ON TIME-DOMAIN AND FREQUENCY-DOMAIN MMSE-BASED TEQ DESIGNS FOR DMT TRANSMISSION

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ABSTRACT
We reconsider the MMSE-based time-domain equalizer (TEQ), bitrate maximizing TEQ (BM-TEQ) and per-tone equalizer design for DMT transmission. The MMSE-TEQ criterion can be formulated as a least-squares (LS) criterion that minimizes a time-domain (TD) error energy. Based on this LS-based TD-MMSE-TEQ, we derive new LS-based frequency-domain (FD) MMSE-TEQ criteria that are intermediate in terms of computational complexity and performance between the TD-MMSE-TEQ and the BM-TEQ. In addition, we show that the BM-TEQ design itself is equivalent to a so-called iteratively-reweighted separable nonlinear LS (SNLLS) problem.

1. INTRODUCTION
In discrete multitone (DMT) based systems, such as asymmetric digital subscriber lines (ADSL), channel impulse responses can be very long, hence a long cyclic prefix (CP, length ν) would be required. A solution to avoid this overhead is to insert a (real) T-input time domain equalizer w (TEQ) before demodulation, which then shortens the channel impulse response to ν + 1 samples. Among the numerous TEQ designs, we will focus on the so-called minimum mean-square error (MMSE)-based TEQ design [1] with a unit energy constraint (UEC) [2, 3] and the recently proposed bitrate maximizing TEQ design (BM-TEQ) [4]. In [5], the alternative per-tone equalizer (PTEQ) scheme is proposed that always performs at least as well as - and usually better than - a TD equalizer receiver while keeping complexity during data transmission at the same level. The PTEQ is a complex MMSE equalizer designed for each tone separately.

The classical MMSE-TEQ criterion can be formulated as a constrained least-squares (CLSLS) criterion that minimizes a time-domain (TD) error energy. Starting from this CLLS-based TD-MMSE-TEQ criterion, we derive new LS-based MMSE-TEQ criteria, that minimize a sum-of-square of frequency-domain (FD) error energies (i.e., after DFT demodulation), rather than a TD error energy; especially the so-called separable nonlinear LS (SNLLS)-based FD-MMSE-TEQ appears a reasonable intermediate in terms of complexity and performance between the TD-MMSE-TEQ and the BM-TEQ. Remarkably, the MM-TEQ criterion itself is found to be equivalent to a so-called iteratively-reweighted SNLLS-based FD-MMSE-TEQ design. As a side result, the LS-based formulations of the TD-MMSE-TEQ, FD-MMSE-TEQ, BM-TEQ and PTEQ design cost functions appear to be closely related, especially when turning each of them into a generalized eigenvalue (GEV) problem

\[ \mathbf{Bw} = \lambda \mathbf{Aw} \]  

where, loosely speaking, \( \mathbf{A} \) is an autocorrelation metric of the received signal \( \mathbf{y} \) and \( \mathbf{B} \) depends on a crosscorrelation metric between transmitted (TX) and received (RX) signal \( \mathbf{x} \) and \( \mathbf{y} \). For an extended version of this paper, we refer to [6].

Notation. The DMT symbol index is \( k \). \( S \) is the set of \( K \) active tones; \( n \) is a tone index, \( N \) is the (D)F matrix size, \( \mathcal{F}_N \) is an \( N \times N \) submatrix of the full DFT matrix, \( \mathcal{F}_H \) with only the \( N \)th rows of the active tones \( \mathcal{F}_S \); the \( n \)-th DFT row is \( \mathcal{F}_N n \). Vectors are set in bold lowercase while matrices are in bold uppercase. A tilde over a variable distinguishes frequency-domain (FD) symbols from time-domain (TD) symbols, e.g., the \( N \times 1 \) TD symbol vector at time \( k \), \( \mathbf{x}_k \). FD vectors or matrices only account for the number of data points, e.g., \( L \) samples or \( K \) DMT symbols, is used to distinguish between a (deterministic) correlation estimate, e.g., \( \Sigma_{y,y}^L = \frac{1}{N} \sum_{k=1}^{N} y_k y_k^T \mathbf{y} \) and the (true) statistical correlation, e.g., \( \Sigma_{y,y} = \delta \{ y^T y \} \) or \( \sigma_{y,y} \). Throughout this paper we only define the statistical correlations.

