ON DESIGNING A WIDEBAND FRACTIONAL DELAY FILTER USING THE FARROW APPROACH

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ABSTRACT

In this paper the problem of designing a wideband FIR fractional delay filter using the Farrow approach is considered. A two-stage design with a half-band linear-phase prefilter optimal in the Chebyshev sense and a short maximally flat Lagrangian fractional delay filter in the Farrow structure is proposed. It offers an economic implementation with the number of multiplications smaller than for one-stage approach used hitherto, provided that the prefilter is a Nyquist filter. A novel, accurate method of computing the coefficients of Farrow subfilters is introduced based on symbolic designing of k-th degree differentiators. The proposed filter is intended for applications with variable fractional delay value.

1. INTRODUCTION

One of the problems presently encountered, e.g., in interfacing digital equipment having different sample rates, lies in designing a wideband fractional delay filter (FDF), fast and numerically inexpensive. It means a filter capable of realization of an arbitrary fractional delay (FD) value during a time slot dependent on a given sample rate. Some comparative results of designing highly accurate wideband FDFs using the windowing technique, with parametric windows, and the Chebyshev approach are presented in [1]. Here we extend this work from the point of view of an economical implementation of a wideband FDF. A means of achieving this is a two-stage approach with a half-band prefilter followed by a FDF in the Farrow structure. This structure allows for updating the filter coefficients efficiently in accordance with the varying FD value. This is because it is built of a bank of linear-phase differentiating subfilters, and the same set of subfilters is used for all possible FD values. As compared with the transversal structure of length *N*, where all *N* coefficients must be changed for updating the filter for a new FD value, the number of coefficients in the Farrow structure is greater than *N*. However, what is gained is numerical savings in the number of arithmetic operations or storage due to lack of need for recalculating the coefficients or using a look-up table for the coefficients calculated in advance.

All these factors are especially important in the case of wideband FDFs, because then the filter is long. Even in the Farrow structure the number of arithmetic operations (multiplications and additions) may exceed the capabilities of a DSP at disposal to realize a real-time implementation. A more economical solution is obtained in Sect.2 by using a two-stage upsample-downsample approach, aimed at shortening the Farrow subfilters. In Sect.3 a novel symbolic method for calculating the coefficients of Lagrangian Farrow subfilters is introduced. Sect.4 compares the numerical costs of the proposed two-stage design with one-stage design [2] having Chebyshev subfilters with optimised number and length.

2. THE STRUCTURE OF PROCESSING

The idea of the upsample-downsample wideband design introduced in [3] is presented in Fig.1. A half-band prefilter produces a two-times oversampled signal from a zero-padded (expanded by a factor of 2) input. This signal is passed through a half-band variable FD (VFD) filter. The result is finally down sampled (compressed by a factor of 2) back down to the baseband frequency. In this arrangement the requirements for the VFD filter are relaxed relative to one-stage approach, allowing for a much smaller number of taps in the Farrow structure. A wideband design is understood here as having the bandwidth of about 0.9π radians per sample (rad/Sa) or wider.

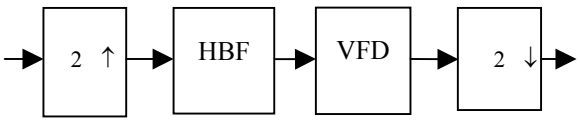

Fig.1. Upsample-downsample wideband VFD filter with linear-phase half-band prefilter.

An efficient two-stage realization of the structure from Fig.1 is shown in Fig.2. Here, in the upper branch, the even-indexed taps of the HBF are combined with odd taps of the VFD filter. It is done conversely in the lower branch. The results are summed up and multiplied by a factor of 2. The processing is performed at the input (low) sampling rate. Thus the operations of expansion and compression at the front-end and back-end of Fig.1 are not implemented.

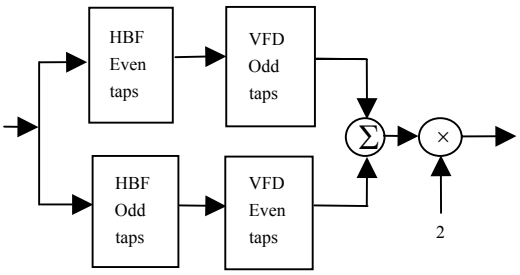

Fig.2. Efficient realisation of structure from Fig.1.

