QUATERNIONIC BUILDING BLOCK FOR PARAUNITARY FILTER BANKS[*)]

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ABSTRACT

This paper presents a new, motivated by the theory of hypercomplex numbers, approach to the design of paraunitary filter banks. Quaternion multiplication matrices related to 4D hyperplanar transformations turn out to be usable in the factorization of orthogonal matrices, as an extension and alternative for commonly met Givens rotations. The corresponding building block is suitable for design parameterization and efficient implementation of lossless lattices with 4 or more channels. Novel quaternion-based mutations of known filter banks are proposed and the theory is supported with design examples.

1. INTRODUCTION

Paraunitary filter banks (PUFBs) and tightly connected lapped orthogonal transforms (LOTs) are very important tools for processing signals in the time-frequency or multiwavelet domain. They are the basis for high quality coding, enhancing and efficient transmission of digital information such as images, sounds, etc.

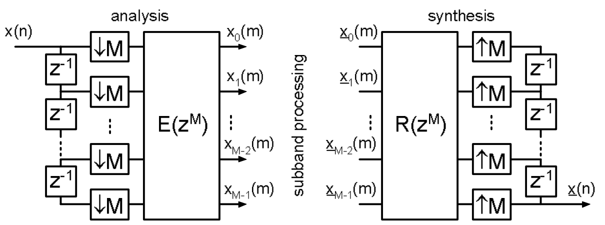

Figure 1: Polyphase representation of M -channel filter bank.

A common task in the design and implementation of PUFB or LOT is the factorization and parameterization of the matrices describing given system. In the polyphase approach shown in Fig. 1, the most important is obviously the analysis polyphase transfer matrix (PTM) $\mathbf{E}(z)$ being paraunitary (lossless), if

$$\mathbf{E}^{H}\left(z^{-1}\right)\mathbf{E}(z)=\mathbf{I}_{M} \qquad (1)$$

The structural enforcing of this equality is essential for perfect reconstruction guaranteed if the synthesis PTM $\mathbf{R}(z)=\mathbf{E}^{\mathrm{H}}\left(z^{-1}\right)$. Additionally, depending on the application, other requirements can arise, such as linear phase of subband filters or supposed regularity. From the other point of view, the minimization of the number of free design parameters is desirable, as it simplifies coefficient optimization. Finally, the computational efficiency and numerical robustness can not be neglected from the implementation perspective. Thus the structures and factorizations of PUFBs, aiming the above-mentioned goals are the subject of active research [7, 10, 11].

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In most approaches to paraunitary systems, orthogonal matrices emerge as building blocks, usually of sizes 4×4 or 8×8 in image applications. So the modeling of such matrices is very important. Well known Givens rotations are commonly employed here, as every $M\times M$ orthogonal matrix can be decomposed into $\binom{M}{2}$ planar rotations [11, 14].

In this contribution, we review some little known mathematical facts, to show that the four-dimensional generalizations of Givens rotations, derived from unit quaternion multiplication matrices, can be used in the factorization of orthogonal matrices as well. Consequently, we propose to treat quaternion multiplication as an alternative and attractive building block for the implementation of lossless lattices. The successful preliminary designs of pure quaternionic PUFBs, confirm the accuracy of the idea and indicate the possibility of developing a new interesting class of systems.

Notations: Matrices and vectors are indicated by bold-faced letters. $\mathbf{I}$ and $\mathbf{J}$ are the identity and reversal matrices, respectively. The transpose is denoted by the subscript *T*, and Hermitian (conjugated) transpose by *H*. The symbol $\lfloor\cdot\rfloor$ indicates the floor function.

2. GIVENS ROTATIONS IN PUFBS

The most classical approach to paraunitary system design [2] consists in representing the FIR PTM as the product.

$$\mathbf{E}(z)=\mathbf{\Theta}_{N}\mathbf{\Lambda}(z)\mathbf{\Theta}_{N-1}\cdots\mathbf{\Theta}_{1}\mathbf{\Lambda}(z)\mathbf{E}_{0} \qquad \mathbf{\Lambda}(z)=\begin{bmatrix} z^{-1} & 0 \\ 0 & \mathbf{I}_{M-1}\end{bmatrix} \qquad (2)$$

It was proved that this factorization is complete (covers all possible PUFBs) even if only the last stage $\mathbf{E}_0$ is general orthogonal matrix (possessing $M(M-1)/2$ degrees of freedom) and each $\mathbf{\Theta}_i$ is sequence of only $M-1$ Givens rotations. This is explained in Fig. 2., where two of several possible layouts of the corresponding lattice structure can be seen.

