COINCIDENCE ANALYSIS OF POINT PROCESSES

Bernard Picinbono and Cherif Bendjaballah

Laboratoire des Signaux & Systèmes
École Supérieure d'Électricité, 91192 Gif sur Yvette (France)
bernard.picinbono@lss.supelec.fr;cherif.bendjaballah@lss.supelec.fr

A point process can be fully described by its coincidence probabilities but the expressions giving counting and life time statistics in terms of coincidence are so complicated that they cannot be used practically. However an analysis limited to bicoincidence yields already important properties of the time behavior of point processes. In particular it explains entirely the bunching or antibunching effect of point processes. In the case of compound Poisson processes, the bicoincidence probability is a correlation function and the bunching effect is simply related to the maximum at the origin of any correlation function. However, there are many other point processes that also show antibunching. Thus the relation between coincidence and correlation must be analyzed more carefully. Similar problems appear when relating the bunching effect to some statistical properties of counting. In order to clarify the question, various examples of point processes are discussed and numerical calculations illustrate the results.

1. INTRODUCTION

Point processes (PP) play an important role in many areas of Physics, Signal Processing, and Information Sciences. They appear on a microscopic scale in the description of particle emission. For example, optical communication at a very low level of intensity requires the use of statistical properties of photons or photoelectrons [1][2]. On the other hand many areas such as traffic problems or computer communications require the use of PP statistics.

There are two usual approaches to describe PPs. In the first counting procedures in one or several non-overlapping time intervals are used. In the second time intervals between points measurements are used. For example the *life time* is the time interval between successive points and it is the basis of the definition of renewal processes characterized by the fact that successive life times are independent random variables (RV). The relationship between these two approaches is a classical topic in the PP theory.

The purpose of this paper is to study another approach in the description of PPs. Its basis is the concept of coincidence probabilities. They can be considered as a limit aspect of counting in which the time intervals of counting are so small that they can only contain one or zero points.

It is not a new concept and coincidence experiments were discussed long time ago in the framework of Nuclear Physics or of Statistical Optics. Furthermore, on a theoretical point of view it is known that a PP can be completely defined by the set of all its coincidence probabilities [3]. Thus we are not at all exploring a new domain. However, the expressions giving counting probabilities or life time statistics in terms of coincidence probabilities are in general so complicated that they cannot be used practically. Thus our aim is to discuss the relationships between bicoincidence, which are the simpler kind of coincidence, and some statistical properties of counting or life time.

The interest of bicoincidence is similar to that of correlation of random signals. The correlation function yields an idea on the time behavior of a signal because it relates properties at two distinct time instants.

The LSS is a joint Laboratory of CNRS and Supelec, associated with the University of Paris XI-Orsay.

There is a same basic idea when introducing coincidence function. This is the case of the bunching or antibunching effect. These effects are related to the fact that the existence of one point of the PP at t modifies its behavior after t.

The analogy between coincidence and correlation is so strong that there exists some confusion in the literature. This especially appears because in the case of compound Poisson processes, which is one of the most important statistical model of PP, coincidence experiments are fully described by a correlation function. In this context the bunching effect has a very simple interpretation. It is simply the consequence that any correlation function has its maximum at the origin. But there are many PPs introducing an antibunching effect and the previous explanation cannot hold. Thus the relation between coincidence and correlation must be analyzed more carefully.

A similar confusion appears in measurements of coincidence. Indeed, still in the case of compound Poisson processes, there is a simple relation between coincidence and life time. Thus, instead of using coincidence device, one can use time interval devices, often much simpler. However this analogy, valid for compound Poisson processes, has no reason to be general, and the exact relationship between coincidence and time measurements is presented in detail.

2. GENERAL CONSIDERATIONS

Consider a time PP, or a random distribution of random time instants t_i, and let $N(t,\tau)$ be the RV equal to the number of points in the time interval $[t,t+\tau[$. We assume that this process is regular, or that there is no accumulation point, which means that a small interval cannot contain more than one point of the process, or more explicitly that $P[N(t,\Delta t)>1]=o(\Delta t)$ and

$$P[N(t,\Delta t)=1]=\lambda(t)\Delta t+o(\Delta t), \qquad (1)$$

where P means the probability and $\lambda(t)$ is the *density* of the PP. As a consequence we have $P[N(t,\Delta t)=0]=1-\lambda(t)\Delta t+o(\Delta t)$. If $\lambda(t)$ is constant, the process is stationary, which is assumed in all what follows. Let us now introduce the coincidence function.

