In this paper, we present a simultaneous approximation of the magnitude and the group delay in FIR digital filters. It is very important for filter design to specify both magnitude and group delay. There are few research works to meet simultaneous specifications on the magnitude and the group delay. This method is based on solving a least squares solution iteratively. At each iteration, the desired responses of the magnitude and the group delay are transformed so that these error responses have equiripple. By using our new algorithm, we obtain a better result in the solution of FIR filter design problem.

1. INTRODUCTION

Recently, the digital filters are designed in the complex domain using the minimax criterion or the least squares criterion[1]-[4]. Most research works minimize the absolute error between the designed and desired frequency responses in the complex domain[1]-[4]. This means the simultaneous approximation of the magnitude and the phase responses. However it is important on signal processing and wave transmission to specify the magnitude and the group delay. In general, allpass filtering is used to equalize the phase response to make the group delay approximately constant. However it is inefficient because the filter coefficients are redundant. It is desired to design a filter that meets the magnitude and the group delay response simultaneously without using allpass filter. Only a few research works have designed FIR filters that meet such a characteristics.

The required length of a linear phase FIR filter is roughly inversely proportional to the absolute transition bandwidth. The group delay becomes large as the transition bandwidth is decreased. Therefore it is desired to design FIR filters with lower group delay while keeping the magnitude response. Calvango et al.[5] have designed such a filters using the multiple criterion optimization. However this algorithm is too complex to design. It may take long time to design.

In this paper, we present a design algorithm of FIR digital filters that approximate the magnitude and the group delay simultaneously and have quasi-equiripple magnitude and group delay errors. Although equiripple solutions are not necessarily optimal, it is true that equiripple solutions often give quite satisfactory results. This method is based on solving a least squares solution iteratively. At each iteration, the desired responses of the magnitude and the group delay are transformed so that these error responses have equiripple. By using our new algorithm, the designed FIR digital filters have quasi-equiripple magnitude error and group delay errors. Of course, it is sometimes impossible for both the magnitude and group delay errors to be equiripple because they are related each other. However the obtained results are better than that of the conventional method. Also since this algorithm only solves a linear equation iteratively, it is very simple and does not need any special optimization algorithm.

In section 2, we show a least squares approximation of FIR digital filters and formulate the design problem. Section 3 gives the transform of the magnitude and group delay errors. Section 4 shows some examples of FIR digital filters to validate the proposed method.

2. FORMULATION OF THE LEAST SQUARES APPROXIMATION

This section gives the least squares approximation of the frequency response and the group delay response in FIR filters. Since it is very difficult to approximate the magnitude response of the digital filters directly, we approximate the frequency response indirectly.

2.1. Frequency Responses

The transfer function of FIR digital filter with order $N$ is defined by

$$H(z) = \sum_{n=0}^{N} h_n z^{-n} \quad (1)$$

Let the desired response and the weighting function be $D(e^{j\omega})$ and $W_M(\omega)$, respectively, the weighted squared error of the frequency response is

$$E^M = \sum_{\ell=0}^{L-1} W_M(\omega_\ell) |H(e^{j\omega_\ell}) - D(e^{j\omega_\ell})|^2 \quad (2)$$

where $\omega_\ell = 2\pi \ell/L (\ell = 0, 1, \ldots, L - 1)$. With the matrix representation, $E^M$ is rewritten by

$$E^M = h^T P^T M P h - 2h^T R e(P^T M d) + d^T M d \quad (3)$$

where $x^T$, $x^1$ and $Re(x)$ denote the transpose, Hermite transpose of $x$ and the real part of $x$, respectively. And,

$$h = [h_0, h_1, \ldots, h_N]^T$$

$$[P]_{\ell n} = e^{-j\omega_\ell n} \quad (\text{for } \ell = 0, 1, \ldots, L - 1, \quad n = 0, 1, \ldots, N)$$