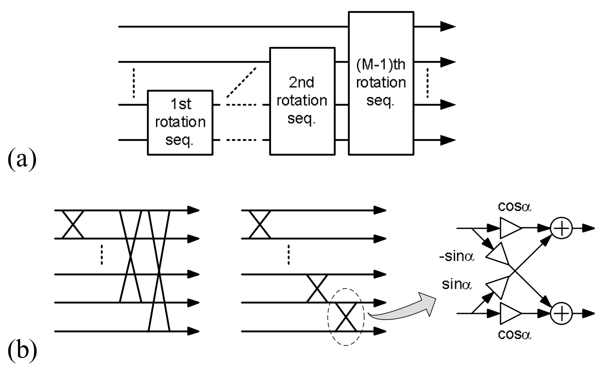


Figure 2: Orthogonal matrix implemented as composition of rotation sequences (a), and possible layouts of single sequence (b).

The basis for this is Givens reduction of a matrix to upper-triangular form. Its details can be easily found [2, 11, 14], so we only draw the main concept of selective entry annihilation via multiplication by an appropriate transposed rotation matrix.

$$\begin{bmatrix} 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & \cos\alpha & \sin\alpha \\ 0 & \cdots & 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \mathbf{R} = \begin{bmatrix} \times & \times & \cdots & \times \\ \vdots & \vdots & \ddots & \times \\ \times & \times & \cdots & \times \\ \times & \times & \cdots & \times \\ 0 & \times & \cdots & \times \end{bmatrix} \tag{3}$$

Repeating this step $\binom{M}{2}$ times, all elements below the diagonal can be zeroed. As the orthogonal matrix $\mathbf{R}$ is reduced, the elements above the diagonal also disappear simultaneously, we are left with $\operatorname{diag}\{\mathbf{I},\pm 1\}$.

Conversely, any special (with the determinant equal +1) orthogonal matrix can be built by accumulating rotations (each parameterized by one angle). The construction of general orthogonal matrix requires additional sign parameters (or equivalently involving Householder reflections) but this can be neglected as not constraining the freedom of filter bank design and simplifying its optimization.

Such approach to the synthesis of orthogonal matrices is exploited in many subsequent works on lossless systems [7, 10, 11]. For example, the factorization of linear phase PUFB

$$\mathbf{E}(z) = \mathbf{G}_N(z)\mathbf{G}_{N-1}(z)\cdots\mathbf{G}_1(z)\mathbf{E}_0$$

$$\begin{aligned} &\mathbf{E}_0 = \operatorname{diag}\{\mathbf{U}_0,\mathbf{V}_0\}\hat{\mathbf{W}} && \mathbf{W} = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \\ &\mathbf{G}_i(z) = \operatorname{diag}\{\mathbf{I},\mathbf{V}_i\}\mathbf{W}\boldsymbol{\Lambda}(z)\mathbf{W} && \\ &\boldsymbol{\Lambda} = \operatorname{diag}\{z^{-1}\mathbf{I},\mathbf{I}\} && \hat{\mathbf{W}} = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{I} & \mathbf{J} \\ \mathbf{J} & -\mathbf{I} \end{bmatrix} \end{aligned} \tag{4}$$

contains arbitrary $M/2 \times M/2$ orthogonal matrices $\mathbf{U}_0$ and $\mathbf{V}_i$, which can be modeled with rotations.

3. CONCEPT OF QUATERNIONS

A quaternion is a 4-dimensional hypercomplex number

$$Q = q_1 + q_2 i + q_3 j + q_4 k \qquad q_{i=1..4} \in R \tag{5}$$

based on three distinct imaginary units i, j, k. Its conjugate and norm are defined similarly to those of complex numbers

$$Q^* = q_1 - q_2 i - q_3 j - q_4 k \tag{6}$$

$$\|Q\| = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2} \tag{7}$$

Alternatively, given quaternion can be represented in 4D polar coordinates with modulus and three angles.

$$\begin{aligned} q_1 &= \mu\cos\phi && \mu = \|Q\| \\ q_2 &= \mu\sin\phi\cos\psi && 0 \le \phi \le \pi \\ q_3 &= \mu\sin\phi\sin\psi\cos\chi && 0 \le \psi \le \pi \\ q_4 &= \mu\sin\phi\sin\psi\sin\chi && 0 \le \chi < 2\pi \end{aligned} \tag{8}$$

In given angle ranges, such representation is unique. It is very important as allowing easy parameterization of the unit quaternions having $\mu = 1$.

