CLOSED-FORM DESIGN OF MAXFLAT *R*-REGULAR FIR *M*TH-BAND FILTERS WITH ARBITRARY GROUP DELAY USING WAVEFORM MOMENTS

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ABSTRACT

*M*th-band filters have been found numerous applications in multirate signal processing systems, filter banks and wavelets. In this paper, a new closed-form expression is presented for the impulse response of the maxflat *R*-regular FIR *M*th-band filters with arbitrary group delay. The filter coefficients are directly obtained by solving a linear system of Vandermonde equations that are derived from the regularity conditions via waveform moments. Moreover, the group delay of the proposed FIR *M*th-band filters can be arbitrarily specified. Finally, some design examples are presented to demonstrate the effectiveness of the proposed maxflat *R*-regular FIR *M*th-band filters.

1. INTRODUCTION

*M*th-band filters are an important class of digital filters and are often used in multirate digital signal processing systems, filter banks and wavelets, and so on. Its impulse response is required to be exactly zero-crossing except for one point. FIR *M*th-band filters with exact linear phase have been studied exhaustively in [1]∼[6]. Among these methods, a closed-form solution is given for the maxflat *R*-regular FIR *M*th-band filters with exact linear phase in [4] and [5], while the minimax solution can be found in [1] and [3]. However, a larger delay results when higher order FIR linear phase filters are needed. This is because the group delay is equal to half the filter order for the exact linear phase FIR filters. In some applications of real-time signal processing, a lower delay is generally needed. For this reason, FIR *M*th-band filters with arbitrary group delay need to be considered. For the design of FIR *M*th-band filters with arbitrary group delay, a closed-form solution is given only for a specific class of the maxflat *R*-regular FIR *M*th-band filters in [2], while the optimal solution with equiripple response in stopband can be found in [8]. The design of a general class of the maxflat *R*-regular FIR *M*th-band filters with arbitrary group delay is still open.

In this paper, we will consider the design of a general class of the maxflat *R*-regular FIR *M*th-band filters with arbitrary group delay. A new closed-form expression for its impulse response is presented. In the proposed method, we derive a linear system of Vandermonde equations from the regularity conditions of the *M*th-band filters via the block-wise waveform moments defined in [7], and then can get a set of filter coefficients by applying the Cramer's formula and Vandermonde's determinant. Moreover, the group delay of the proposed FIR *M*th-band filters can be arbitrarily specified. Finally, some examples are designed to demonstrate the effectiveness of the proposed maxflat *R*-regular FIR *M*th-band filters with arbitrary group delay.

2. FIR *M*TH-BAND FILTERS

Let the transfer function $H(z)$ of a FIR digital filter of length N be

$$H(z) = \sum_{n=0}^{N-1} h_n z^{-n}, \tag{1}$$

where h_n are real filter coefficients. When $H(z)$ is designed as an *M*th-band filter, its impulse response is required to be exactly zero-crossing except for one point K, i.e.,

$$h_{K+kM} = \begin{cases} \frac{1}{M} & (k=0) \\ 0 & (k=\pm1,\pm2,\cdots) \end{cases}, \tag{2}$$

where K and M are integers, and K corresponds to the desired group delay in the passband. In the case of FIR filters with exact linear phase, the filter coefficients have to be symmetric, i.e., $h_n = h_{N-1-n}$, and then, its group delay equals $K = (N-1)/2$. Hence, K increases with an increasing filter length N. It results in a larger delay when higher order FIR filters are needed. In some applications of real-time signal processing, a lower delay is generally required. Here, we consider the design of FIR *M*th-band filters with arbitrary group delay K, resulting in an approximate linear phase response in the passband.

*M*th-band filter is required to be lowpass, and the desired frequency response is given by

$$H_d(e^{j\omega}) = \begin{cases} e^{-jK\omega} & \text{(in passband)} \\ 0 & \text{(in stopband)} \end{cases}. \tag{3}$$

Let a noncausal shifted version of $H(z)$ be $\hat{H}(z) = z^K H(z)$, i.e., $\hat{h}_n = h_{n+K}$. By substituting the time-domain condition of Eq.(2) into Eq.(1), we have

$$\hat{H}(z) = z^K H(z) = \frac{1}{M} + \sum_{\substack{n=0 \\ \neq K+kM}}^{N-1} h_n z^{K-n}, \tag{4}$$

and then

$$\hat{H}_d(e^{j\omega}) = \begin{cases} 1 & \text{(in passband)} \\ 0 & \text{(in stopband)} \end{cases}. \tag{5}$$

It can be seen from Eq.(4) that the frequency response of $\hat{H}(z)$ always satisfies

$$\sum_{k=0}^{M-1} \hat{H}(e^{j(\omega+\frac{2k\pi}{M})}) \equiv 1, \tag{6}$$

which means that the sum of the responses at the frequency points $\omega_k = \omega + 2k\pi/M$ for $k = 0,1,\cdots,M-1$ keeps constant, regardless of what the filter coefficients h_n are.

