ON THE DESIGN OF TWO-DIMENSIONAL POLAR SEPARABLE FILTERS

Michael Felsberg

Computer Vision Laboratory, Linköping University
SE-58183 Linköping, Sweden
phone: +46 13 282460, fax: +46 13 138526, email: mfe@isy.liu.se
web: www.isy.liu.se/˜mfe

ABSTRACT

In this paper we present a new approach to the design of polar separable 2D filters. The novelty lies in the existence of an analytic description of the filter in both domains, the spatial and the frequency domain. That means no numeric optimisation becomes necessary and derivatives of the filters can be computed analytically. The method is based on a series of Poisson filters, which is interpreted in terms of a z-transform. The resulting radial filters are then combined with spherical harmonics. We show several examples, among these a new 2D quadrature filter.

1. INTRODUCTION

The topic of this paper is the design of polar separable filters. In contrast to many former publications on this topic, we do not address numerical filter optimisation schemes, as for instance in [12] (see also [10]), but a fully symbolic approach. Our proposed method allows to derive filters which can be represented analytically in both domains, the spatial domain and the frequency domain.

Polar separable filters are especially useful to extract features of two-dimensional signals which are rotation invariant and to estimate the signal orientation. Examples for rotation invariant features are the local phase and the local frequency which can be extracted by steerable filters [9] or by spherical quadrature filters based on the monogenic signal [6]. Edges [3] and corners [11] are further examples for features which are independent of the absolute orientation.

Several methods in image and image sequence processing make use of polar separable filters for the purpose of orientation estimation, see for instance [10, 4, 13]. Hence, polar separable filters are an important tool in two-dimensional signal processing. The main drawback of these filters is, however, that their (inverse) Fourier transform is difficult or impossible to compute in closed form. Those filters which have closed form solutions (see e.g. [5]) are often not sufficiently band-selective.

Most practically useful filters, as for instance the lognormal filter [10], cannot be represented in analytic form in the spatial domain. In most applications, however, we do not require exactly that particular filter – a sufficiently similar filter would do the same job. Therefore, a filter approximation scheme which produces filters with similar amplitude responses, but with a closed form analytic description in the spatial domain would be useful.

But why do we want to have a closed form description in the spatial domain? There can be several cases where this is useful, for instance as it comes to spatial derivatives of filter responses [8], certain theoretic derivations like bandwidth products, etc.. Also the numerical optimisation of filter masks can be improved by defining error measures in both domains simultaneously.

We propose a new approach for radial symmetric filter approximation in terms of a series of Poisson filters [5] in section 2. The coefficients of this series can be computed in terms of an inverse z-transform. In section 3 we combine this Poisson filter series with spherical harmonics, in order to approximate arbitrary polar separable filters. Some examples for particular filters are derived in section 4.

This work was supported by DFG grant FE 583/1-2.

2. POISSON FILTER SERIES

As first step, we focus on radial symmetric filters. These are approximated by a series of Poisson filters.

The Poisson filter kernels form a family of lowpass filters comparable to Gaussian kernels. They even form a linear scale-space [7]. In the two-dimensional case, the amplitude response of the Poisson kernel is given as

$$P(\mathbf{u},s) = \exp(-2\pi|\mathbf{u}|s) \ , \tag{1}$$

where $\mathbf{u} = [u_1\, u_2]^T$ is the 2D frequency vector, $|\mathbf{u}| = \sqrt{\mathbf{u}^T\mathbf{u}}$ its norm, and s indicates the scale of the filter, i.e., defines its upper band limit. The convolution kernel which corresponds to (1) is given according to [14] as

$$\begin{aligned} p(\mathbf{x},s) &= \int_{\mathrm{R}^2} P(\mathbf{u},s)\exp(i2\pi\mathbf{x}^T\mathbf{u})\,d\mathbf{u} \\ &= \frac{s}{2\pi(|\mathbf{x}|^2+s^2)^{3/2}} \ . \end{aligned} \tag{2}$$

