrices, which is often called Schur product. As in the constant 2-D filter design, the general form weighting matrix \( W \) leads to a non-linear minimization problem that is rather difficult to solve. By re-writing the error function (8) as

\[ J(\tilde{\mu}_i, \tilde{v}_i) = \| W \odot (\hat{H} - \sum_{i=1}^{K} \tilde{\mu}_i \tilde{v}_i^T) \|^2 \] (9)

if we assume the weighting function \( W(\omega, p) \) is separable, i.e.,

\[ W(\omega, p) = w_1(\omega)w_2(p) \] (10)

with

\[ w_1(-\omega) = w_1(\omega) \]

\[ w_2(-p) = w_2(p) \]

then the weighting matrix can be expressed as

\[ W = w_1w_2^T \]

where

\[ w_1 = [w_1(\omega_1)] = \begin{bmatrix} w_1(1) \\ \vdots \\ w_1(L) \end{bmatrix}, \quad w_2 = [w_2(p_m)] = \begin{bmatrix} w_2(1) \\ \vdots \\ w_2(M) \end{bmatrix} \]
i.e.,

\[ W(l,m) = w_1(l)w_2(m) \].

If we let

\[ \hat{H} = W \odot \hat{H} \]

whose SVD generates

\[ \hat{H} \approx \sum_{i=1}^{K} \tilde{\mu}_i \tilde{v}_i^T. \] (11)

It is easy to prove that the vectors \( \tilde{\mu}_i \) and \( \tilde{v}_i \) also satisfy the symmetries stated in the theorem (6). Since

\[ W \odot \sum_{i=1}^{K} \tilde{\mu}_i \tilde{v}_i^T = (w_1w_2^T) \odot \sum_{i=1}^{K} \tilde{\mu}_i \tilde{v}_i^T \]

\[ = \sum_{i=1}^{K} (w_1 \odot \tilde{\mu}_i) (w_2 \odot \tilde{v}_i)^T \] (12)
if we set
\[ \begin{align*}
\hat{\mu}_i &= w_1 \odot \hat{\mu}_i \\
\hat{\nu}_i &= w_2 \odot \hat{\nu}_i
\end{align*} \tag{13} \]
then the optimal vectors \( \hat{\mu}_i \) and \( \hat{\nu}_i \) in (8) can be easily determined as
\[ \begin{align*}
\hat{\mu}_i &= \hat{\mu}_i \odot w_1 \\
\hat{\nu}_i &= \hat{\nu}_i \odot w_2
\end{align*} \tag{14} \]
where the notation \( \odot \) denotes the element-wise division between two vectors. It should be noticed here that the resulting vectors (14) minimize the weighted decomposition error (8). Furthermore, in the VFD filter case, since our interested region (2) excludes “don’t care band”, thus
\[ w_1(p) \neq 0 \quad w_2(p) \neq 0 \]
which makes the element-wise divisions (14) feasible. Our computer simulations have verified the effectiveness of the separable weighting function in the WLS-SVD design for suppressing the error peaks and shaping the design errors almost uniformly.

### 4. DESIGN EXAMPLE

In this section, we present a design example to illustrate the effectiveness of the proposed WLS-SVD-based design methods, and compare the new approach with the existing WLS design.

The variable design specification is given by (1) with \( \alpha = 0.9 \), which has been typically used in the literature [3, 4, 5, 6]. In addition, the weighting functions \( w_1(\omega) \) and \( w_2(p) \) are chosen as
\[ \begin{align*}
w_1(\omega) &= \begin{cases} 0.3693 & \text{for } |\omega| \in [0, 0.55\pi] \\
0.4882 & \text{for } |\omega| \in (0.55\pi, 0.85\pi] \\
1 & \text{for } |\omega| \in [0.85\pi, (\alpha + \delta)\pi] \end{cases} \tag{15} \\
w_2(p) &= \begin{cases} 0.6535 & \text{for } |p| \in [0, 0.4] \\
1 & \text{for } |p| \in [0.4, 0.5] \end{cases} \tag{16} \]

such that the error distribution \( e(p) \)— from the WLS-SVD is almost flat as shown in Fig. 1. To construct the complex matrix \( \hat{H} \), we sample
\[ \omega \in \left[-(\alpha + \delta)\pi, (\alpha + \delta)\pi\right] \]
with step size \( (\alpha + \delta)\pi/100 \), and sample
\[ p \in [-0.5, 0.5] \]
with step size 1/30, thus the size of \( \hat{H} \) is 201-by-31. Here, the small number \( \delta, \delta = 0.0014 \), is added for suppressing the error jump around the edge frequencies \( \omega = \pm \alpha \pi \) at some extent. Then the proposed WLS-SVD is performed, which results in the normalized decomposition error given in Table 1, where the normal SVD errors are also listed. Finally, the resulting vectors \( \hat{\mu}_i \) and \( \hat{\nu}_i \) \( (i = 1, 2, \cdots, 6) \) from the WLS-SVD are approximated separately. The orders \( N \) for the 6 sub-filters are \( \{25, 40, 25, 32, 18, 19\} \), and the degrees of the 1-D polynomials are \( \{5, 4, 5, 4, 4, 4\} \).