3. DEFINITION OF WAVEFORM MOMENTS

For a non-negative integer r, the rth waveform moment around zero for $\hat{h}_n$ is defined by

$$m_r = \sum_{n=-K}^{N-K-1} n^r \hat{h}_n. \quad (7)$$

Note that $\hat{h}_n$ ranges from $-K$ to $(N-K-1)$. It is known in [7] and [5] that the rth waveform moment m_r describes the rth differential coefficient of the frequency response $\hat{H}(e^{j\omega})$ at $\omega = 0$;

$$m_r = j^r \frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\bigg|_{\omega=0}. \quad (8)$$

In [7] and [5], the blockwise waveform moment around zero has been also defined by

$$m_r(i) = \sum_{k=N_l(i)}^{N_u(i)} (kM+i)^r \hat{h}_{kM+i}, \quad (9)$$

where $0 \le i \le M-1$ and

$$\begin{cases} N_l(i) = -\lfloor \frac{K+i}{M} \rfloor \\ N_u(i) = \lfloor \frac{N-K-i-1}{M} \rfloor \end{cases}. \quad (10)$$

Note that $\lfloor x \rfloor$ is the largest integer not greater than x. It follows from the definition of $m_r(i)$ in Eq.(9) that

$$\frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\bigg|_{\omega=\frac{2k\pi}{M}} = (-j)^r \sum_{i=0}^{M-1} m_r(i) e^{-j\frac{2ik\pi}{M}}, \quad (11)$$

i.e., the blockwise waveform moments describe the derivative behavior of the frequency response $\hat{H}(e^{j\omega})$ at the frequency points $\omega_k = 2k\pi/M$ $(0 \le k \le M-1)$. It is clear in Eq.(11) that the rth derivatives of the frequency response $\hat{H}(e^{j\omega})$ at $\omega_k = 2k\pi/M$ are the M-point DFT (Discrete Fourier Transform) of the blockwise waveform moments $m_r(i)$. Thus, we have by the inverse transform

$$m_r(i) = \frac{j^r}{M} \sum_{k=0}^{M-1} \frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\bigg|_{\omega=\frac{2k\pi}{M}} e^{j\frac{2ik\pi}{M}}. \quad (12)$$

It is clear that the blockwise waveform moments $m_r(i)$ bridge between the time and frequency domains by Eqs.(9) and (12).

4. CLOSED-FORM DESIGN

M-band wavelets have been studied in [4] as a generalization of 2-band wavelets, since they help to zoom in onto narrow band high frequency components of a signal, while simultaneously having a logarithmic decomposition of frequency channels. In the construction of M-band wavelet bases, a regular M-band scaling filter needs to be designed firstly. It is known in [4] that this M-band scaling filter can be obtained from Mth-band filters. Therefore, the design of Mth-band filters satisfying the R-regularity condition will be required. In [4] and [5], a closed-form solution is given for the maxflat R-regular FIR Mth-band filters with exact linear phase. In particular, the closed-form solution given in [5] is more general than that in [4], and is derived via the blockwise waveform moments defined in [7]. For the design of R-regular FIR Mth-band filters with arbitrary group delay, however, a closed-form solution is given in [2] only for a specific class of the maxflat R-regular FIR Mth-band filters with $K = kM-1$ and $N = MR-1$, where $1 \le k \le R-1$. Thus, K is restricted to several special integers and then the group delay cannot be arbitrarily specified. In the following, we will describe how to design a general class of the maxflat R-regular FIR Mth-band filters with arbitrary group delay.