The HBF used further in experiments is a linear-phase lowpass FIR filter optimal in the Chebyshev sense, instead of a suboptimal design with the Dolph-Chebyshev window used primarily in [3]. This filter is easily designed in MATLAB by the remez.m file. The input data for this filter are passband and stopband edge frequencies, f_p and f_s, respectively. Here we assume that $f_s + f_p = 1/2$ and that f_p and f_s are symmetrically located at the edges of the transition band centred at $f_c = 1/4$. The weights for passband and stopband are chosen depending on the design specification. The most favourable situation occurs when the maximal (total peak) value of the complex approximation error magnitude is imposed. Then one can use equal weights for

the Chebyshev filter passband and stopband ripples. The resulting HBF is then a Nyquist filter with every second sample equal to zero except the central sample whose value is ½. Realisation of this symmetrical filter of length $N = 4K - 1$, where K is an integer, needs K different coefficients, thus only K multiplications.

The second filter in the two-stage approach here is the Lagrangian VFD filter realised in the Farrow structure.

3. THE FARROW APPROACH

The idea introduced by Farrow is the following. The FDF transfer function of FIR type

$$H_{N\beta}^{(0)}(z) = \sum_{n=0}^{N-1} h_{N\beta}^{(0)}[n] z^{-n} \tag{1}$$

with $\beta = D + d$, where $D \triangleq (N-1)/2$ is the transport delay and N is the filter length, is expressed as a polynomial of the fractional delay d, where $|d| \le 0.5$. In the case of Lagrangian filters, firstly the impulse response samples $h_{N\beta}^{(0)}[n], \quad n = 0,1,...,N-1$, i.e. the coefficients of the transversal structure of the filter, are expressed as polynomial functions of d of degree M:

$$h_{N\beta}^{(0)}[n] = \sum_{k=0}^{M} c_k[n] d^k \tag{2}$$

Next, by substituting (2) to (1) the transfer function (1) is rewritten as

$$H_{N\beta}^{(0)}(z) = \sum_{k=0}^{M} \sum_{n=0}^{N-1} c_k[n] z^{-n} d^k = \sum_{k=0}^{M} C_k(z) d^k \tag{3}$$

where

$$C_k(z) = \sum_{n=0}^{N-1} c_k[n] z^{-n} \tag{4}$$

are the transfer functions of Farrow subfilters whose impulse responses are $c_k[n]$ of length N each. The resulting structure is shown in Fig.3. The specific feature of this structure is that the coefficients $c_k[n]$ of subfilters $C_k(z), \quad k = 0,1,...,M$ do not depend on d. Thus, they can be designed once, off-line. The fractional delay parameter steers M multipliers in the lower branch of the overall VFD. Hence the filter can be updated very efficiently with d varying from one input signal sample to another without changing the coefficients of subfilters. This is contrary to the standard transversal structure of the overall filter, where each filter coefficient has to be changed according to the varying fractional delay value.

We have dealt here with the Farrow FIR subfilters $C_k(z)$ of equal length N. For strictly Lagrangian FDFs we have N=M+1 with M being the degree of the approximating polynomial in d. Generally, however, N and M can be set independently. Thus, the number of subfilters, M+1, can be different than N.

It is important to guarantee beforehand the symmetry of subfilters. For Lagrangian FDFs this is achieved by rearrangement of the filter's transfer function as above on the basis of the impulse response expressed as a function of d. The more general approach allowing for Lagrangian subfilters of different length and for non-polynomial based Farrow subfilters, e.g. optimal in the Chebyshev sense or designed by using the windowing technique, is based on computing the impulse response of each subfilter individually. The idea of the latter approach is based on the Taylor series expansion of the frequency response

$$H_{\beta}^{(0)}(e^{j\omega}) = \exp(-j\omega\beta), \quad |\omega| < \pi \tag{5}$$

of the ideal FDF, aimed at decomposition of this filter into a differentiating filters' bank [2] [5]. In order to do this let us define the frequency response of the k-th degree linear-phase differentiating filter as given by

$$H^{(k)}(e^{j\omega}) = (j\omega)^k e^{-j\omega D} \tag{6}$$

where $H^{(k)}(e^{j\omega})$, k=0,1,2,… , is independent of the fractional delay d. Thus the total delay of this filter is $\beta = D$ and the phase response is linear.