2. MMSE-TEQ, BM-TEQ AND PTEQ: LS PROBLEMS

2.1 CLLS-based TD-MMSE-TEQ design
One of the earliest presented TEQ designs is the MMSE-based TEQ [1]: it minimizes the time-domain (TD) MSE between the output of the TEQ, \( y_{TEQ} = \mathbf{y} w \), with \( \mathbf{w} \) the T-tap TEQ, \( \mathbf{f} \) the sample index and \( \mathbf{y}_f = [ y_f \cdots y_{f-2N+1} ]^T \) a vector of RX samples¹, and the BM-TEQ design itself is equivalent to a so-called target impulse (TIR) b of length \( \nu + 1 \) (with \( \nu \) the CP length), which is fed with a vector of TX samples \( \mathbf{x}_f = [ x_{f-\nu} \cdots x_{f-1}]^T \):  

\[ \min_{\mathbf{w}} \| \mathbf{y}_f - \mathbf{w} \mathbf{x}_f \|^2 \]  

To avoid the trivial solution \( \mathbf{w} = 0, b = 0 \), a nontriviality constraint is added [2]. We focus on the particular choice of a so-called unit energy constraint (UEC) on \( \mathbf{w} \) [3]:

\[ \mathbf{w}^T \Sigma_{y,y}^L \mathbf{w} = 1 \]  

with the autocorrelation matrix \( \Sigma_{y,y}^L = \delta \{ y^T y \} \). This constrained TD-MMSE-TEQ criterion (2) forces the joint channel-TEQ impulse response to have a main energy window of \( \nu + 1 \) samples. A deterministic constrained linear least-squares (CLSLS) based TD-MMSE-TEQ criterion, equivalent to (2), is given by:

\[ \min_{\mathbf{w}} \frac{1}{L} \sum_{f=1}^{L} \| y_f^T \mathbf{w} - x_f^T \mathbf{b} \|^2 \]  

s.t. \( \mathbf{w}^T \Sigma_{y,y}^L \mathbf{w} = 1 \)  

with \( L \) the total number of available data samples and \( \Sigma_{y,y}^L \) an estimate of \( \Sigma_{y,y} \) as clarified earlier on this page in the paragraph on the adopted notation. Using the so-called orthogonality condition

¹The RX signal \( \mathbf{y} \) and vector \( \mathbf{y}_f \) depend on a synchronization delay \( \Delta \), which we do not mention explicitly here.
[1, 2] eliminating \( b \) and defining \( \Sigma_k^2 = \mathcal{E} \{ x_k x_k^T \} \) and \( \Sigma_{L,XY} = \mathcal{E} \{ x_k y_k^T \} \), (4) reduces to:

\[
\min_w w^T \left[ \Sigma_{L,XY}^2 - \Sigma_{L,XY}^2 \Sigma_{L,XY}^2 \right] w \quad \text{s.t.} \quad w^T \Sigma_{L,XY}^2 w = 1 \tag{5}
\]

The solution is seen to be the dominant GEV of (1) with the matrix pair

\[
(B, A) = \left( \Sigma_{L,XY}^2 \Sigma_{L,XY}^2 \right)^{-1} \Sigma_{L,XY}^2 \Sigma_{L,XY}^2 \tag{6}
\]

### 2.2 CLLS-based FD-MMSE-TEQ design

The TD-MMSE-TEQ (2) is sample-based and minimizes a TD MSE. In this and the next section, we develop new frequency-domain (FD) MMSE-TEQ criteria that account for the DMT block transmission structure, including the CP, and minimize a sum of FD MSEs. Especially the FD-MMSE-TEQ criterion, developed in Section 2.3, appears a useful intermediate in terms of complexity and performance between the TD-MMSE-TEQ on one hand, and the PTEQ and the BM-TEQ on the other hand (see Section 3).