In order to derive an approximation of an arbitrary radial filter, we define a finite linear combination of Poisson filters according to

$$F(|\mathbf{u}|) = \sum_{s=0}^{S} c_s P(\mathbf{u},s) = \sum_{s=0}^{S} c_s \exp(-2\pi|\mathbf{u}|s) \ , \tag{3}$$

where c_s indicate the linear coefficients and $S < \infty$. Substituting $z = \exp(2\pi|\mathbf{u}|)$, (3) becomes the z-transform [1] of c_s, assuming that $c_s = 0$ for $s > S$:

$$G(z) = \sum_{s=0}^{S} c_s z^{-s} \ , \tag{4}$$

where $G(z) = F(\log(z)/(2\pi))$.

The z-transform is a very common tool in 1D signal processing, but in higher dimensions it is less common. Interpreting the series (3) as a z-transform allows us to use all known properties of the z-transform for our purpose. However, one should always keep in mind that the relation between frequency response and the z-transform of a sequence differ in the classical 1D case and in our case: we do not obtain the frequency response by substituting $z = \exp(i2\pi u)$ but by substituting $z = \exp(2\pi|\mathbf{u}|)$. Hence, we do not look at the unit circle in the z-domain, but at the line $z > 1 \in \mathrm{R}$.

Before we make use of the z-transform, we have to consider the region of convergence. Since $\exp(2\pi|\mathbf{u}|) \in [1,\infty)$ for all $\mathbf{u}$, we have to assure a convergence radius (w.r.t. $1/z$) of at least one. Since our series is finite, convergence becomes trivial. This changes if we consider infinite linear combinations. Since we want to write down closed form solutions, the infinite case is irrelevant.

One of the main properties of the z-transform is that it maps sequences to polynomials in $1/z$. This allows us to formulate constraints on the radial filter in terms of polynomials, which is straightforward. For instance, we can set the amplitude response and its derivative to zero at the origin and at infinity. The resulting Poisson filter series consists of at least three non-zero components at scales 1, 2, and 3 (see section 4)

Another useful property of the z-transform are the limit theorems, which allow to compute the series coefficients directly from the z-transform:

$$\begin{aligned} c_0 &= \lim_{z\to\infty} G(z) \\ c_1 &= \lim_{z\to\infty} z(G(z)-c_0) \\ c_2 &= \lim_{z\to\infty} z^2(G(z)-c_0-c_1z^{-1})\dots \end{aligned} \tag{5}$$

This can also be very useful for the approximation of radial functions which are known analytically in the Fourier domain. However, this topic is out of the scope of this paper.

3. GENERAL POLAR SEPARABLE FILTERS

Up to now we have only considered radial filters, i.e., filters where the radial impulse response and the radial amplitude response are related by the (zeroth order) Hankel transform [2]. For 2D filters which also contain angular variations, the more general theorem of Stein and Weiss [14] must be applied, leading to a kth order Hankel transform of the radial function:

Theorem (Stein and Weiss). The Fourier transform of a 2D function $g(x_1,x_2) = g_0(r)\exp(im\phi)$, where m is an integer and $x_1 + ix_2 = r\exp(i\phi)$, is given as

$$G(u_1,u_2) = 2\pi(-i)^m \exp(im\theta)\int_0^\infty g_0(r)J_m(2\pi qr)r\,dr \ , \tag{6}$$

where $u_1 + iu_2 = q\exp(i\theta)$ and $J_m(\cdot)$ is the Bessel function of mth order.

The angular functions $\exp(im\theta)$ are the spherical harmonics of order m for Euclidean 2D space. Due to the theory of Fourier series, one can approximate any angular function (in L_2 sense) using the spherical harmonics. The Fourier coefficients are denoted d_m in the following.