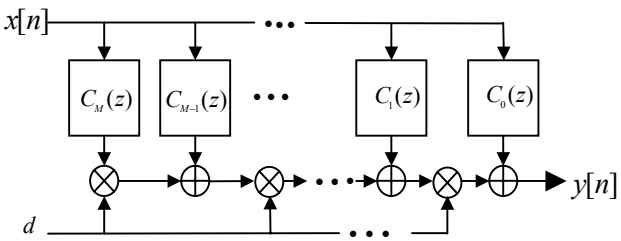

Fig.3. Farrow structure with variable fractional delay d steering multipliers in the lower branch.

Next we have to perform the Taylor series expansion of the frequency response (5). This expansion has the form [5]

$$\begin{aligned} H_{\beta}^{(0)}(e^{j\omega}) &= \sum_{k=0}^{\infty} \frac{(-d)^k}{k!} (j\omega)^k e^{-j\omega D} = \\ &= \sum_{k=0}^{\infty} \frac{(-d)^k}{k!} H^{(k)}(e^{j\omega}) \end{aligned} \tag{7}$$

That is, the FDF is expanded into a series of differentiating filters (6) of different degrees of differentiation.

In practice the length of the Taylor series has to be limited into a finite number of terms (here $M+1$). Using differentiating filters of length N, the frequency response in (7) is approximated as follows

$$H_{N\beta}^{(0)}(e^{j\omega}) = \sum_{k=0}^{M} \frac{(-1)^k}{k!} H_N^{(k)}(e^{j\omega}) d^k = \sum_{k=0}^{M} C_k(e^{j\omega}) d^k \tag{8}$$

where M is the largest degree of differentiation, and

$$C_k(e^{j\omega}) = \frac{(-1)^k}{k!} H_N^{(k)}(e^{j\omega}) \tag{9}$$

Consequently, the FDF can be directly implemented in the Farrow structure from Fig.3. Thus the design problem of FDF resolves itself to designing a bank of M+1 differentiators, with degree of differentiation k=0,1,...,M .

However, there is a difficulty with designing linear-phase Lagrangian differentiators (thus maximally accurate around $\omega = 0$) individually. It lies in that only for the first degree differentiators closed-form formulae for the coefficients exist [6], where these coefficients can be calculated sufficiently accurate even for long FIR approximations. For higher degree differentiators the applicability of the available algorithms is limited to FIR differentiators of length smaller than about 20. This appears insufficient for wideband applications and even for some

half-band designs where high accuracy of approximation is of prime importance. This problem is mastered here by using symbolic computations in MATLAB. These computations are based on the following relationship between the impulse responses: $h_{N\beta}^{(k)}[n]$ of a k-th degree differentiator, and $h_{N\beta}^{(0)}[n]$ of an FDF, of the same length, and the same total delay β

$$h_{N\beta}^{(k)}[n] = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}\beta^k} h_{N\beta}^{(0)}[n] \tag{10}$$

The coefficients of the Lagrangian FDF of length N can be written in the form of symbolic expressions in terms of the variable d (see the function h=fdfsymbf(N) in Fig.4). These expressions can be differentiated symbolically to the form of a set of N polynomials in d (see the function h=ddksymbf(N) in Fig.4) whose values, for the linear-phase condition (d=0), are sufficiently accurate even for very long and high degree differentiators. Short MATLAB m-files for doing this for Lagrangian FDFs are given in Fig.4. Similar program functions can be readily written for other methods of approximation, for example for FDFs designed using the windowing technique.

```
function h=fdfsymbf(N)
syms d h h1
D=(N-1)/2;%transport delay
for n=1:N; h(n)=1; for k=1:N
if k~=n h(n)=h(n)*(D+d-(k-1))/(n-k); end; end; end
for n=1:N; h1(n)=expand(h(n)); end; h=h1;
```

```
function [dh,dhh]=ddksymbf(N,k,h)
% k - the degree of differentiation
syms dh
if k==0;    dh=h; end;
if k~=0; dhh=((-1)^k)*diff(h,k); end;
for n=1:N; dh(n)=polyval(sym2poly(dhh(n)),0); end
```

Fig.4. MATLAB programs for the generation of symbolic expressions (polynomials in d) for the coefficients of Lagrangian VFD filter: function h=fdfsymbf(N), and for linear-phase differentiators: function [dh,dhh]=ddksymbf(N,k,h).