First, we rewrite (2) on a per-DMT-symbol basis:

\[
\min_w \mathcal{E} \left\{ \left\| Y_k w - X_k b \right\|^2 \right\} \quad \text{s.t.} \quad w^T \Sigma_k^2 w = 1 \tag{7}
\]

The Toeplitz matrix \( Y_k \) has size \( N \times T \); its first column and row are given by \( [y_k, 0, \ldots, y_{N-1}]^T \) and \( [y_k, y_{k+1}, \ldots, y_{T-1}] \), respectively, with \( y_k = y_k(N+N) \). The matrix \( X_k \), which incorporates the CP, has size \( N \times (N+1) \) and is columnwise circulant with first column \( [y_0, \ldots, y_{N-1}]^T \). The first term \( Y_k w \) in (7) convolves the k-th DMT RX symbol with the TEQ and is \( N \times 1 \) TEQ output vector that feeds to the RX DFT. The second term \( X_k b \) is the convolution of the k-th DMT TX symbol and the TIR \( b \).

In a second step, \( X_k \) is extended with \( N - v - 1 \) columns to an \( N \times N \) circulant matrix \( X_k C \), where \( b \) is zero-padded accordingly and the DFT-based decomposition of the circulant matrix \( X_k C = \mathcal{F}_N X_{k, D} X_k \), with \( X_{k, D} = \text{diag}(x_k) \) and \( X_k X = \text{N x 1} \) DMT TX symbol vector, is plugged in:

\[
X_k b = X_k C \begin{bmatrix} b \\ 0 \end{bmatrix} = \mathcal{F}_N X_{k, D} \begin{bmatrix} b \\ 0 \end{bmatrix} \tag{8}
\]

Thirdly, the cost function and constraint (7) are transformed to the FD and only the active tones \( \mathcal{J} \) are considered:

\[
\min_{w,b} \mathcal{E} \left\{ \left\| \mathcal{F}_N \mathcal{E} \left\{ \left\| Y_k w - X_k b \right\|^2 \right\} \right\} \quad \text{s.t.} \quad w^T \Sigma_k^2 w = 1, \quad b = \mathcal{F}_N \begin{bmatrix} b \\ 0 \end{bmatrix} \quad \text{real w and b} \tag{9}
\]

where \( Y_k = \mathcal{F}_N Y_k \) is the dft of \( \tilde{Y}_k \) (with \( \tilde{Y}_k \) the \( N \times 1 \) DMT TX symbol vector) and \( \Sigma_k^2 = \mathcal{E} \left\{ \left\| Y_k w - \tilde{Y}_k \right\|^2 \right\} \). The first term of the error vector \( \tilde{Y}_k \) in (9) corresponds to the RX DFT output at the tones \( \mathcal{J}_k \): \( \mathcal{F}_N(Y_k w) = (\mathcal{F}_N Y_k) w = \tilde{Y}_k w = \tilde{Y}_k \)

\[
\min_{w,b} \mathcal{E} \left\{ \left\| Y_k w - X_k b \right\|^2 \right\} \quad \text{s.t.} \quad w^T \Sigma_k^2 w = 1 \quad \text{real w and b} \tag{10}
\]

which can either be computed as \( 1 \) the DFT of the TEQ output \( Y_k w \) or \( 2 \) as a linear combination \( w \) of the sliding DFT of the \( k \)-th DMT RX symbol, \( Y_k = \mathcal{F}_N Y_k \) (see [4] for details). The second constraint in (10) comes from the original TD-MMSE-TEQ design that imposes channel shortening by means of a TIR \( b \) of length \( v + 1 \). If we drop the constraints on \( b \) in (10) and instead optimize

\[
\min_{w,b} \mathcal{E} \left\{ \left\| Y_k w - X_k b \right\|^2 \right\} \quad \text{s.t.} \quad w^T \Sigma_k^2 w = 1 \quad \text{real w} \tag{11}
\]

we obtain an FD-MMSE-TEQ criterion in the (typically) real TEQ \( w \) and the complex vector \( b \) instead of \( b \). The optimum solution for the unconstrained \( b \) follows from the so-called orthogonality condition and is a vector with as entries \( b_n \) the inverses of the unbalanced MMSE-based (uMMSE) FEQs \( \sigma_n^2 \) which are in fact the optimal choice of FEQs for a given \( w \) [4, 7]:

\[
\gamma_n^2 \mathcal{M}_{b,\mathcal{M}} = \frac{\sigma_n^2}{\sigma_{n,\mathcal{M}} w} = \frac{1}{b_n} \tag{12}
\]

where \( \sigma_n^2 = \mathcal{E} \left\{ \left\| \tilde{Y}_k \right\|^2 \right\} \) is the variance of \( \tilde{Y}_k \) and where the denominator is the crosscorrelation \( \mathcal{E} \left\{ \tilde{Y}_k \tilde{Y}_n^H \right\} \) between the RX DFT output and the TX symbol on tone \( n \). It follows from (11) that this crosscorrelation is equal to \( \sigma_n^2 \tilde{Y}_k w \) with \( \sigma_n^2 = \mathcal{E} \left\{ \tilde{Y}_k \tilde{Y}_n^H \right\} \) the \( 1 \times T \) crosscorrelation vector of \( \tilde{Y}_k \) and the \( n \)-th sliding DFT output \( \tilde{Y}_n = \mathcal{F}_N Y_k \) (see [4] for details). Solving (12) then optimizes the sum-square energy between the DFT outputs \( \tilde{Y}_n w \) and the scaled desired symbols \( \sigma_n^2 w \). A deterministic CLLS-based FD-MMSE-TEQ criterion, equivalent with (12) is given by:

\[
\min_{w,B} \mathcal{R} \left\{ \sum_{k=1}^K \left| \tilde{Y}_k w - X_k b \right|^2 \right\} \quad \text{s.t.} \quad w^T \Sigma_k^2 w = 1 \quad \text{and real w} \tag{13}
\]

where \( K \) is the number of available DMT symbols. Due to the similarity between the CLLS-based FD-MMSE-TEQ criterion (12) and the CLLS-based TD-MMSE-TEQ (4), it comes as no surprise that (12) reduces to a GEV problem (1) that is closely related to (6):

\[
(B, A) = \mathcal{R} \left\{ \sigma_k^2, \Sigma_k^2 \right\} \quad \mathcal{R} \left\{ \sigma_k^2, \Sigma_k^2 \right\} \tag{14}
\]

The \( N_0 \) rows of \( \Sigma_k^2 \) are the above defined crosscorrelation vectors \( \sigma_{n,k}^2 \) with \( \Sigma_k^2 = \mathcal{R} \left\{ \sigma_{n,k}^2, \sigma_{n,k}^2 \right\} \) the autocorrelation matrix of the \( n \)-th sliding DFT output; \( \Sigma_k^2 = \mathcal{E} \left\{ \tilde{Y}_k \tilde{Y}_k^H \right\} \) is the autocorrelation matrix of the DMT TX symbol vector; the second equality assumes independent symbols \( \tilde{Y}_k \), such that \( \Sigma_k \) is diagonal with diagonal elements \( \sigma_n^2 \); the \( \mathcal{R} \)-operators ensure a real TEQ.

The complex LS-based MMSE-TEQ [5] is closely related to the CLLS-based FD-MMSE-TEQ (14) when only one tone \( n \) is considered. It follows from (15) that the real PTEQ for tone \( n, w_n \), is the dominant eigenvector of

\[
(B, A) = \mathcal{R} \left\{ \sigma_{n,k}^2, \sigma_{n,k}^2 \right\} \quad \mathcal{R} \left\{ \sigma_{n,k}^2, \sigma_{n,k}^2 \right\} \tag{16}
\]

In case of a complex \( w_n \), the \( \mathcal{R} \)-operators should be dropped. In this case the matrix \( B \) becomes rank-one and the dominant eigenvector of (16) (up to a scaling) is seen to be given by (6)

\[
w_n = \sigma_{n,k}^2 \tilde{Y}_n^H \tag{17}
\]

This is exactly the solution of the LS-based MMSE-TEQ criterion of [5]:

\[
\min_{w_n} \mathcal{R} \left\{ \sum_{k=1}^K \left| \tilde{Y}_n w_n - \tilde{Y}_k w_n \right|^2 \right\} \tag{18}
\]