In our framework, we can now approximate any polar separable 2D filter by the outer product of the Poisson filter series and the Fourier series:

$$F(\mathbf{u}) \approx \sum_{s=0}^{S}\sum_{m=0}^{M} c_s d_m \exp(im\theta)\exp(-2\pi|\mathbf{u}|s) \ . \tag{7}$$

Note that substituting $\tilde{z} = \exp(i\theta)$ leads to the z-transform w.r.t. the angular coordinate, i.e., the angular part of the 2D amplitude response corresponds to the frequency response in the 1D case. Furthermore, the 1D impulse response corresponds to the coefficients of the Fourier series and 1D FIR filters correspond to 2D filters based on a finite number of spherical harmonics.[1]

In order to obtain an analytic expression in the spatial domain, the inverse Fourier transforms of all combinations $\exp(im\theta)\exp(-2\pi|\mathbf{u}|s)$ with non-zero coefficients $c_s d_m$ are required. Using the Stein and Weiss theorem and exchanging spatial and Fourier domain, one can compute these kernels – however, the algebraic manipulations might become tedious for large m. For $m \in \{0,\dots,3\}$ the analytic closed form expressions are [5]

$$\begin{aligned} m=0 &\ : \ p(\mathbf{x},s) \\ m=1 &\ : \ i\frac{x_1+ix_2}{2\pi(|\mathbf{x}|^2+s^2)^{3/2}} \\ m=2 &\ : \ (x_1+ix_2)^2\frac{s(2s^2+3|\mathbf{x}|^2)-2(s^2+|\mathbf{x}|^2)^{3/2}}{2\pi|\mathbf{x}|^4(|\mathbf{x}|^2+s^2)^{3/2}} \\ m=3 &\ : \ i(x_1+ix_2)^3 \\ &\qquad \frac{12s^2|\mathbf{x}|^2+8s^4+3|\mathbf{x}|^4-8s(s^2+|\mathbf{x}|^2)^{3/2}}{2\pi|\mathbf{x}|^6(|\mathbf{x}|^2+s^2)^{3/2}} . \end{aligned}$$

4. EXAMPLE FILTERS

In this section we present a few examples in order to illustrate how the theory from section 2 and section 3 can be applied in practice.

We start with the derivation of a radial bandpass filter. The minimum requirements for a bandpass filter are that its value and its derivative are zero for zero frequency and at infinite frequency. Therefore, we have

$$\begin{aligned} F(0) = 0 \quad &\text{and} \quad \lim_{|\mathbf{u}|\to\infty} F(|\mathbf{u}|) = 0 \\ F'(0) = 0 \quad &\text{and} \quad \lim_{|\mathbf{u}|\to\infty} F'(|\mathbf{u}|) = 0 \ . \end{aligned}$$

The first two conditions lead to

$$G(1) = 0 \quad \text{and} \quad \lim_{z\to\infty} G(z) = 0 \tag{8}$$

respectively. For the derivative of the radial amplitude response, we obtain ($q = |\mathbf{u}|$)

$$F'(q) = \frac{\partial}{\partial q}G(\exp(2\pi q)) = 2\pi z G'(z)$$

and hence,

$$G'(1) = 0 \quad \text{and} \quad \lim_{z\to\infty} zG'(z) = 0. \tag{9}$$

Plugging in (4), we obtain from (8) and (9), left,

$$\sum_{s=0}^{S} c_s = 0, \qquad c_0 = 0, \quad \text{and} \quad \sum_{s=0}^{S} s c_s = 0.$$

[1] 1D IIR filters implemented by recursive filters correspond to repeated application of the 2D filter to the signal. This observation might lead to new adaptive filtering schemes – however, this is out of the scope of this paper.

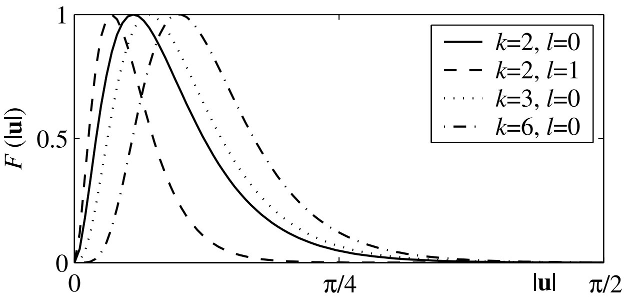

Figure 1: Examples of radial bandpass filters according to (11).