2.1 Coincidence

In coincidence measurements two (or more than two) small time intervals at different time instants are considered and a coincidence event appears when there is one point of the PP in each of these intervals. The bicoincidence are described by a *coincidence function* $c(t,t')$, $t\neq t'$, defined by

$$P\{[N(t,\Delta t)=1].[N(t',\Delta t')=1]\}=c(t,t')\Delta t\Delta t'+o(\Delta t,\Delta t'). \qquad (2)$$

Since the PP is stationary this function depends only on $\tau=t-t'$ and for $\tau\neq 0$ we define $c(.)$ by $c(\tau)=c(-\tau)=c(t,t-\tau)$. In general the RVs $N(t,\Delta t)$ and $N(t',\Delta t')$ become uncorrelated for large values of τ, which is expressed as

$$\lim_{\tau\to\infty} c(\tau)=\lambda^2. \qquad (3)$$

As (2) implies $\tau\neq 0$, we state $c(0)=\lim_{\tau\to 0} c(\tau)$.

Note that for Poisson processes the variables $N(t,\Delta t)$ and $N(t',\Delta t')$ are independent. This yields $c(\tau)=\lambda^2$ for any τ and this is the simplest example of coincidence function.

According to (1) the RVs $N(t,\Delta t)$ and $N(t',\Delta t')$ take only the values 0 or 1 for small intervals Δt and $\Delta t'$, which yields

$$\mathsf{E}[N(t,\Delta t)N(t',\Delta t')] = c(\tau)\Delta t\Delta t' + o(\Delta t,\Delta t'), \tag{4}$$

where E means the expectation value. From this relation it is tempting to interpret $c(\tau)$ as a correlation function. However it is important to note that, despite its appearance, $c(\tau)$ *is not a correlation function*. Indeed a correlation function $\gamma(\tau)$ must satisfy some constraints, and more precisely must be positive-definite, which is characterized by the fact that its Fourier transform $\Gamma(f)$ is positive (power spectrum). As a consequence $|\gamma(\tau)| \le \gamma(0)$. This has no reason to be true for $c(\tau)$ and we shall present below various examples of PP where $c(\tau) > c(0)$. It is even possible, as seen later, to have PPs for which there exists τ_0 such that $c(\tau)=0$ for $|\tau|<\tau_0$.

2.2 Bunching Effect

The coincidence function has a direct application in the description of the *bunching* (or antibunching) effect in a PP. It is related with the fact that the presence of a point of the process at $t=0$ can modify the probability $P[N(t,\Delta t)=1]$. Of course this effect does not occur for a Poisson PP, because it is memoryless.

However, since the fact that there is a point at $t=0$ is not a statistical event, we shall introduce a small interval $\Delta t'$ at $t=0$ and use the conditional probability

$$\widehat{F}(t,\Delta t,\Delta t') \overset{\triangle}{=} P\{[N(t,\Delta t)=1] \mid [N(0,\Delta t')=1]\}. \tag{5}$$

The bunching effect is described by

$$F(t,\Delta t) \overset{\triangle}{=} \lim_{\Delta t'\to 0} \widehat{F}(t,\Delta t,\Delta t'). \tag{6}$$

The conditional probability $\widehat{F}(t,\Delta t,\Delta t')$ can be expressed as

$$\widehat{F}(t,\Delta t,\Delta t') = \frac{P\{[N(t,\Delta t)=1].[N(0,\Delta t')=1]\}}{P\{[N(0,\Delta t')=1]\}} \tag{7}$$

and it results from (1) and (2) that

$$F(t,\Delta t) = \frac{c(t)}{\lambda}\Delta t \overset{\triangle}{=} b(t)\Delta t. \tag{8}$$

This introduces the bunching function $b(t)$ related to the coincidence function $c(t)$ by $c(\tau)=\lambda b(\tau)$. Finally note that (3) yields

$$\lim_{t\to\infty} b(t) = \lambda. \tag{9}$$

For a Poisson process we have $b(t)=\lambda$. Then we shall say that there is a bunching effect at t if $b(t)>\lambda$, and an antibunching effect in the opposite case. Note that the bunching effect is not necessarily an intrinsic property of a PP because for the same PP it can exist instants t with bunching effect and other with antibunching effect.