It is known in [4] that an Mth-band filter is said to be R-regular if it has

$$H(z) = (1 + z^{-1} + \cdots + z^{-(M-1)})^R Q(z), \quad (13)$$

where $Q(z)$ is a FIR filter of length L. $Q(z)$ is used for $H(z)$ to satisfy the time-domain condition in Eq.(2). Thus, the minimal length of $Q(z)$ is $L = R$, in general, then the filter length of $H(z)$ becomes

$$N = (M-1)R + R = MR. \quad (14)$$

Deponding on the positin of K, the filter length N may degrade to $N = MR-1$, when $h_0 = 0$ if $K = kM$ $(1 \le k \le R-1)$ or $h_{MR-1} = 0$ if $K = kM-1$ $(1 \le k \le R-1)$. It is known in [4] that Eq.(13) is equivalent to

$$\frac{\partial^r H(e^{j\omega})}{\partial \omega^r}\bigg|_{\omega=\frac{2k\pi}{M}} = \frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\bigg|_{\omega=\frac{2k\pi}{M}} = 0, \quad (15)$$

for $k = 1, 2, \cdots, M-1$ and $r = 0, 1, \cdots, R-1$. It can be obtained from Eq.(6) that

$$\hat{H}(e^{j\omega}) = 1 - \sum_{k=1}^{M-1} \hat{H}(e^{j(\omega + \frac{2k\pi}{M})}), \quad (16)$$

then we have from Eq.(15),

$$\begin{cases} \hat{H}(e^{j\omega})\big|_{\omega=0} = 1 & (r=0) \\ \frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\bigg|_{\omega=0} = 0 & (r = 1, 2, \cdots, R-1) \end{cases}, \quad (17)$$

which means that the magnitude response $|\hat{H}(e^{j\omega})|$ and group delay $\hat{\tau}(\omega)$ of $\hat{H}(z)$ satisfy at $\omega = 0$

$$\begin{cases} |\hat{H}(e^{j\omega})|\big|_{\omega=0} = 1 & (r=0) \\ \frac{\partial^r |\hat{H}(e^{j\omega})|}{\partial \omega^r}\bigg|_{\omega=0} = 0 & (r = 1, 2, \cdots, R-1) \end{cases}, \quad (18)$$

and

$$\frac{\partial^r \hat{\tau}(\omega)}{\partial \omega^r}\bigg|_{\omega=0} = 0 \quad (r = 0, 1, \cdots, R-2). \quad (19)$$

From the relationship between $H(z)$ and $\hat{H}(z)$ in Eq.(4), we have

$$\begin{cases} |H(e^{j\omega})|\big|_{\omega=0} = 1 & (r=0) \\ \frac{\partial^r |H(e^{j\omega})|}{\partial \omega^r}\bigg|_{\omega=0} = 0 & (r = 1, 2, \cdots, R-1) \end{cases}, \quad (20)$$

and

$$\begin{cases} \tau(\omega)\big|_{\omega=0} = K & (r=0) \\ \frac{\partial^r \tau(\omega)}{\partial \omega^r}\bigg|_{\omega=0} = 0 & (r = 1, 2, \cdots, R-2) \end{cases}. \quad (21)$$

It is seen in Eqs.(20) and (21) that $H(z)$ has both a flat magnitude reponse and a flat group delay at $\omega = 0$.

By using the regularity conditions in Eqs.(15) and (17), we have from Eq.(12)

$$m_r(i) = \begin{cases} \frac{1}{M} & (r=0) \\ 0 & (r = 1,2,\cdots,R-1) \end{cases}. \quad (22)$$

From definition of the blockwise waveform moment in Eq.(9), we can derive a system of linear equations as follows;

$$\begin{cases} \sum_{k=N_l(i)}^{N_u(i)} \hat{h}_{kM+i} = \frac{1}{M} & (r=0) \\ \sum_{k=N_l(i)}^{N_u(i)} (kM+i)^r \hat{h}_{kM+i} = 0 & (r=1,2,\cdots,R-1) \end{cases}. \quad (23)$$

Since $\hat{h}_n = h_{n+K}$ and the minimal length is $N = MR$ for the R-regular FIR Mth-band filters in general, we have

$$\begin{cases} \sum_{k=0}^{R-1} h_{kM+i} = \frac{1}{M} & (r=0) \\ \sum_{k=0}^{R-1} (kM+i-K)^r h_{kM+i} = 0 & (r=1,2,\cdots,R-1) \end{cases}. \quad (24)$$