$$d = [D(e^{j\omega_0}), D(e^{j\omega_1}), \ldots, D(e^{j\omega_{L-1}})]^T$$

$$W_M = \text{diag}[W_M(\omega_0), W_M(\omega_1), \ldots, W_M(\omega_{L-1})]$$

A least squares solution is obtained by setting $\partial E^M/\partial h = 0$

$$h = (P^T M P)^{-1} R e(P^T M d) \quad (4)$$

This is a normal least squares solution.
2.2. Group Delay

The group delay response $\tau(\omega)$ of the polynomial $H(z)$ in (1) is expressed by

$$\tau(\omega) = \text{Re} \left\{ \frac{dH(z)/dz}{H(z)} \right\}_{z=e^{j\omega}} \tag{5}$$

where

$$[zdH(z)/dz]_{z=e^{j\omega}} = \sum_{n=1}^{N} nh_n e^{-jn\omega}$$

Also, we denote $H(e^{j\omega})$ as

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)}$$

The group delay response of $H(e^{j\omega})$ is written by

$$\tau(\omega) = \text{Re} \left\{ \frac{1}{|H(e^{j\omega})|} \sum_{n=1}^{N} nh_n e^{-jn\omega} \right\}$$

$$= \frac{1}{|H(e^{j\omega})|} \sum_{n=1}^{N} nh_n \cos(n\omega + \theta(\omega)) \tag{6}$$

Let the desired delay response be $\tau_d(\omega)$, the squared error of the group delay response is

$$E^\tau = \sum_{k=0}^{L-1} W^\tau(\omega_k) |\tau(\omega_k) - \tau_d(\omega_k)|^2 \tag{7}$$

where $W^\tau(\omega)$ is the weighting function for group delay, which is 1 in the interest band and 0 otherwise. Note this weighting function only defines the approximated and non-approximated bands.

The error on the discrete frequency point is expressed by

$$E^\tau(\omega_k) = \frac{1}{|H(e^{j\omega_k})|} \sum_{n=1}^{N} nh_n \cos(n\omega_k + \theta(\omega_k)) - \tau_d(\omega_k)$$

The above equation is written in the matrix form as

$$E^\tau = \mathbf{P}^\tau \mathbf{h} - \mathbf{d}_\tau \tag{8}$$

where

$$\mathbf{E}^\tau = [E^\tau(\omega_0), E^\tau(\omega_1), \ldots, E^\tau(\omega_{L-1})]^T$$

$$\mathbf{d}_\tau = [\tau_d(\omega_0), \tau_d(\omega_1), \ldots, \tau_d(\omega_{L-1})]$$

$$[\mathbf{P}]^\tau_{\ell n} = \frac{1}{|H(e^{j\omega_n})|} \cos(n\omega_\ell + \theta(\omega_\ell))$$

for \( \ell = 0, 1, \ldots, L-1 \), \( n = 1, 2, \ldots, N \)

where $\hat{H}(e^{j\omega})$ and $\hat{\theta}(\omega)$ are $H(e^{j\omega})$ and $\theta(\omega)$ that were obtained at the previous iteration. Therefore the objective function is

$$E^\tau = \mathbf{h}^T \mathbf{P}^\tau \mathbf{W}^\tau \mathbf{P}^T \mathbf{h} - 2\mathbf{h}^T \mathbf{P}^T \mathbf{W}^\tau \mathbf{d}_\tau + \mathbf{d}_\tau^T \mathbf{W}^\tau \mathbf{d}_\tau \tag{9}$$

Thus by solving (9) iteratively while changing $\hat{H}(e^{j\omega})$ and $\hat{\theta}(\omega)$, a least squares solution can be obtained.