To evaluate the final VFD filter design accuracy, the normalized root-mean-squared error between the desired and actual variable frequency responses defined by
\[ e_2 = \left[ \frac{\int_{-\alpha \pi}^{\alpha \pi} e(\omega, p)^2 d\omega dp}{\int_{-\alpha \pi}^{\alpha \pi} |H_d(\omega, p)|^2 d\omega dp} \right]^{1/2} \times 100\% \tag{17} \]
and the maximum absolute error
\[ e_{max} = 20 \log_{10} \max \{|e(\omega, p)|, \omega \in [-\alpha \pi, \alpha \pi], p \in [-0.5, 0.5]\} \tag{18} \]
(dB) are used, where
\[ e(\omega, p) = H(\omega, p) - H_d(\omega, p). \]
In evaluating \( e_2 \) and \( e_{max} \), defined above, the frequency
\[ \omega \in [-\alpha \pi, \alpha \pi] \]
is uniformly sampled at the step size \( \alpha \pi/200 \), and the fractional-delay
\[ p \in [-0.5, 0.5] \]
is uniformly sampled at the step size 1/60 such that the discrete points in-between the ones used for decomposition can also be checked.

To compare the proposed WLS-SVD design with the WLS method [4] in terms of design accuracy and filter complexity, the design errors and the numbers of total coefficients are listed in Table 2, which shows that the WLS-SVD design requires much less filter coefficients (188) than the WLS method (488) to achieve much higher design accuracy, especially the maximum absolute error \( e_{max} \) of the variable frequency response is significantly reduced. Moreover, the maximum deviation \( e_{max} \) of the VFD response from the WLS-SVD design is 0.00026, which is much smaller than that of the WLS method (0.00214).

Fig. 2 and Fig. 3 show the passband VFD response and its absolute errors. From the design results it can be concluded that the WLS-SVD method can achieve considerably accurate designs with much less filter complexity than the existing WLS design method.

### 5. CONCLUSION

The main contributions of the paper can be summarized as follows.

- A WLS-SVD method has been proposed for extending the normal SVD-based design to more general case where one seeks to suppress the design error peaks and make the design errors almost uniformly distributed. Therefore, the normal SVD-based design can be considered as a special case of the WLS-SVD design. However, it should be noted that the WLS-SVD design can suppress the error peaks, but the total squared error \( e_2 \) is increased slightly. If one wants to minimize the total squared error between the desired and actual variable frequency responses, then the normal SVD-based design is preferred. On the other hand, if one wants to suppress the error peaks in some region and get almost uniform design errors, then the proposed WLS-SVD design is preferred;
Table 1: SVD and WLS-SVD Errors (%)

<table>
<thead>
<tr>
<th>Term (K)</th>
<th>SVD</th>
<th>WLS-SVD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>44.154394</td>
<td>59.215601</td>
</tr>
<tr>
<td>2</td>
<td>9.447473</td>
<td>13.330488</td>
</tr>
<tr>
<td>3</td>
<td>0.111424</td>
<td>0.133022</td>
</tr>
<tr>
<td>4</td>
<td>0.008167</td>
<td>0.009410</td>
</tr>
<tr>
<td>5</td>
<td>0.000494</td>
<td>0.000559</td>
</tr>
<tr>
<td>6</td>
<td>0.000025</td>
<td>0.000001</td>
</tr>
<tr>
<td>7</td>
<td>0.000001</td>
<td>0.000000</td>
</tr>
<tr>
<td>8</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Table 2: Design Errors and Filter Complexities

<table>
<thead>
<tr>
<th>Method</th>
<th>$\varepsilon_2$ (%)</th>
<th>$\varepsilon_{\text{max}}$ (dB)</th>
<th>$\varepsilon_{p\text{Max}}$</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>SVD</td>
<td>0.000467</td>
<td>-91.66</td>
<td>0.00033</td>
<td>185</td>
</tr>
<tr>
<td>WLS-SVD</td>
<td>0.000555</td>
<td>-98.29</td>
<td>0.00020</td>
<td>188</td>
</tr>
<tr>
<td>WLS [4]</td>
<td>0.000675</td>
<td>-91.83</td>
<td>0.00214</td>
<td>488</td>
</tr>
</tbody>
</table>

- A typical design example has been given to illustrate that the WLS-SVD-based method can achieve better design accuracy with much less filter complexity than the existing WLS method.

REFERENCES