We rewrite Eq.(24) in matrix form as

$$\mathbf{Vh} = \mathbf{u}, \quad (25)$$

where $\mathbf{h} = [h_i, h_{M+i}, \cdots, h_{(R-1)M+i}]^T$, $\mathbf{u} = [1/M, 0, \cdots, 0]^T$, and

$$\mathbf{V} = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ (i-K) & (M+i-K) & \cdots & \cdots & ((R-1)M+i-K) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ (i-K)^{R-1} & (M+i-K)^{R-1} & \cdots & \cdots & ((R-1)M+i-K)^{R-1} \end{bmatrix}.$$

It should be noted that $\mathbf{V}$ is the Vandermonde matrix with distinct elements. Therefore, there is always a unique solution. By using the Cramer's rule and Vandermonde's determinant, a closed-form solution can be obtained by

$$h_{kM+i} = \frac{(-1)^k \prod_{\substack{n=0 \\ \neq k}}^{R-1} (nM-K+i)}{M^R k!(R-k-1)!}. \quad (26)$$

Once M, K and R are given, then a set of filter coefficients can be easily obtained by using Eq.(26). From Eq.(26), we have the following relation between the filter coefficients;

$$\begin{aligned} h_{kM+i} &= \frac{((k-1)M-K+i)(k-R)}{(kM-K+i)k} h_{(k-1)M+i} \\ &= \frac{((k+1)M-K+i)(k+1)}{(kM-K+i)(k+1-R)} h_{(k+1)M+i}. \end{aligned} \quad (27)$$

It is found that the filter coefficients can be efficiently calculated by using Eq.(27) instead of Eq.(26). Also, Eq.(27) will lead to a new efficient implementation of the proposed filters.

5. DESIGN EXAMPLES

In this section, we consider the design of the maxflat R-regular FIR Mth-band filters with $M = 7$ and $R = 10$. The filter length is $N = 70$ generally. We first set $K = 25$ and get a set of filter coefficients by Eq.(26). The resulting magnitude response and group delay are shown in the solid line in Fig.1 and Fig.2, respectively, and its impulse response is shown in Fig.3(a). We have also designed five other filters with $K = 28, 34, 35, 41, 44$. Their magnitude responses and group delays are shown also in Fig.1 and Fig.2. The impulse responses of $K = 28$ and $K = 34$ are shown in Fig.3(b) and Fig.3(c), respectively. When $K = 35, 41$ and 44, their impulse responses are the time-reversed versions of $K = 34, 28$ and 25, so are omitted here. It is then seen in Fig.1 that the filters of $K = 35, 41$ and 44 have the same magnitude responses as that of $K = 34, 28$ and 25. As shown in Fig.2, two filters with $K = 34$ and $K = 35$ have a constant group delay at all frequencies, i.e., their phase responses are exactly linear. Thus the two filters have a symmetric impulse response, as shown in Fig.3(c). It is seen in Fig.2 that the group delays of $K = 25, 28, 34$ are symmetric with that of $K = 44, 41, 35$ about $(N-1)/2 = 34.5$, respectively. Since $h_0 = 0$ when $K = 28, 35$ and $h_{69} = 0$ when $K = 34, 41$ by the time-domain condition of Eq.(2), it is noted that these filters have the actual filter length $N = 69$. From the above results, we can conclude that the maxflat R-regular FIR Mth-band filters with arbitrary group delay can be easily designed.

6. CONCLUSION

In this paper, a general class of the maxflat R-regular FIR Mth-band filters with arbitrary group delay have been considered. A new closed-form expression for its impulse response has been presented. The filter coefficients are directly obtained by solving a linear system of Vandermonde equations that are derived from the regularity conditions of the Mth-band filters via the blockwise waveform moments. Moreover, the group delay of the proposed FIR Mth-band filters can be arbitrarily specified. Finally, some design examples have been presented to demonstrate the effectiveness of the proposed maxflat R-regular FIR Mth-band filters with arbitrary group delay.

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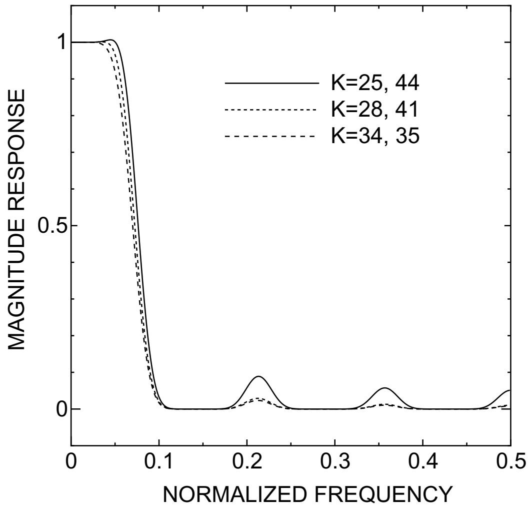

Figure 1: Magnitude responses of the maxflat *R*-regular FIR *M*th-band filters.

