ABSTRACT
Empirical Mode Decomposition (EMD) has recently been introduced as a local and fully data-driven technique aimed at decomposing nonstationary multicomponent signals in “intrinsic” AM-FM contributions. Although the EMD principle is appealing and its implementation easy, performance analysis is difficult since no analytical description of the method is available. We will here report on numerical simulations illustrating the potentialities and limitations of EMD in two signal processing tasks, namely detrending and denoising. In both cases, the idea is to make use of partial reconstructions, the relevant modes being selected on the basis of the statistical properties of modes that have been empirically established.

1. INTRODUCTION AND MOTIVATION
Empirical Mode Decomposition (EMD) has been recently pioneered by Huang et al. [2] for adaptively decomposing signals in a sum of “well-behaved” AM-FM components. The technique has already received some attention in terms of both applications [1, 2, 7, 9, 11] and interpretation [3, 4, 5, 8]. The method consisting in a local and fully data-driven splitting of a (possibly nonstationary) signal in fast and slow oscillations, the purpose of this paper is to investigate further its capabilities in terms of detrending and denoising.

2. BASICS OF EMPIRICAL MODE DECOMPOSITION

2.1 Principle
Empirical Mode Decomposition (EMD) [2] is a technique which has been designed primarily for obtaining AM-FM type representations in the case of signals which are oscillatory (possibly nonstationary and/or generated by a nonlinear system), in some automatic, fully data-driven, way. In a nutshell, the starting point of EMD is to consider oscillatory signals at the level of their local oscillations and to formalize the idea that:

“signal = fast oscillations superimposed to slow oscillations,”

and to iterate on the slow oscillations component considered as a new signal.

2.2 Algorithm
More precisely, if we look at the evolution of a signal \( x(t) \) between two consecutive local extrema (say, two minima occurring at times \( t_- \) and \( t_+ \)), we can heuristically define a (local) “high-frequency” part \( \{d(t), t_- < t < t_+\} \). This detail \( d(t) \) corresponds to the oscillation terminating at the two minima and passing through the maximum which necessarily exists in between them. For the picture to be complete, we also identify the corresponding (local) “low-frequency” part \( m(t) \), or local trend, so that we have \( x(t) = m(t) + d(t) \) for \( t_- \leq t \leq t_+ \). Assuming that this is done in some proper way for all the oscillations composing the entire signal, we get what is referred to as an Intrinsic Mode Function (IMF) as well as a residual consisting of all local trends. The procedure can then be applied to this residual, considered as a new signal to decompose, and successive constitutive components of a signal can therefore be iteratively extracted. Given a signal \( x(t) \), the effective algorithm of EMD can therefore be summarized as the following main loop [2]:

1. identify all extrema of \( x(t) \);
2. interpolate between minima (resp. maxima), ending up with some “envelope” \( \varepsilon_{\text{min}}(t) \) (resp. \( \varepsilon_{\text{max}}(t) \));
3. compute the average \( m(t) = (\varepsilon_{\text{min}}(t) + \varepsilon_{\text{max}}(t))/2 \);
4. extract the detail \( d(t) = x(t) - m(t) \);
5. iterate on the residual \( m(t) \).

In practice, the above procedure has to be refined by a sifting process, an inner loop that iterates steps (1) to (4) upon the detail signal \( d(t) \), until this latter can be considered as zero-mean according to some stopping criterion. Once this is achieved, the detail is considered as the effective IMF, the corresponding residual is computed and only then, step (5) applies. Eventually, the original signal \( x(t) \) is first decomposed through the main loop as

\[
x(t) = d_1(t) + m_1(t),
\]

and the first residual \( m_1(t) \) is itself decomposed as

\[
m_1(t) = d_2(t) + m_2(t),
\]

so that

\[
x(t) = d_1(t) + m_1(t) = d_1(t) + d_2(t) + m_2(t) = \ldots = \sum_{k=1}^{\infty} d_k(t) + m_k(t).
\]
2.3 Interpretations

Modes and residuals have been heuristically introduced on “spectral” arguments, but this must not be considered from a too narrow perspective. First, the decomposition makes no assumption about the harmonic nature of oscillations, and it can thus guarantee a compact representation (i.e., with fewer modes than a Fourier or wavelet decomposition) in situations involving nonlinear oscillations. Second, it is worth stressing the fact that, even in the case of harmonic oscillations, the high vs. low frequency discrimination mentioned above applies only locally and corresponds by no way to a predetermined sub-band filtering. Indeed, selection of modes rather corresponds to an automatic and adaptive (data-driven) time-variant filtering.