4. EXPERIMENTS

In experiments we have designed wideband FDFs based on specifications given in [2], in order to have a platform to compare the results of the two-stage approach proposed here, with one-stage approaches used in [2]. For this aim the total peak error of approximation (TPE) is defined

$$\mathrm{TPE} = \max_{\omega \in <0,\mathrm{MBW}>} \left| H_{\beta}^{(0)}(e^{j\omega}) - H_{N\beta}^{(0)}(e^{j\omega}) \right| \tag{11}$$

where $\mathrm{MBW} \in (0,\pi)$ stands for the measurement bandwidth. Other measures of quality of the FDF used further are: the magnitude response peak error in the linear scale

$$\mathrm{MPE} = \max_{\omega \in <0,\mathrm{MBW}>} \left\{ \left| H_{\beta}^{(0)}(e^{j\omega}) \right| - \left| H_{N\beta}^{(0)}(e^{j\omega}) \right| \right\} \tag{12}$$

and phase delay response peak error

$$\mathrm{PDPE} = \max_{\omega \in <0,\mathrm{MBW}>} \frac{\arg H_{\beta}^{(0)}(e^{j\omega}) - \arg H_{N\beta}^{(0)}(e^{j\omega})}{\omega} \tag{13}$$

Example 1.

The design specification is the following. Filter bandwidth B=0.9π =MBW and TPE=0.0042 in MBW for all possible fractional delay values $d \in [-0.5, 0.5]$.

The numerical costs for one-stage design [2] (further denoted as case A) and for the proposed two-stage design (case B), both using the Farrow structure, are given in Table 1 in terms of a number of multiplications. (The number of additions is about the same as the number of multiplications.) In case A the solution is a filter of length N=39 with M+1=7 Farrow subfilters optimised separately to different lengths N_i, from 39 to 7, as in Table 2. This filter needs 76 multiplications.

Table 1. A comparison of designs in cases: A and B.

Example	No of Mult. A	No of Mult. B	Delay D A	Delay D B
1	76	60	19	16,5
2	250	227	37,5	38,5
3	57	69	13,5	15,5

Table 2. Design A (one-stage) after [2].

Example	Specification (linear scale)	N max length	M+1 No of subfil-ters	N_i length of subfilters
1	TPE ≤ 0.0042	39	7	39, 39, 19, 33, 15, 21, 7
2	TPE $\le 10^{-5}$	76	10	74, 48, 76, 48, 66, 40, 52, 28, 34, 14
3	MPE ≤ 0.01 PDPE ≤ 0.001	28	6	28, 10, 28, 12, 20, 6

The two-stage upsample-downsample design (case B, see Table 3)) fulfils the specification with TPE=0.00376, thus with a reserve, by using a half-band symmetric Chebyshev Nyquist prefilter of length N=59. The VFD filter is the Lagrangian (maximally flat) filter in the Farrow structure including M+1=7 subfilters of length N_i=11 each. This two-stage design needs 60 multiplications instead of 76 for one-stage, and introduces transport delay D=16.5 sample intervals, also smaller. The total delay is $\beta = 16.5 + d$. Table 1 summarises these results.

Table 3. Design B (two-stage) proposed here.

Example	Achieved (linear scale)	N HBF length	M+1 No of subfilters	N_i length of subfilters
1	TPE ≤ 0.00376	59	7	11
2	TPE $< 10^{-5}$	133	13	25
3	MPE ≤ 0.095 PDPE ≤ 0.00097	55	7	11

The performance of this filter for a few chosen FD values is presented in Fig.5. Note that the maximal value of phase delay response ripples is much smaller (here about 40 times) than the maximal value of the group delay response ripples and their shape is different. The group delay ripples are smallest for low frequencies and increase towards the edge of the MBW, where also the complex error magnitude ripples are greatest. This effect can be countered by using longer filters, but the price paid is a greater numerical cost of processing per sample.