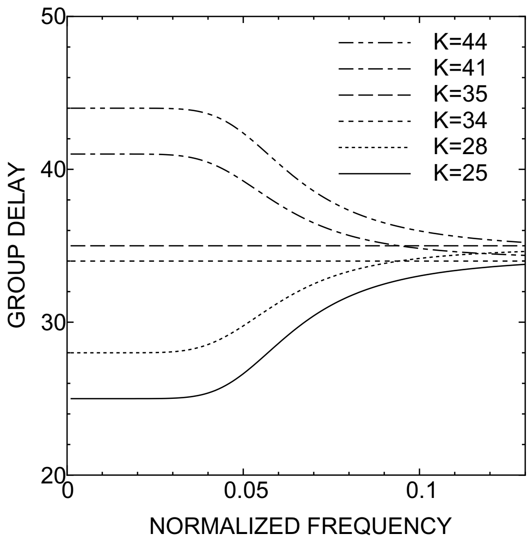

Figure 2: Group delays of the maxflat *R*-regular FIR *M*th-band filters.

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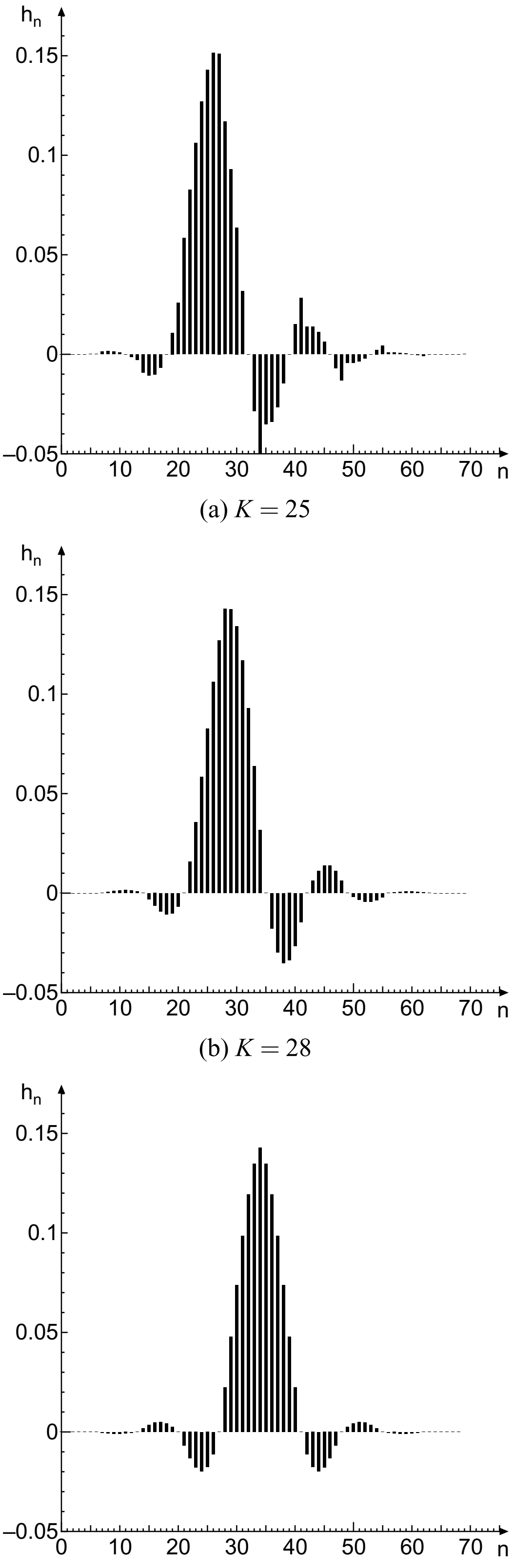

Figure 3: Impulse responses of the maxflat *R*-regular FIR *M*th-band filters. (a) $K = 25$, (b) $K = 28$, (c) $K = 34$.