Quaternion addition is performed in the same manner as in the case of complex numbers. But multiplication is no longer commutative, though still being associative.

$$\begin{gathered} i^2 = j^2 = k^2 = ijk = -1 \\ ij = -ji = k \qquad jk = -kj = i \qquad ki = -ik = j \end{gathered} \tag{9}$$

So factor ordering is significant in the quaternion products, as $PQ \neq QP$.

Quaternions are not a new invention, as their history begins in the XIX century. They are commonly used to represent 3D geometrical transformations. In DSP, quaternions are quite popular in image processing as the colour information can be consistently identified with a hypercomplex signal. Also, there were attempts to use them and their mutation - biquaternions in filter design [8, 9, 12, 13]. But the results were not sufficiently spectacular to attract wider attention.

4. QUATERNION MULTIPLICATION MATRICES

4.1 Quaternion multiplication in matrix form

It is convenient to treat quaternions as four-element vectors.

$$Q = q_1 + q_2 i + q_3 j + q_4 k \quad \Rightarrow \quad Q = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^T \tag{10}$$

In this case, quaternion addition corresponds to that of vectors, and multiplication can be represented in two forms as matrix-vector product

$$R = PQ = \underbrace{\begin{bmatrix} p_1 & -p_2 & -p_3 & -p_4 \\ p_2 & p_1 & -p_4 & p_3 \\ p_3 & p_4 & p_1 & -p_2 \\ p_4 & -p_3 & p_2 & p_1 \end{bmatrix}}_{\mathbf{M}^+(P)} \times \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \underbrace{\begin{bmatrix} q_1 & -q_2 & -q_3 & -q_4 \\ q_2 & q_1 & q_4 & -q_3 \\ q_3 & -q_4 & q_1 & q_2 \\ q_4 & q_3 & -q_2 & q_1 \end{bmatrix}}_{\mathbf{M}^-(Q)} \times \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \tag{11}$$

with $\mathbf{M}^+$ standing for the left-side multiplication matrix and $\mathbf{M}^-$ for the right-side one.

It can be easily verified that the transpositions of the matrices are equivalent to taking the conjugate quaternions.

$$\mathbf{M}^+\left(Q^*\right) = \mathbf{M}^+(Q)^T \qquad \mathbf{M}^-\left(Q^*\right) = \mathbf{M}^-(Q)^T \tag{12}$$

All rows and columns have the same norm, so the matrices can be easily normalized. The resulting unit quaternion multiplication matrices have many interesting properties [3], and hence all following considerations are restricted to them, unless stated otherwise.

Both matrices are members of the group SO(4) of all special orthogonal 4×4 matrices. Moreover, $\mathbf{M}^+(Q)$ is symplectic orthogonal, i.e.

$$\mathbf{M}^+(Q)^{-1} = \hat{\mathbf{J}}_4 \mathbf{M}^+(Q)^T \hat{\mathbf{J}}_4 \qquad \hat{\mathbf{J}}_4 = \begin{bmatrix} 0 & \mathbf{I}_2 \\ -\mathbf{I}_2 & 0 \end{bmatrix} \tag{13}$$

They can be identified as four-dimensional analogues of Givens rotations, well suited to the decompositions of different structured matrices [3]. The best known result concerns 3×3 rotation matrices, but here we go toward higher dimensions.