3. MODE MANIPULATIONS

By construction, the number of extrema decreases when going from one residual to the next, thus guaranteeing that the complete decomposition is achieved in a finite number of steps (typically, at most $O(\log_2 N)$ for $N$ data points). Moreover, the whole decomposition being only based on elementary subtractions, it obviously allows for a perfect reconstruction of the initial signal $x(t)$, given the collection of details $\{d_k(t), k = 1, \ldots K\}$ and the residual $m_K(t)$.

This property of perfect reconstruction, together with the spectral interpretation outlined above, suggests to achieve partial reconstructions only, so as to selectively remove slow or fast oscillations (detrending or denoising, respectively).

3.1 Detrending

In the case where the analyzed signal $x(t)$ consists in a slowly varying trend superimposed to a fluctuating process $y(t)$, the trend is expected to be captured by IMFs of large indices (+ the final residual). Detrending $x(t)$, which corresponds to estimating $y(t)$, may therefore amount to computing the partial, fine-to-coarse, reconstruction

$$\hat{y}_D(t) = \sum_{k=1}^{D} d_k(t),$$

where $D$ is the larger IMF index prior contamination by the trend. Each of the IMFs $\{d_k(t); k = 1, \ldots D\}$ being zero-mean, a rule of thumb for choosing $D$ is to observe the evolution of the (standardized) empirical mean of $\hat{y}_D(t)$ as a function of a test order $d$, and to identify for which $d = D$ it departs significantly from zero. An example of this approach is given in Figure 1, where a 7000 data point segment of a Heart-Rate Variability signal is considered.

3.2 Statistics

The procedure outlined above is a rough approach that can be improved upon when a more precise model can be advocated for the signal + noise mixture. To this end, a detailed knowledge of IMFs statistics in noise only situations can help identifying the significance of a given mode. This idea, which has been pioneered by Wu and Huang [10], can be followed in two directions, namely detrending as in the previous section (by keeping only those modes which are identified as noise) and denoising (by removing them).

Previous EMD studies have considered in some detail white Gaussian noise (iGn) [3, 5, 6] as a versatile class for broadband noise with no dominant frequency band. What has been shown in this context is that EMD acts spontaneously as a dyadic filterbank [3, 5, 6, 10]. Furthermore, the expected IMFs log-variance has been shown to admit a simple linear model controlled by the Hurst exponent $H$ of the considered process:

$$\log_2 V_H[k] = \log_2 V_H[2] + 2(H - 1)(k - 2) \log_2 \rho_H$$

for $k \geq 2$, with $\rho_H \approx 2$.

As far as the variability of this quantity is concerned, a quantitative yet empirical appreciation can be gained from the upper part of Figure 2 where, in 3 typical cases ($H = 0.2$, 0.5 and 0.8), the experimental mean, median and various confidence intervals have been reported, together with the model (4). This series of simulations (which has been carried out on 10000 realizations of 2048 data points in each case) evidences larger and larger fluctuations for modes of larger and larger indices, in agreement with (and generalization of) the findings reported in [10] for the only case of white noise. (Interestingly, it has to be remarked that the skewed (marginal) distribution of these “modegrams” reveals a better agreement when fitting the linear model (4) with the median rather than the mean of the realizations.)