### 2.3 SNLSS-based FD-MMSE-TEQ design

An alternative (suboptimal) FD criterion is obtained by minimizing the sum-square energies at the FEQ output instead of the DFT output:


\[
\min_{w_d} \left\{ \left\| \text{diag}(d) \hat{Y}_d w - \hat{x}_d \right\|^2 \right\} \quad \text{with real } w
\]

(19)

where the FEQ output error vector \( \hat{e}_d \) depends on both the real TEQ and complex FEQ parameters, \( \theta = [w_d^T \hat{d}_0^T]^T \). The UEC constraint of (12) has been dropped as the criterion (19) has no trivial solution anymore. This criterion corresponds to an SNLLS criterion [8, 9]:

\[
\min_{w_d} \frac{1}{K} \sum_{k=1}^{K} \left\{ \left\| \text{diag}(d) \hat{Y}_d w - \hat{x}_d \right\|^2 \right\} \quad \text{with real } w
\]

(20)

which we call an SNLLS-based FD-MMSE-TEQ criterion. The separability property follows from the fact that the error \( \hat{e}_d \) is nonlinear in \( \theta \), whereas the TEQ \( w_d \) and FEQs \( d \) appear linearly. Solving (19) as a linear problem in \( d \), while keeping \( w_d \) fixed, results in the (biased) MMSE FEQs for the given \( w \) [4, 7]:

\[
d_{\text{MMSE}} = w^T \Sigma_{d}^{\frac{1}{2}} y
\]

(21)

where the numerator is equal to the complex conjugate of the denominator of the uMMSE FEQ (13) and where the denominator is the autocorrelation of the DFT output, i.e.,

\[
\mathcal{E} \left\{ \left[ \hat{Y}_k n, w \right]^2 \right\} = w^T \Sigma_{d}^{\frac{1}{2}} y
\]

It will be shown in Section 2.4 that the SNLLS problem (20) can be solved iteratively with a sequence of GEV problems (1). As will be shown in the simulations of Section 3, this SNLLS-based FD-MMSE-TEQ design consistently outperforms the CLLS-based TD-MMSE-TEQ design and closely approaches the BM-TEQ performance.

### 2.4 Bitrate maximizing FD-MMSE-TEQ design

The bitrate maximizing TEQ (BM-TEQ), originally presented in [4], is the solution to the following constrained nonlinear optimization problem in \( \theta = [w_d^T \hat{d}_0^T]^T \):

\[
\max \sum_{c \in C} \log_2 \left( 1 + \frac{\text{SNR}_{c, \theta}}{\Gamma_c} \right)
\]

with SNR\( _{c, \theta} = \mathcal{E} \left\{ \left| \hat{Y}_k n, w \right|^2 \right\} / \mathcal{E} \left\{ \left| d_{k, n} x_n w - x_n \right|^2 \right\} \)

(22)

subject to \( d_{k, n} = d_{k, n}^{\text{prev}} \forall n \in \mathcal{N}_d \)

(23)

with \( \theta = [w_d^T \hat{d}_0^T]^T \), i.e., maximizing the number of bits per DMT symbol (given a certain SNR gap \( \Gamma_c \) between \( \text{SNR}_{c} \) and the SNR required to achieve Shannon capacity, typically assumed to be independent of the equalizer [4]), over the joint TEQ-FFEQ parameters \( \theta \), subject to the use of uMMSE FEQs (24) (see also (13)), which render the subchannel SNR model in (23) exact [4]. It has been shown in [6], based on (22-24), that this optimization criterion is equivalent to the following iteratively reweighted SNLLS-based bitrate maximizing FD-MMSE-TEQ (IR-SNLLS-based BM-FD-MMSE-TEQ) criterion (explained below) [10]:

\[
\min_{\theta} \frac{1}{K} \sum_{k=1}^{K} \left\{ \left\| \text{diag}(\hat{Y}_k \theta) e_k \right\|^2 \right\}
\]

(25)

with

\[
\hat{e}_{k, n, \theta} = d_{k, n} y_{k, n} w - \hat{x}_{k, n}
\]