2.3 Relation to Life Time

Let L_k be the life time of order k, or the RV equal to the distance between a point t_i of the PP and the kth point of this process posterior to t_i. Because of the assumption of stationarity, the probability distribution of L_k does not depend on t_i, and then we can assume that $t_i=0$. For all the processes considered here the RVs L_k are continuous and characterized by their probability density function (PDF) $f_k(t)$. The quantity $f_k(t)\Delta t$ is by definition the probability to have one point of the PP in $[t,t+\Delta t[$ and $k-1$ points in $[0,t[$, conditionally to one point at $t=0$. But it results from (8) that $b(t)\Delta t$ is the probability to have one point in $[t,t+\Delta t[$ conditional to one point at 0. As a result we have

$$b(t) = \frac{c(t)}{\lambda} = \sum_{k=1}^{\infty} f_k(t). \tag{10}$$

This equation yields the relationship between the coincidence function and the set of PDF of the life times of any order. It is clear that it will play an important role for all the processes defined from their life times, and this is especially the case of renewal processes.

From this relation we can also deduce the expression of the coincidence function in terms of counting probabilities defined by $p_n(t)=P[N(0,t)=n]$. For this it suffices to use the expression of the life times in terms of these probabilities [4][5].

2.4 First-Order Approximation

In some specific situations it is possible to approximate (10) by its first term, or to write $b(t)\approx f_1(t)$. In this case coincidence measurements are equivalent to life time measurements. Thus, instead of using a coincidence device, it is possible to use a time to amplitude converter to reach the coincidence function $c(t)$.

Let us discuss briefly some conditions allowing this approximation. Let τ_0 be the radius of variation of the coincidence function $c(t)$. This means that for $t>\tau_0$ $c(t)\approx\lambda^2$, which, according to (3), is its asymptotic value. For a stationary PP the mean value of the distance between two points is the inverse of the density λ. Then if $\lambda\tau_0 << 1$ the probability to have more than one point in the interval $[0,t[$ with $t<\tau_0$ is very low. As a consequence in the domain of variation of $c(t)$ the bunching function $b(t)$ is approximately equal to the PDF of the life time of order one. We shall see that there are PPs where this can always be satisfied, and this is especially the case of compound Poisson processes. On the other hand there are PPs where this approximation can never be used and this is discussed in the following.

3. POISSON-TYPE PROCESSES

3.1 Pure Poisson Processes

It is of course the simplest example of application of the previous results. As indicated above, the bunching function is $b(t)=\lambda$. Furthermore the PDF of the life time of order k is

$$f_k(t) = u(t)\lambda\exp(-\lambda t)\frac{(\lambda t)^{k-1}}{(k-1)!} \tag{11}$$

where $u(t)$ is the unit step function. This leads immediately to (10).

3.2 Compound Poisson Processes

Compound Poisson processes [6], sometimes called doubly stochastic processes [7], are Poisson PP in which the density $\lambda(t)$ is a stationary random function . They play an important role in many areas of Physics or Information Sciences and especially in optical communications. Indeed it can be shown that the PP of the detection of photons is a compound Poisson process with a random density proportional to the random intensity of the optical field [1].

Let $A(t)$ be a positive random signal and denote by A and $\gamma(\tau)$ its mean and its correlation function respectively. The random intensity (or the random density) of the PP can be expressed as $\lambda(t)=\alpha A(t)$ where α is a non-random modulation factor. In order to calculate the coincidence function we start from (2). Conditionally to a value of $A(t)$, the RVs $N(t,\Delta t)$ and $N(t',\Delta t')$ are independent and characterized by (1). Taking the expectation we find

$$c(t) = \alpha^2\mathsf{E}[A(s)A(s-t)] = \alpha^2[\gamma(t)+A^2]. \tag{12}$$

This expression leads to various comments. First we note that for regular processes the correlation function $\gamma(t)$ tends to zero when t tends to infinity. As $\lambda=\alpha A$, this yields (3). Second, and more important, the coincidence function of a compound Poisson process has necessarily a maximum at the origin. This comes from the same property known for correlation functions. This is in relation with the bunching effect. Indeed, (8) yields

$$b(t) = \frac{\alpha}{A}\,\gamma(t) + \lambda \tag{13}$$

and there is a bunching effect if $b(t) > \lambda$ or $\gamma(t) > 0$. This is frequently realized. However, even if this is not the case for some correlation functions, the fact that $\gamma(t)$ is a continuous function with a positive maximum at the origin implies that there exists a t_0 such that if $|t| < t_0$, then $\gamma(t) > 0$. Then there is necessarily a bunching effect in the neighborhood of the origin. Therefore a PP with antibunching effect for small t cannot be a compound Poisson process.