2.3. Design Formula

In order to design FIR digital filters that specify the magnitude and group delay simultaneously, the objective function is represented by a linear combination of the frequency response and group delay errors.

$$E_m = \alpha E_m^M + E_m^\tau = h^T Q \mathbf{h} - 2h^T \mathbf{q} + q \tag{10}$$

where $\alpha$ is the weighting function trading-off the magnitude versus the group delay errors and $m$ is the iteration number. $\mathbf{Q}$ and $\mathbf{q}$ are the matrix and vector, respectively which are written by linear combination of (3) and (9). The minimization of this objective function is achieved when $\partial E/\partial \mathbf{h} = 0$, namely

$$\mathbf{h} = \mathbf{Q}^{-1} \mathbf{q} \tag{11}$$

This least squares solution is solved iteratively while changing $\hat{H}(e^{j\omega})$ and $\hat{\theta}(\omega)$ that is the phase response of $H(e^{j\omega})$. Since we are interested in the magnitude and the group delay approximation, the phase response of the filter is “do not care” response because the group delay is already approximated. Then the desired frequency response $D(e^{j\omega})$ is transformed while keeping the desired magnitude response $|D(e^{j\omega})|$ as

$$D(e^{j\omega}) = |D(e^{j\omega})| \frac{H(e^{j\omega})}{H(e^{j\omega})} \tag{12}$$

This means that the phase response of the filter designed at previous iteration is used as the phase response of the desired frequency response.

Finally we summarize the design algorithm for a least squares approximation.

[DESIGN ALGORITHM 1]

1. Decide N, M, and the desired frequency response $D(\omega)$ and the desired group delay $\tau_d(\omega)$

2. Set $\hat{\theta}(\omega) = 0$, $H(\omega) = 1$.

3. The filter coefficients are obtained by solving (10)

4. If $|E_m^M - E_m^{M-1}|/E_m^M << \epsilon$ and $|E_m - E_m^{n-1}|/E_m << \epsilon$ (set $\epsilon = 10^{-3}$), then terminate, otherwise go to Step 5

5. Set $\hat{\theta}(\omega) = \theta(\omega)$ and $H(\omega) = H(\omega)$

6. Set $D(e^{j\omega}) = |D(e^{j\omega})| \frac{H(e^{j\omega})}{H(e^{j\omega})}$ and return to Step 3

3. TRANSFORM OF DESIRED RESPONSE

In this section, we present a new method to design the FIR digital filters that minimize the errors of the magnitude and group delay simultaneously. This method is based on solving the least squares method iteratively while transforming the desired response of the magnitude and the group delay so that these errors between the original desired and the designed frequency responses become equiripple. This algorithm consists of two iteration loops, a loop for a least squares approximation as mentioned before, another is a main loop that transforms the desired response for a least squares approximation. Furthermore, this algorithm does not need the special optimization algorithm and only solve the least squares method iteratively.
3.1. Transform of the Desired Response

Let the frequency response and the group delay response of the designed filters at m-th iteration be \( H_m(e^{j\omega}) \) and \( \tau_m(\omega) \), respectively, the magnitude and group delay errors between the original desired and the designed filter are written by

\[
E_m^M(\omega) = W_m(\omega)|H_m(e^{j\omega}) - |D_0(e^{j\omega})| \]
\[
E_m^\tau(\omega) = \tau_m(\omega) - \tau_d(\omega) \quad (13)
\]

where \( D_0(e^{j\omega}) \) is the original desired magnitude response that is 1 at the passband and 0 at the stopband in general. Also \( \tau_d(\omega) \) is the original desired group delay that is constant in the interest band in general. Next we find the local maximum point \( \omega_{\text{max}}^M(k) \) and the local minimum point \( \omega_{\text{min}}^M(k) \) of \( |E_m^M(\omega)| \), and set the local maximum value \( \delta_m^M(k) = |E_m^M(\omega_{\text{max}}^M(k))| \). Similarly to the magnitude response, we find the local maximum point \( \omega_{\text{max}}^\tau(k) \) and the local minimum point \( \omega_{\text{min}}^\tau(k) \) of \( |E_m^\tau(\omega)| \) and set the local maximum value \( \delta_m^\tau(k) = |E_m^\tau(\omega_{\text{max}}^\tau(k))| \). It is obvious that the local maximum and minimum points appear alternately as \( 0 < \omega_{\text{max}}^M(1) < \omega_{\text{min}}^M(1) < \omega_{\text{max}}^M(2) \cdot \cdot \cdot \). To make the error be equiripple, we take the averages of the local maximum magnitude errors and the local maximum group delay errors, respectively as follows