(26)

\[
\hat{Y}_{k, \theta} = \frac{\left( \text{SNR}_{k, \theta} + \Gamma_k \right)^2}{\sigma_{k, n, \theta}^2} \left( \frac{\text{SNR}_{k, \theta} + 1}{\sigma_{k, n, \theta}^2} \right)^2
\]

(27)

\[
\text{SNR}_{k, \theta} = \frac{1}{K} \sum_{k=1}^{K} \left\{ \left\| \hat{e}_{k, n, \theta} \right\|^2 \right\}
\]

(28)

\[
\rho_{k, n, \theta}^2 = \frac{\sqrt{\sigma_{k, n, \theta}^2 w^T \Sigma_{k, n, \theta}^2 y}}{\Sigma_{k, n, \theta}^2 y}
\]

(29)

The SNLLS-based FD-MMSE-TEQ (19) is indeed an unweighted version of, hence closely related to the IR-SNLLS-based BM-FD-MMSE-TEQ (25).

IR-LS problems such as (25) are weighted LS problems where the weights \( \hat{Y}_{k, \theta} \) depend on the LS errors \( e_k \), here: via the subchannel SNR (28), hence on the optimization parameters \( \theta \). They are typically solved as a sequence of weighted LS problems (here: a SNLLS problem) where the weights in each iteration are computed with the parameter estimates from the previous iteration, \( \theta_{\text{prev}} \). According to [10], convergence occurs provided that the weights are bounded and non-increasing in the absolute value of the LS errors. For a non-convex cost function, the IR-LS algorithm leads to a local optimum.

According to [8, 9], an SNLLS problem, such as the FD-MMSE-TEQ criterion (19) and (25), are -as the IR-LS problem- also solved iteratively by alternately updating the parameters \( w \) and \( d \). An iteration step for the IR-SNLLS-based BM-FD-MMSE-TEQ criterion then consists of the computation of (1) the weights, \( \hat{Y}_{k, \theta_{\text{prev}}} \), (2) estimates of the biased MMSE FEQs (21), \( d_{k, \theta} \), which are the solutions of (25) for a fixed \( w_{\text{prev}} \) and (3) a new BM-TEQ estimate \( w \):

\[
w = \mathcal{R} \left\{ \sum_{n \in \mathcal{N}_d} \hat{y}_n \left| d_{n}^2 \Sigma_{k, n, y}^2 \right\}^{-1} \right\}
\]

(30)

with \( \hat{y}_n = \hat{Y}_{k, \theta_{\text{prev}}} \) and \( d_{n} = d_{k, n} \), which is (similar to the PTEQ \( w_n (17) \)) the solution of a GEV problem with rank-one matrix \( B \):

\[
(B, A) = \left( \Sigma_{k, n, y}^2, \Sigma_{k, n, y}^2 \right)
\]

(31)

For a complex TEQ, the \( \mathcal{R} \)-operators must be omitted. The iterations for solving the SNLLS-based FD-MMSE-TEQ (19) do not include the first step, i.e., the weights \( \hat{Y}_{k, \theta_{\text{prev}}} \) always equal 1. Note that other solution strategies for SNLLS problems exist: in [8, 9], it is argued that step (3), which solves for \( w \) keeping \( d \) fixed can be better replaced by, e.g., a much faster converging Gauss-Newton updating step of the joint parameter vector \( \theta \).

### 2.5 Relation between the LS cost functions

Throughout the text, each LS problem has been shown to be equivalent to a GEV problem (1), with the SNLLS-based criteria giving rise to an iterative sequence of GEV problems. Table 1 summarizes the encountered matrix pairs \((B, A)\) (for real-valued TEQ and PTEQ designs) and shows that the \( A \) matrices are closely related autocorrelation matrices of the RX signal \( y \), while the \( B \) matrices are closely related, often low-rank, matrices determined by a crosscorrelation metric between the TX and RX signal \( x \) and \( y \), respectively. Complex TEQs or PTEQs are obtained by omitting the \( \mathcal{R} \)-operators in Table 1.