Moreover it is seen that it is possible to change the density of the process without changing the shape of its coincidence function. For this purpose it suffices to modify the modulation parameter α. This is especially important for applying the first order approximation introduced above. Indeed, as the density λ is αA, it is always possible to reach the condition $\lambda t_0 << 1$ by using sufficiently small α. In this operation the shape of $c(t)$ given by (12) does not change. However this property has no reason to be general.

3.3 Dead Time Effects

Dead-time is a common phenomenon in PPs. To any PP P_1 the dead time effect associates another PP P_2 according to various possible mechanisms. In the most common, called *output dead time*, to any point t_j of P_2, which by construction is a non-erased point of P_1, we associate an interval $[t_j, t_j + D[$ such that all the points of P_1 belonging to this interval are erased. In the case where P_1 is Poisson, the PDFs $f_k(t)$ are given by

$$f_k(t) = u(t-kD)\mu\exp[-\mu(t-kD)]\,\frac{[\mu(t-kD)]^{k-1}}{(k-1)!} \tag{14}$$

where μ is the density of the Poisson process P_1. According to (10) the sum of these functions is equal to the bunching function $b(t)$, This sum is presented in Figure 1 and is calculated for $\mu = 1$ and $D = 1, 2, 3$.

This figure leads to the following comments. For large values of t the bunching functions tend to become equal to λ, the density of the process P_2. This density can easily be obtained from the density μ of P_1 by the relation $\lambda = \mu/(1+\mu D)$. When $\mu = 1$ this leads to the asymptotic values 1/2, 1/3, 1/4, which clearly appear in the figure.

However this asymptotic behavior is reached much more rapidly for $D = 1$ than for $D = 3$. This is quite normal because increasing of D increases the effect of the dead time. On the opposite for $D = 0$ we come back to a Poisson process where $c(t) = \lambda^2$.

According to (10) and (14) these three functions are equal to zero for $t < D$, which is an obvious result of the dead time effect. Furthermore, as $f_k(kD) = 0$ for $k > 1$, the bunching function is continuous. Similarly the derivatives satisfy $f'(kD) = 0$, except for $k = 2$. Thus there is a discontinuity of the derivative at the point $t = 2D$. All those properties appear clearly in the figure.

Finally it is clear that $b(t)$ is equal to $f_1(t)$ for $0 < t < 2D$. This means that the first order approximation presented above yields exactly the bunching function in this interval but not at all for $t > 2D$. Furthermore as $f_1(t) = 0$ for $0 < t < D$, there is an antibunching effect. This is an obvious consequence of the dead time effect making impossible the existence of too close points.

There is another mechanism of dead time called *input dead time*. In this case each point t_i of P_1 introduces an interval $[t_i, t_i + D[$ such that any point of P_1 belonging to this interval is suppressed and the points of P_2 are those of P_1 that are not suppressed. As a consequence any point t_i of P_1 remains a point of P_2 if and only if there is no point of P_1 in the interval $]t_i - D, t_i]$. The obvious consequence is that the coincidence function $c(t)$ defined by (2) is

$$c(t) = 0 \;\text{ for }\; |t| \le D \;,\; c(t) = \lambda^2 \;\text{ for }\; |t| > D. \tag{15}$$

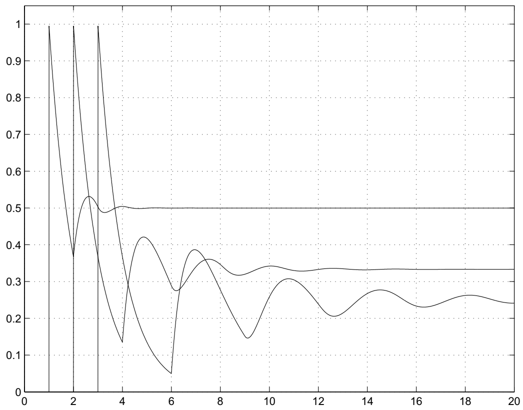

Figure 1: Bunching function for Poisson process with output dead-time.

It is clear that this is not a correlation function and this is the simplest example showing this fact. There is obviously an antibunching effect for $t < D$ for the same reason as in the previous mechanism.

3.4 Poisson Process with Regular Erasing

Consider a stationary Poisson process P_1 of density μ and let us construct the PP P_2 by erasing regularly every second point. Then if the points of P_1 are t_i, those of P_2 are t_{2i+1}. It is clear that the density of P_2 is $\lambda = \mu/2$ and that it cannot be a Poisson process because of the memory effect of the erasing procedure. This is verified by the calculation of the coincidence function.