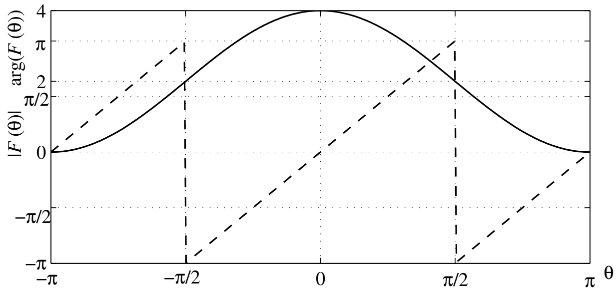

Figure 2: Amplitude response (solid) and phase response (dashed) of (12).

The right equation in (9) becomes trivial. The first non-trivial solution for smallest possible S is obtained for $S = 3$:

$$c_3 = c_1 \quad \text{and} \quad c_2 = -2c_1 .$$

Choosing $c_1 = 1$, we thus obtain the convolution kernel

$$f(\mathbf{x}) = p(\mathbf{x},1) - 2p(\mathbf{x},2) + p(\mathbf{x},3). \qquad (10)$$

In the 2D case, this expression becomes

$$f(\mathbf{x}) = 2\pi^{-1}((1+|\mathbf{x}|^2)(4+|\mathbf{x}|^2)(9+|\mathbf{x}|^2))^{-\frac{3}{2}} \cdot \\ [3((1+|\mathbf{x}|^2)(4+|\mathbf{x}|^2))^{\frac{3}{2}} - 4((1+|\mathbf{x}|^2)(9+|\mathbf{x}|^2))^{\frac{3}{2}} \\ +((4+|\mathbf{x}|^2)(9+|\mathbf{x}|^2))^{\frac{3}{2}}].$$

In general, constraining the first k derivatives being zero, results in a filter series of the form

$$f(\mathbf{x}) = \sum_{s=0}^{k} (-1)^s \left(\begin{array}{c} k \\ s \end{array} \right) p(\mathbf{x}, s+1) \ , \qquad (11)$$

which follows from some basic, but tedious calculations.

Similar bandpass filters with lower centre frequency can be obtained by shifting the scale parameters. This can be shown my exploiting the semigroup property of the Poisson kernel [7] or directly by setting $c_0, \ldots, c_l = 0$. The only change in (11) is then to replace $p(\mathbf{x}, s+1)$ with $p(\mathbf{x}, s+l+1)$.

In Fig.1, the radial amplitude responses for some examples are shown.

The filter design becomes much more interesting if we allow angular variations. According to the series (7), angular variations can be considered independently of the radial series, leading to Fourier series. We will give just one example for the latter, the reader is referred to the literature for further examples.

A particularly interesting case is obtained if we plug the binomial coefficients $\left(\begin{array}{c} 2 \\ m-1 \end{array} \right)$ into the angular part of (7):

$$F(\theta) = \exp(i\theta) + 2\exp(i2\theta) + \exp(i3\theta). \qquad (12)$$

The amplitude response and the phase response are plotted in Fig. 2. The amplitude response of the filter shows a behaviour similar to a quadrature filter or to the Hilbert transform. Assuming an intrinsically 1D signal[2] with a compatible orientation, i.e., a signal varying in the horizontal direction, the filter response is related to the analytic signal. Using two different imaginary units for the Fourier transform and the series in (12), or equivalently, considering (12) as two components of a vector, and computing the amplitude w.r.t. this vector, results in the analytic signal. The phase response of (12) directly represents the orientation in double angle representation [10].