The lower part of Figure 2 precises further how the relative confidence intervals can be given a semi-analytical form as a function of the IMF index: the quasi-linear dependences reported in the diagram allow for a parameterization of the curve $T_H[k]$ corresponding to a chosen confidence interval according to a functional relationship of the form

$$\log_2 (\log_2 (T_H[k]/W_h[k])) = a_H k + b_H,$$

where $W_H[k]$ stands for the $H$-dependent variation of some IMF mean energy, considered as a variance estimator. In accordance with what has been said previously, the best linear

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Figure 1: Detrending of a Heart-Rate Variability signal. Left: standardized empirical mean of the fine-to-coarse EMD reconstruction, evidencing $D = 5$ as the change point. Top right: original signal. Middle right: estimated trend obtained from the partial reconstruction with IMFs 6 to 9 + the residual. Bottom right: detrended signal obtained from the partial reconstruction with IMFs 1 to 5.
fit is obtained when choosing for \( W_H[k] \) the median of the IMFs energy over the realizations, which is in this case very close from the model \( V_H[k] \). The parameters \( a_H \) and \( b_H \) that are used as ingredients for modelling the confidence intervals can be deduced from simulation results, and their values are reported in the following Table:

<table>
<thead>
<tr>
<th>( H )</th>
<th>( a_H(95%) )</th>
<th>( b_H(95%) )</th>
<th>( a_H(99%) )</th>
<th>( b_H(99%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.46</td>
<td>-2.43</td>
<td>0.45</td>
<td>-1.95</td>
</tr>
<tr>
<td>0.5</td>
<td>0.47</td>
<td>-2.45</td>
<td>0.46</td>
<td>-1.92</td>
</tr>
<tr>
<td>0.8</td>
<td>0.50</td>
<td>-2.33</td>
<td>0.50</td>
<td>-1.83</td>
</tr>
</tbody>
</table>

### 3.3 Denoising

The considerations above can be used for denoising a signal embedded in fGn of known Hurst exponent \( H \), based on the empirically observed energy \( W_H[k] \) of the IMFs \( d_k[n] \).

In practice, \( W_H[1] \) can be estimated as

\[
\hat{W}_H[1] = \sum_{n=1}^{N} d_n^2[n], \quad (6)
\]

and the subsequent values of \( W_H[k] \) follow as

\[
\hat{W}_H[k] = C_H \rho_H^{2(1-H)k}, \quad k \geq 2, \quad (7)
\]

where \( C_H = \hat{W}_H[1]/\beta_H \) and \( \beta_H \) values can be found in [6].

Given these results, a possible strategy for denoising a signal corrupted by fGn (with known \( H \)) is as follows:

1. assuming that IMF 1 captures mostly noise, estimate the noise level in the signal + noise mixture by computing \( \hat{W}_H[1] \) as in (6);
2. estimate the “noise only” model from (6) and (7);
3. estimate the corresponding model for the chosen confidence interval from (5) and the Table;
4. compute the EMD of the signal + noise mixture and compare the IMF energies with the confidence interval used as a threshold;
5. compute a partial reconstruction by keeping only the residual and those IMFs whose energy exceeds the threshold.

A toy example of the EMD approach to denoising is given in Figure 3, in the case of an oscillatory low frequency waveform embedded in fGn with \( H = 0.3 \).

This Figure suggests of course that a dual strategy can be used for detrending a fGn-type noise process by computing the complementary partial reconstruction based on only those IMFs whose energy is below the threshold. In this respect, the HRV example used in Section 3.1 can be revisited from a more quantitative perspective. Indeed, the inspection of the signal spectrum (top of Figure 4) suggests that a fGn model is qualitatively admissible in the mid-frequency range, with a spectral exponent \( 2H - 1 \approx 0.79 \), leading to \( H \approx 0.9 \). The resulting detrending (which can be compared with profit to Figure 4) has been obtained by letting \( H = 0.9 \) and using a confidence interval of only 95% because of the approximate relevance of the fGn model.

### 4. CONCLUDING REMARKS

Empirical Mode Decomposition (EMD) is an appealing new technique for adaptively decomposing signals in a sum of AM-FM modes. Because the selection of these modes is fully data-driven and very local in time, EMD paves the way for new automatic approaches to detrending and denoising in nonstationary situations. In this perspective, we have reported here on exploratory quantitative results that demonstrate effective and potential usefulness of EMD-based techniques. The current status of EMD, which still lacks from
solid theoretical grounds, imposed to conduct the present study on the basis of extended numerical experiments. In order to elaborate on our present findings, extended studies are necessary, both in terms of experiments (in particular, comparisons with existing competing approaches), and theory developments.

Software. The Matlab codes used in this study are available, and they can be downloaded for free from the URL: perso.ens-lyon.fr/patrick.flandrin/emd.html.

REFERENCES