The probability appearing in (2) requires that the number of points of P_1 in the interval $[t + \Delta t, t'[$ is odd, and this easily yields

$$c(t) = \lambda^2[1 - \exp(-4\lambda t)]. \tag{16}$$

This function is obviously not a correlation function. Furthermore we have a simple example where the first order approximation cannot be applied. Indeed the coincidence function $c(t)$ is not proportional to λ^2 and its shape also depend on λ.

The bunching function $b(t)$ given by (8) is $b(t) = \lambda(1 - \mathrm{e}^{-4\lambda t})$, and $b(t) \le \lambda$, which characterizes an antibunching effect.

Let us now verify how the bunching function is constructed from the PDFs of life times. The DDP $f_k(t)$ appearing in (10) is obtained from the structure of the process P_1 and is given by

$$f_k(t) = 2\lambda\exp(-2\lambda t)\,\frac{(2\lambda t)^{2k-1}}{(2k-1)!}\;. \tag{17}$$

It is easy to verify that (10) is satisfied. In order to illustrate this expression, some results are displayed on Figure 2 where $\lambda = 1$. The upper curve is the coincidence function given by (16) and the other curves are the PDFs of lifetimes f_{2k+1} for $0 \le k \le 4$. We have here a good example where the first-order approximation cannot be used. Indeed (16) cannot be approximated by (17) with $k = 1$ even by using small value of λ. The only property common to these functions is that they have the same behavior in the neighborhood of the origin as seen on the figure.

4. RENEWAL PROCESSES

4.1 General Results

A stationary renewal process is characterized by the fact that the distances L between successive points (life time of order one) are IID random variables. Let $f(t)$ be their common PDF. As the life time of order k is a sum of k IID RVs, its PDF $f_k(t)$ is simply the convolution-power of order k of $f(t)$, or

$$f_k(t) = \underbrace{f(t)\star f(t)\star\ldots\star f(t)}_{k} \stackrel{\triangle}{=} f^{\star k}(t). \tag{18}$$

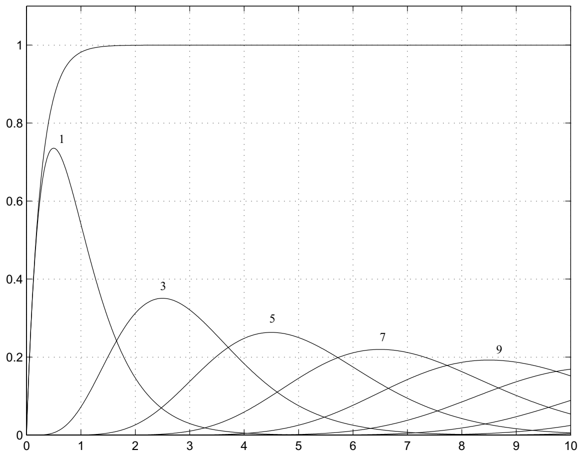

Figure 2: Regular Erasing. Bunching and life time PDFs for Poisson process.

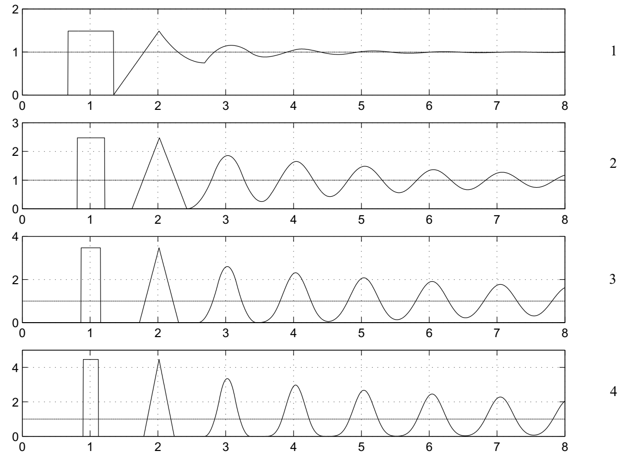

Figure 3: Bunching function for renewal process with uniform distribution.

The PDF $f(t)$ is not completely arbitrary. In practice it has a finite mean value m which is the mean distance between successive points of the PP. As a consequence its density λ is $\lambda = 1/m$.