For non-horizontal orientations, we need a further component for obtaining the analytic signal. Consider the 2D signal $s(\mathbf{x}) = v(\mathbf{x}^T \mathbf{n}_0)$, where $v(\cdot)$ is a 1D signal and $\mathbf{n}_0$ a normalised 2D vector representing the orientation of the signal. Applying (12) to $s(\mathbf{x})$ results in a response which can be separated into one part that depends on $v(\cdot)$ and one part that depends on $\mathbf{n}_0 = [\cos\theta_0 \ \sin\theta_0]^T$. Considering the latter one, we obtain for the

even part of $v(\cdot)$	$2\cos(2\theta_0) + i2\sin(2\theta_0)$
odd part of $v(\cdot)$	$\cos\theta(2\cos(2\theta_0) + i2\sin(2\theta_0))$.

Hence, for non-horizontal signals, we have to add a response for the odd part of $v(\cdot)$ with the angular shape $\sin\theta(2\cos(2\theta_0) + i2\sin(2\theta_0))$. By one line of calculus, we obtain the second filter

$$F_2(\theta) = i\exp(i\theta) - i\exp(i3\theta) \ . \qquad (13)$$

In total, we obtain six real-valued responses from the filters $F(\theta)$ and $F_2(\theta)$.

In a simple experiment, we combined the angular filters $F(\theta)$ and $F_2(\theta)$ with the radial bandpass obtained from $k = 2$ and $l = 1$ and applied this filter to the test image in Fig. 3. Taking the quadratic norm of the responses, we obtain the magnitude response in Fig. 4.

The filter response also includes information about the orientation of the signal, see Fig. 5, and the local phase, see Fig. 6.

5. CONCLUSION

We presented a novel method for designing polar separable 2D filters. By exploiting the z-transform which maps the radial frequency response into the space of polynomials, one can easily put certain constraints on the frequency response. In a similar way, the angular factor of the frequency response can be controlled by means of a Fourier series. Combining both series, polar separable filters with known analytic description in both domains, the spatial domain and the frequency domain, are obtained. We presented several examples, among these one for a novel 2D quadrature filter. The properties of the latter will become subject of our future research.

[2] The spectrum is concentrated on a line; also called simple signal [10].

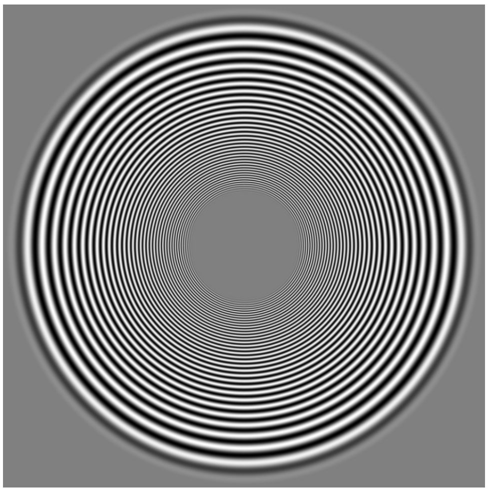

Figure 3: Test image for quadrature filter [10].

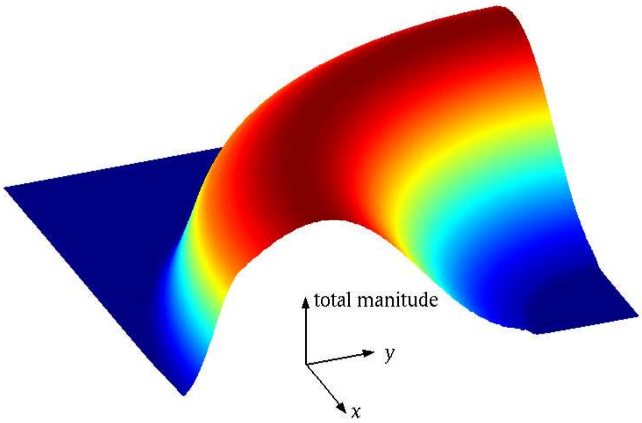

Figure 4: Magnitude of the responses of (12,13) in one quadrant of Fig. 3.