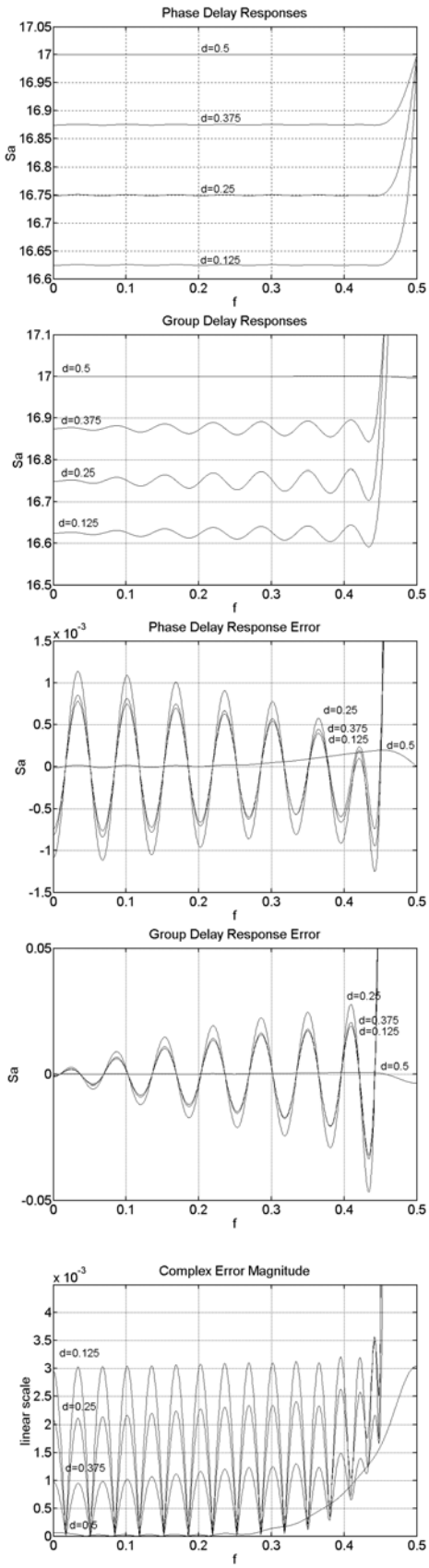

Fig.5. The performance of VFD filters from Example 1; $f = \omega/(2\pi)$.

Example 2.

In this example the requirement for the TPE is very tight. It is limited to the value of 0.00001 in the 90% MBW.

Here also a smaller number of multiplications are needed for the two-stage approach than for the one-stage design with separate optimization of subfilters (cf. Table 1) but the transport delay is greater by one sample interval.

Example 3.

In the third example the specification is different. Here, instead of TPE, the MPE (12), thus the magnitude response peak error, and the PDPE (13), thus the phase delay response peak error, are specified. These errors are of different values: MPE= δ_1 =0.01 and PDPE= δ_2 =0.001, respectively. In this case an improved optimal one-stage design with simultaneous rather than separate optimisation of subfilters outperforms the two-stage design in both: the number of multiplications and the transport delay value (see Tables 2 and 3). The main reason is that in this example the prefilter cannot fulfil the Nyquist condition in order to meet the requirements. This condition is fulfilled only when the weights for passband and stopband ripples are of equal values ($\delta_p = \delta_s$). In Example 3 these weights have to be unequal (1/10 for passband and 1 for stopband) in order to fulfil the specification for very small phase delay total peak error in MBW.

5. CONCLUSIONS

The conclusions are the following.

1. In the two-stage VFD filter design the number of multiplications is generally smaller than in one-stage design with subfilters optimised separately, provided that the half-band prefilter is a Nyquist filter.
2. The one-stage optimal minimax design is the most economical solution when the maximal allowable errors are independently specified for the magnitude as well as the delay responses.
3. The phase delay response ripples are generally smaller than the group delay response ripples for the same filter.
4. The coefficients of long (N>20) Lagrange FIR differentiators can be accurately and readily obtained in a symbolic way from the coefficients of VFD filter of the same length.

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