The bunching or coincidence functions defined by (10) take the form

$$b(t) = \frac{c(t)}{\lambda} = \sum_{k=1}^{\infty} f^{\star k}(t). \tag{19}$$

It is convenient to express this result in the Laplace domain. Let $F(s)$ be the generating function of the life time, or the Laplace transform of $f(t)$, given by

$$F(s) = \int_0^{\infty} f(t)\exp(-st)dt = \mathsf{E}[\exp(-sL)]. \tag{20}$$

In the right complex plane $\mathrm{Re}(s) > 0$ we have $|F(s)| < 1$. As a consequence the Laplace transform of $b(t)$ is

$$B(s) = \sum_{k=1}^{\infty} F^k(s) = \frac{F(s)}{1-F(s)}\ . \tag{21}$$

This shows that in the s-domain the coincidence or bunching functions can very easily be deduced from the generating function $F(s)$.

From this expression we find again (9). To prove it, we apply the limit expression

$$\lim_{t\to\infty} b(t) = \lim_{s\to 0} sB(s) = \lim_{s\to 0} \frac{sF(s)}{1-F(s)}\ , \tag{22}$$

For small values of s we have $F(s) = 1 - ms + o(s)$ and $m = 1/\lambda$. This yields (9).

The counting probabilities $p_k(t)$ of a renewal process can be calculated from the PDF $f(t)$ by solving the so-called renewal equation. However the calculation is in general very complicated, and the coincidence approach is easier to use.

In order to illustrate the results let us consider the case where L is uniformly distributed in a given interval. More precisely suppose that the PDF of L is equal to $1/2a$ in the interval $[1-a, 1+a]$ and zero otherwise. For $a = 1$ we obtain a uniform distribution in the interval $[0,2]$. On the contrary when $a << 1$ the RV L is almost equal to 1, and this occurs for example in the jitter effect in communication systems. Let us now present some properties of $b(t)$ of (19). It is possible to show that the derivative of $b(t)$ is continuous, except at the five points $1-a, 1+a, 2-2a, 2, 2+2a$. For $a = 1/3$, $1+a = 2-2a$, and there remain four points of discontinuity.

It is easy to verify that the PDF given by (18) is symmetric with respect to k and equal to zero outside the interval $[k(1-a), k(1+a)]$. As a consequence there is no overlapping between $f_{k-1}(t)$ and $f_k(t)$ if $(n-1)(1+a) < n(1-a)$, or $a < 1/(2n-1)$. Furthermore if a satisfies this inequality the PDFs $f_i(t)$ do not overlap for $i \leq k$ and there is an overlapping effect if $i > k$.

These properties appear clearly in Figure 3. The figure 3.1 is calculated for $a = 1/3$. There is no overlapping between f_1 and f_2 and the four discontinuities appear clearly. The last figures are calculated for $a = 1/5$, $a = 1/7$ and $a = 1/9$ respectively. In the last figure there is no overlapping between f_1, f_2, f_3, f_4 and f_5 and there are five points of discontinuity.

Note also that all these curves tend to the asymptotic value 1 but the speed of convergence decrease with a. For small values of a there are some oscillations that disappear when t increases and at the limit of $a = 0$ the RVs become deterministic and $c(t)$ is a set of Dirac functions at the time instants k.

5. CONCLUSION

In this paper we have defined the bicoincidence function of a stationary point process. This function yields important information on the time behavior of the process. In particular it describes completely bunching and antibunching as well. In some cases it is a correlation function, especially for compound Poisson processes. However, we emphasize that this property is not general and cannot hold for point processes with antibunching. Various examples of such processes are presented and numerical results are in complete agreement with the theoretical analysis.

REFERENCES

[1] B. Saleh and M. Teich, *Fundamentals of Photonics*. New-York: Wiley, 1991.

[2] C. Bendjaballah, *Introduction to Photon Communication*. Heidelberg: Springer Verlag, 1995.

[3] O. Macchi, "The coincidence approach to stochastic point processes," *Adv. Appl. Prob.*, vol. 7, pp. 83-122, 1975.

[4] J. A. McFadden, "On the lengths of intervals in a stationary point process," *Journ. of Royal Stat. Soc.*, vol. B 24, pp. 364-382, 1962.

[5] B. Picinbono "Time intervals and counting in point processes," *IEEE Trans. Inf. Theory*, to appear 2004.

[6] B. Picinbono, *Random Signals and Systems*. Englewood Cliffs, N.J: Prentice Hall, 1993.

[7] D. L. Snyder and M. I. Miller, *Random Point Processes in Time and Space*. New-York: Springer Verlag, 1991.