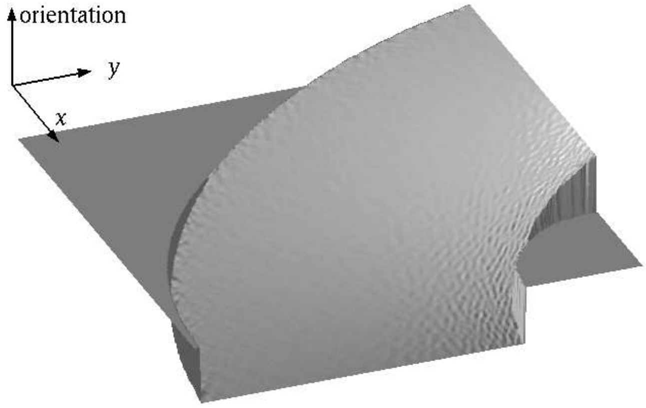

Figure 5: Signal orientation extracted from the filter response.

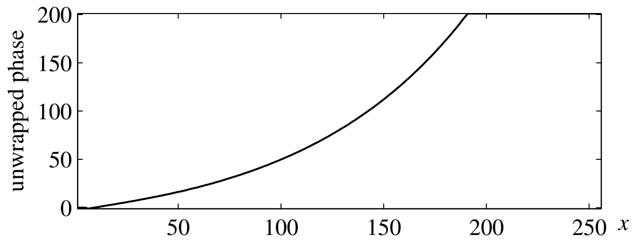

Figure 6: Unwrapped phase of the response for one line.

REFERENCES

[1] BRACEWELL, R. N. *The Fourier Transform and its Applications*, 2nd ed. McGraw-Hill, 1986.

[2] BRACEWELL, R. N. *Two-Dimensional Imaging*. Prentice Hall Signal Processing Series. Prentice Hall, Englewood Cliffs, 1995.

[3] CANNY, J. A computational approach to edge detection. *IEEE Transactions on Pattern Analysis and Machine Intelligence PAMI-8*, 6 (November 1986), 255–274.

[4] DANIELSSON, P., AND SEGER, O. Rotation invariance in gradient and higher derivative detectors. *Computer Vision, Graphics, and Image Processing 49*, 2 (February 1990).

[5] FELSBERG, M. *Low-Level Image Processing with the Structure Multivector*. PhD thesis, Institute of Computer Science and Applied Mathematics, Christian-Albrechts-University of Kiel, 2002. TR no. 0203, available at `http://www.informatik.uni-kiel.de/reports/2002/0203.html`.

[6] FELSBERG, M., AND SOMMER, G. The monogenic signal. *IEEE Transactions on Signal Processing 49*, 12 (December 2001), 3136–3144.

[7] FELSBERG, M., AND SOMMER, G. The monogenic scale-space: A unifying approach to phase-based image processing in scale-space. *Journal of Mathematical Imaging and Vision* (2004). accepted for publication in a future issue.

[8] FLEET, D. J., AND JEPSON, A. D. Computation of Component Image Velocity from Local Phase Information. *Int. Journal of Computer Vision 5*, 1 (1990), 77–104.

[9] FREEMAN, W. T., AND ADELSON, E. H. The design and use of steerable filters. *IEEE Transactions on Pattern Analysis and Machine Intelligence PAMI-13*, 9 (September 1991), 891–906.

[10] GRANLUND, G. H., AND KNUTSSON, H. *Signal Processing for Computer Vision*. Kluwer Academic Publishers, 1995. ISBN 0-7923-9530-1.

[11] HARRIS, C., AND STEPHENS, M. A combined corner and edge detector. In *4th Alvey Vision Conference* (September 1988), pp. 147–151.

[12] KNUTSSON, H. *Filtering and Reconstruction in Image Processing*. PhD thesis, Linköping University, Sweden, 1982. Diss. No. 88.

[13] KNUTSSON, H., AND ANDERSSON, M. What's so good about quadrature filters? In *ICIP* (Barcelona, Spain, September 2003).

[14] STEIN, E., AND WEISS, G. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, New Jersey, 1971.