\[
\delta_m^M = \text{mean}(\delta_m^M(k))
\]
\[
\delta_m^\tau = \text{mean}(\delta_m^\tau(k)) \quad (14)
\]

These values are candidates for equiripple at next iteration. In order to obtain the equiripple error response, \( E_m^M(\omega) \) and \( E_m^\tau(\omega) \) are transformed by

\[
R_m^M(\omega) = \frac{E_m^M(\omega) - \delta_m^M}{\delta_m^M(k+1) W_m(\omega)} \quad (15)
\]

for \( \omega_{\text{min}}^M(k) \leq \omega < \omega_{\text{min}}^M(k+1) \)

\[
R_m^\tau(\omega) = \frac{E_m^\tau(\omega) - \delta_m^\tau}{\delta_m^\tau(k+1)} \quad (16)
\]

for \( \omega_{\text{min}}^\tau(k) \leq \omega < \omega_{\text{min}}^\tau(k+1) \)

With this procedure, \( R_m^M(\omega) \) and \( R_m^\tau(\omega) \) have equiripple responses with weighting function. This procedure to scale the error response \( |E_m^M(\omega)| \) is shown in Fig. 1.

By adding the transformed error response \( R_m^M(\omega) \) and \( R_m^\tau(\omega) \) to the original desired function \( D_0(\omega) \) and \( \tau_d(\omega) \), respectively, new desired functions with equiripple magnitude and group delay error responses are obtained by

\[
D_{m+1}(\omega) = (R_m^M(\omega) + |D_0(\omega)|) |H_m(\omega)|
\]
\[
\tau_{m+1}(\omega) = R_m^\tau(\omega) + \tau_d(\omega) \quad (18)
\]

In this case, the phase of the filter designed in Algorithm 1 is used as the phase response of the desired frequency response. With these new desired responses, a least squares approximation is solved by Algorithm 1.

3.2. Algorithm

Finally, we summarize the overall algorithm to approximate the magnitude and group delay responses simultaneously.

[ DESIGN ALGORITHM 2 ]

1. Decide \( N, M \) and the desired magnitude response \( D_0(\omega) \) and the desired group delay \( \tau_d(\omega) \).
2. The filter is designed by the least squares method using Algorithm 1.
3. Calculate the magnitude and the group delay errors.
4. Find the local maximum and minimum points of the error responses and calculate the average of the local maximums.
5. If \( |\delta_{m-1}^M - \delta_m^M|/\delta_m^M < \epsilon \) and \( |\delta_{m-1}^\tau - \delta_m^\tau|/\delta_m^\tau < \epsilon \) (set \( \epsilon = 10^{-5} \)), then the algorithm terminates. Otherwise, go to next step.
6. Transform the errors so that these error responses become equiripple.
7. Add the transformed errors to the original desired response and return to 2.

In this algorithm, step 2 derives an initial guess for iteration and need some iterations in Algorithm 1.

We have confirmed that the algorithm converges very quickly and the maximum error monotonically decreases through considerable experiences. Remarks: the overall objective function is defined by

\[
E_m = \alpha |E_m^M + E_m^\tau|
\]

where \( \alpha \) is the weighting function. When \( \alpha \) becomes large, the magnitude error is minimized more than the group delay error. However, the dimension of the magnitude is quite different with that of the group delay. Then \( \alpha \) does not have evident relationship with the magnitude and the group delay errors. Therefore we need some trials to determine \( \alpha \) when the specification is given.