4.2 Parameterization of 4 x 4 orthogonal matrices

Quaternions are especially suited to the parameterization of 4×4 orthogonal matrices. Namely, every matrix belonging to SO(4), can be represented as a product of left and right unit quaternion multiplication matrices.

$$\underset{\mathbf{R}\in SO(4)}{\forall} \ \underset{P,Q \in \text{unit quat.}}{\exists} \ \mathbf{R} = \mathbf{M}^+(P)\cdot\mathbf{M}^-(Q) \tag{14}$$

This fact is rather little known, though several different forms of its proof were published [1, 3, 6]. One can check that the product (14) is commutative and the factors can be transposed. The de-composition is unique up to sign, as $(-P,-Q)$ is also a valid solution. Neglecting these trivial transformations, there are six actual degrees of freedom, identified with the angles in the polar representations (8) of P and Q. This does not differ from 2D Givens approach.

It should be mentioned that in the biquaternion algebra exploited in [12, 13], multiplication is commutative and there is only one, but non-orthogonal multiplication matrix useless in our context.

4.3 Parameterization of general orthogonal matrices

Quaternion multiplication matrices can also be used in the factorization of arbitrary $M \times M$ $(M > 4)$ orthogonal matrix. The algorithm is very similar to the reduction described in Sec. 2. But now, the main tool is 4D Givens rotation matrix built by embedding $\mathbf{M}^{+}$ in the identity matrix. The case of $\mathbf{M}^{-}$ is also valid but we omit it for discussion clarity. One step consists in the multiplication of given matrix $\mathbf{R}$ by transposed 4D rotation matrix.

$$\begin{bmatrix} 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & q_1 & q_2 & q_3 & q_4 \\ 0 & \cdots & 0 & -q_2 & q_1 & q_4 & -q_3 \\ 0 & \cdots & 0 & -q_3 & -q_4 & q_1 & q_2 \\ 0 & \cdots & 0 & -q_4 & q_3 & -q_2 & q_1 \end{bmatrix} \mathbf{R} = \begin{bmatrix} \times & \times & \cdots & \times \\ \vdots & \vdots & \ddots & \times \\ \times & \times & \cdots & \times \\ \times & \times & \cdots & \times \\ 0 & \times & \cdots & \times \\ 0 & \times & \cdots & \times \\ 0 & \times & \cdots & \times \end{bmatrix} \tag{15}$$

The last four rows are replaced by their linear combinations. The angles of the rotation can be selected to zero three elements at once. The angle determination is directly related to the conversion of quaternion from rectangular (5) to polar representation (8).

To eliminate the subsequent entries of the current column, we repeat the above operation using a new properly modified rotation matrix. If the number of remaining elements is less than 4, the transition to common 2D rotations is appropriate. Thus we have a novel mixed 2D / 4D Givens reduction. Following this way, we can reduce the first column to obtain

$$\mathbf{\Theta}^{T}_{0,0}\mathbf{\Theta}^{T}_{0,1}\cdots\mathbf{\Theta}^{T}_{M-2-2\beta(M)}\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{R}' \end{bmatrix} \tag{16}$$

It may be shown that $\beta(M) = \lfloor (M-1)/3 \rfloor$ quaternionic, and $M - 1 - 3\beta(M)$ 2D Givens rotations are needed.

As $\mathbf{R}'$ is an orthogonal matrix of reduced size, it can be factorized in the above manner as well. But if it is 4×4 matrix, the result from Sec. 4.2 can be exploited and the factorization is completed with the product of two rotation matrices (one based on $\mathbf{M}^{-}$). Finally, the reduced matrix can be expressed as

$$\mathbf{R} = \underbrace{\mathbf{\Theta}_{0,M-2-2\beta(M)}\cdots\mathbf{\Theta}_{0,0}}\,\underbrace{\mathbf{\Theta}_{1,M-3-2\beta(M-1)}\cdots\mathbf{\Theta}_{1,0}}\cdots\underbrace{\mathbf{\Theta}_{M-4,+}\mathbf{\Theta}_{M-4,-}} \tag{17}$$

In total there are $\beta(M)\left(M - \tfrac{3}{2}\left(\beta(M)+1\right)\right)+1$ 4D rotations, and $\lfloor (M-2)/3 \rfloor + 2\lfloor (M-3)/3 \rfloor$ 2D Givens rotations. An example of a lattice derived this way can be seen in Fig. 3.

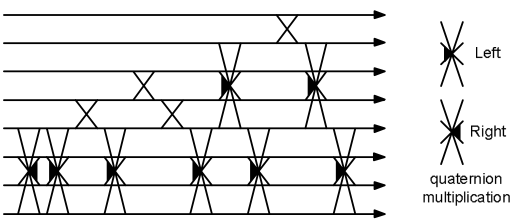


Figure 3: Lattice for 8×8 orthogonal matrix mixing 4D and 2D Givens rotations.