### 3. SIMULATIONS

Figure 1 shows bitrate performance plots for the considered equalizer designs with 32 taps (both real and complex TEQs and PTEQs are considered). The FD-SNLLS-based TEQ and IR-SNLLS-based BM-TEQ have been computed using the iterative Gauss-Newton algorithm suggested in Section 2.4. The bitrate is depicted for 8 downstream CSA loops with strong front-end filtering to separate up- and downstream transmission (see [4] for details). All simulations use the same synchronization delay \( \Delta \), which is determined by the first sample index of the channel impulse response window of \( v+1 \) samples with maximum energy. The noise in Figure 1a is a superposition of AWGN noise at -140dBm/Hz, residual echo and near-end crosstalk from 24 ADSL disturbers. In Figure 1b, severe RFI (7 RFIs with carrier frequencies 540, 650, 680, 760, 790, 840 and 1080kHz; the first two RFIs have a power of -30dBm, the remaining five have a power of -50dBm) is added. RFI, especially ingress from AM radio stations, can be an important interferer in ADSL. It is
Table 1: Real-valued TEQ/PTEQ designs as a GEV problem $Bw = \lambda Aw$. Complex equalizers are obtained by omitting $\Re$-operators.

<table>
<thead>
<tr>
<th>Design</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>CLLS-based TD-MMSE-TEQ</td>
<td>$\Sigma L_{xy}^{-1} (\Sigma L_{xy})^{-1} \Sigma L_{xy}$</td>
<td>$\Sigma L_{xy}$</td>
</tr>
<tr>
<td>CLLS-based FD-MMSE-TEQ</td>
<td>$\Re { \Sigma_{K,n,y} (\Sigma_{K,n,k}^{\dagger})^{-1} \Sigma_{K,n,y} }$ = $\Re { \Sigma_{K,n,y}^{2} } = \sum_{n \in \mathcal{N}} \Re { \Sigma_{K,n,y}^{2} }$</td>
<td>$\Re { \Sigma_{K,n,y}^{2} }$</td>
</tr>
<tr>
<td>SNLLS-based FD-MMSE-TEQ</td>
<td>$\Re { \sum_{n \in \mathcal{N}} d_{k,n}^{2} \sigma_{k,n,y}^{2} d_{k,n}^{\dagger} } \times \Re { \sum_{n \in \mathcal{N}} d_{k,n}^{2} \sigma_{k,n,y}^{2} }$</td>
<td>$\Re { \sum_{n \in \mathcal{N}} d_{k,n}^{2} \sigma_{k,n,y}^{2} }$</td>
</tr>
<tr>
<td>IR-SNLLS-based BM-FD-MMSE-TEQ</td>
<td>$\Re { \sum_{n \in \mathcal{N}} d_{k,n}^{2} \sigma_{k,n,y}^{2} d_{k,n}^{\dagger} } \times \Re { \sum_{n \in \mathcal{N}} d_{k,n}^{2} \sigma_{k,n,y}^{2} }$</td>
<td>$\Re { \sum_{n \in \mathcal{N}} d_{k,n}^{2} \sigma_{k,n,y}^{2} }$</td>
</tr>
<tr>
<td>LS-based MMSE-PTEQ</td>
<td>$\sigma_{k,n,y}^{-2} \Re { \sigma_{k,n,y}^{2} }$</td>
<td>$\Re { \Sigma_{K,n,y}^{2} }$</td>
</tr>
</tbody>
</table>

Figure 1: Bitrate performance of the considered TEQ and PTEQ designs for 8 CSA loops. From left to right: TD-MMSE-TEQ, real and complex CLLS-based FD-MMSE-TEQ, real and complex SNLLS-based FD-MMSE-TEQ, real and complex BM-TEQ, real and complex PTEQ. (a) Without RFI. (b) With RFI.

clear from Figure 1b that in this RFI case, the BM-TEQ and PTEQ can effectively mitigate RFI and outperform the suboptimal TEQ designs. The SNLLS-based FD-MMSE-TEQ consistently outperforms the CLLS-based FD-MMSE-TEQ and the CLLS-based FD-MMSE-TEQ and closely approaches the BM-TEQ performance. The CLLS-based FD-MMSE-TEQ performs worse than the TD-MMSE-TEQ; apparently, it makes more sense to minimize the sum-square FEQ output energies than the sum-square FFT output energies.

REFERENCES