On the other hand, we can only minimize the magnitude error while keeping the group delay error constant. Fixing \( \delta_m^\tau \) in (14), the above objective function is minimized. In this case, it is
important to select suitable $\alpha$ because the magnitude error $E_m^\alpha$ is minimized while making the group delay error $E_\tau^\alpha$ small. From considerable experiences, the magnitude error is minimized while changing $\alpha$ as follows.

$$\alpha_m = \begin{cases} \alpha_{\text{max}} & \beta \geq \alpha_{\text{max}} \\ \beta & \text{otherwise} \end{cases}$$  \hspace{1cm} (19)$$

where $\beta = \frac{E_\tau^\alpha}{E_m^\alpha}$ and $\alpha_{\text{max}}$ and $\alpha_{\text{min}}$ are the maximum and minimum values of $\alpha$. We set $\alpha_{\text{max}} = 180$ and $\alpha_{\text{min}} = 30$.

4. DESIGN EXAMPLE

The specifications of FIR digital filter is $N = 30$.

$$|D_0(\omega)| = \begin{cases} 1 & 0 \leq |\omega| \leq 0.12\pi \\ 0 & 0.24\pi \leq |\omega| \leq \pi \end{cases}$$

$$\tau_\alpha(\omega) = 12 \text{ for } 0 \leq |\omega| \leq 0.12\pi$$

The weighting function is set to $W_M(\omega) = 1$ in passband, $W_M(\omega) = 8$ in stopband and $W_M(\omega) = 0$ in transition band. This specification is same as [5] for comparison.

Fig.2 shows the magnitude responses in the passband(a), stopband(b) and overall(c), and the group delay response(d). The solid line, dashed line, dash dotted line and dotted line correspond to the proposed method with $\alpha = 1$, $\delta_{\text{max}} = 0.1\pi$, $\delta_{\text{max}} = 0.439$, and $\delta_{\text{max}} = 0.5755$. The later two results are for comparison with [5].

As shown these figures, the designed responses are almost equiripple in magnitude and group delay responses. Fig.2(e) and (f) show the convergence response of the maximum error of both the magnitude and the group delay, respectively, when $\alpha = 1$. As shown in these figures, the maximum errors monotonically decrease.

Table 1 shows the magnitude error in the passband and the stopband, group delay error and the number of iterations. The values in bracket are the results in [5]. As shown this table, the proposed method has comparable results to [5].

Table 1: Comparison with the conventional method in Example 1

<table>
<thead>
<tr>
<th>Filter</th>
<th>$\alpha = 1$</th>
<th>$\delta = 0.1$</th>
<th>$\delta = 0.439$</th>
<th>$\delta = 0.575$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum PB error</td>
<td>0.0716</td>
<td>0.0623</td>
<td>0.0452</td>
<td>0.0396</td>
</tr>
<tr>
<td>Maximum PB error</td>
<td>0.00896</td>
<td>0.0079</td>
<td>0.00446</td>
<td>0.00096</td>
</tr>
<tr>
<td>Maximum PB error</td>
<td>0.001622</td>
<td>0.100</td>
<td>0.4393</td>
<td>0.5755</td>
</tr>
<tr>
<td># of iterations</td>
<td>77</td>
<td>61</td>
<td>67</td>
<td>57</td>
</tr>
</tbody>
</table>

PB: Passband, SB: Stopband, () indicates the result in [5]

5. CONCLUSION

In this paper, we proposed a new design method of the FIR digital filters that approximate the magnitude and the group delay responses simultaneously. The proposed method solves only a least squares approximation while transforming the desired magnitude and group delay responses to obtain the equiripple characteristics. With this method, FIR digital filters can be easily designed without a special optimization algorithm. In spite of a very simple algorithm, its results are better than the conventional methods.

6. REFERENCES