4.4 Connection to Givens rotations

As orthogonal, the matrices $\mathbf{M}^{+}$ and $\mathbf{M}^{-}$ can be factorized into six Givens rotations. This is done in quite interesting way ('c' - cosine and 's' - sine).

$$\mathbf{M}^{+}(Q) = \tag{18}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{c}\chi & -\mathrm{s}\chi \\ 0 & 0 & \mathrm{s}\chi & \mathrm{c}\chi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{c}\psi & -\mathrm{s}\psi & 0 \\ 0 & \mathrm{s}\psi & \mathrm{c}\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{c}\phi & -\mathrm{s}\phi & 0 & 0 \\ \mathrm{s}\phi & \mathrm{c}\phi & 0 & 0 \\ 0 & 0 & \mathrm{c}\phi & -\mathrm{s}\phi \\ 0 & 0 & \mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{c}\psi & \mathrm{s}\psi & 0 \\ 0 & -\mathrm{s}\psi & \mathrm{c}\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{c}\chi & \mathrm{s}\chi \\ 0 & 0 & -\mathrm{s}\chi & c\chi \end{bmatrix}$$

$$\mathbf{M}^{-}(Q) = \tag{19}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{c}\chi & -\mathrm{s}\chi \\ 0 & 0 & \mathrm{s}\chi & \mathrm{c}\chi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{c}\psi & -\mathrm{s}\psi & 0 \\ 0 & \mathrm{s}\psi & \mathrm{c}\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{c}\phi & -\mathrm{s}\phi & 0 & 0 \\ \mathrm{s}\phi & \mathrm{c}\phi & 0 & 0 \\ 0 & 0 & \mathrm{c}\phi & \mathrm{s}\phi \\ 0 & 0 & -\mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{c}\psi & \mathrm{s}\psi & 0 \\ 0 & -\mathrm{s}\psi & \mathrm{c}\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{c}\chi & \mathrm{s}\chi \\ 0 & 0 & -\mathrm{s}\chi & c\chi \end{bmatrix}$$

These equations correspond to the following lattice structure

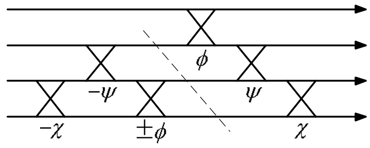


Figure 4: Givens decomposition of quaternion multiplication.

The angles of Givens rotations are simply the same as in polar quaternion representation (8).

5. PARAUNITARY SYSTEMS WITH QUATERNIONIC BUILDING BLOCKS

5.1 Implementational advantages of new building block

The preceding sections show that the quaternion multiplication matrices can replace Givens rotations, but the real profits offered by the quaternionic approach must be indicated. They are related to the implementation and concern very important issues.

Firstly, the quaternion product can be computed very efficiently with only 8 real multiplications [4]. Thus the complexity related to the product (14) is comparable with that of the multiplication by the corresponding 4×4 orthogonal matrix. This is not a case of Givens rotations, whose are too complex to be useful in the implementations of multichannel systems [10].

To have a quaternion multiplication matrix it is sufficient to store only 4 numbers. Thus two quaternions occupy less memory than the 16 elements of the direct form of the equivalent orthogonal matrix.

The numerical robustness of quaternionic arithmetic is noteworthy. Due to constant column norm [14], quaternion matrices can easily be re-orthogonalized after quantization to retain losslessness.

The quaternion arithmetic is well suited to hardware realizations in VLSI and FPGA. It can be implemented using distributed arithmetic, with only adders, shifters and ROMs [9]. Moreover, there exists a four-dimensional extension of the CORDIC algorithm to the quaternion rotations [5].

5.2 Quaternionic mutations of existing structures

The idea of modeling 4×4 orthogonal matrices with the quaternion multiplications, directly applies to many known structures of PUFBs, giving equivalent lattices with better implementational properties.

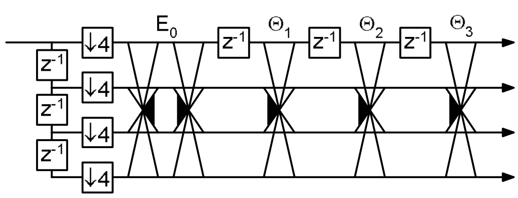


Figure 5: Pure quaternionic structure for 4 channel PUFB.

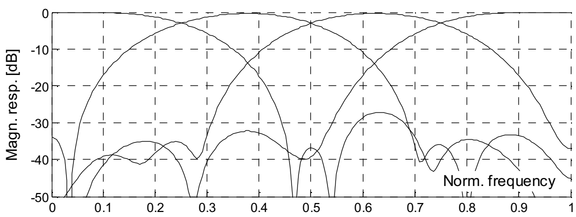
As an example, let us consider, shown in Fig. 5, quaternionic equivalent of the 4-channel system described by (2). The transition to quaternionic lattice is performed simply: by building $\mathbf{E}_0$ as in Sec. 4.2, and assuming $\mathbf{\Theta}_i$ to be the unit quaternion multiplication matrices. The proof of the equivalence starts with the decomposition of all $\mathbf{\Theta}_i$ in terms of Givens rotations, described in Sec. 4.4. Then starting at the right end of the structure, three rotations from one block (at the left side of the dashed line in Fig. 4), not affecting the line with the delays, can be coalesced with the previous block. Thus every stage can be reduced to sequence of Givens transformations, and $\mathbf{E}_0$ remains a general orthogonal matrix, exactly as it is in (2).

Another area of the utilization of the quaternionic building block are linear-phase PUFBs and LOTs. In many factorizations, e.g. in (4) for 8 channels, there are 4×4 orthogonal matrices, which can be modeled via quaternion multiplications.

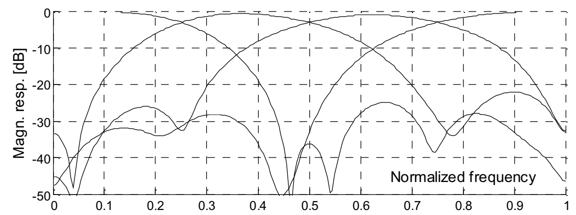
Unfortunately, the application of quaternions does not reduce the number of design parameters. But there are the prerequisites suggesting that the developing of new quaternion-specific factorizations can be fruitful. Namely we conclude that the structure in Fig. 5 retains good properties (Fig. 6b) even if the first stage E_0 is constrained to be right quaternion multiplication matrix (3 degree of freedom are renounced). This maneuver looks applicable to other schemes as well.

6. DESIGN EXAMPLES

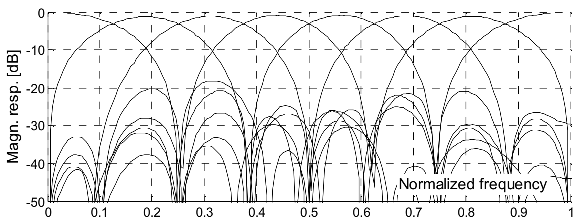
To validate our approach, we design several filter banks using quaternionic building blocks. The coefficient optimization was performed with standard Matlab routines, and the robust genetic algorithm already used successfully in filter design [8]. The goals of the optimization were typical, i.e. coding gain (AR(1) process with $\rho = 0.95$) and stopband attenuation. Here we only show the results obtained for the structures considered in Sec. 5. The achieved performance is comparable with those reported by other researchers.



a) 16 taps, Cod. Gain: 8.33 dB, Stopband Att. ≥ 27 dB



b) 16 taps, Cod. Gain: 8.31 dB, Stopband Att. ≥ 22 dB



c) 16 taps, Cod. Gain: 9.27 dB, Stopband Att. ≥ 18 dB

Figure 6: Freq. resp. of quaternionic PUFBs: 4 channels as in Fig. 5 (a), and after constraining E_0 (b); 8 channel linear-phase (c)

7. SUMMARY

Probably this is the first truly successful use of hypercomplex numbers in digital filtering structures. Quaternion multiplication turns out to be an alternative factorization tool and very good component for the implementation of typical PUFBs. Known structures can be easily converted to pure quaternionic form, but the development of new quaternion-oriented factorizations seems to be an appropriate direction for future works. The transition from lattices to lifting schemes is also interesting.

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