PULSE SHAPE DESIGN USING ITERATIVE PROJECTIONS

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ABSTRACT

In this paper, the pulse shape design for various communication systems including PAM, FSK, and PSK is considered. The pulse is designed by imposing constraints on the time and frequency domains constraints on the autocorrelation function of the pulse shape. Intersymbol interference, finite duration and spectral mask restrictions are a few examples leading to convex sets in L^2. The autocorrelation function of the pulse is obtained by performing iterative projections onto convex sets. After this step, the minimum phase or maximum phase pulse producing the autocorrelation function is obtained by cepstral deconvolution.

1. INTRODUCTION

The problem of pulse shape design often comes up in communication systems including PAM, FSK, and PSK with the challenge of utilizing the bandwidth efficiently while having a low complexity receiver. One way is to use a suboptimal demodulator using a matched filter for complexity reduction and defining constraints on the spectrum, intersymbol interference, and duration of the pulse. Each of these conflicting constraints are convex sets in L^2, which are known to provide a useful base in optimization problems and lay the ground for the method of projection onto convex sets [1–5]. This approach was previously used for designing pulse shapes for digital communication systems [6]. However, the difficulty of associating the matched filter output to the corresponding time-domain signal still remains, which is a similar problem to phase retrieval [4, 7]. This information corresponds to a non-convex set in L^2. To avoid this problem, the pulse shape design is considered in two stages in this paper. In the first step, the autocorrelation function of the pulse is obtained by performing orthogonal projections onto convex sets corresponding to intersymbol interference, finite duration and spectral mask constraints. This approach leads to a globally convergent algorithm. In the second stage, the minimum phase or maximum phase pulse producing the autocorrelation function is obtained by cepstral deconvolution.

2. PROJECTIONS ONTO CONVEX SETS

In order to design a pulse shape satisfying the requirements, we use a well-known numerical method called Projection Onto Convex Sets (POCS), defined on the Hilbert space ℓ^2 or L^2. It is an iterative method which is based on making successive projections onto closed and convex sets. A set C is convex if it satisfies:

$$\forall x, y \in C,\ 0 \leq \alpha \leq 1 \Longrightarrow \alpha x + (1-\alpha)y \in C \qquad (1)$$

The criteria of bandlimitedness. finite duration, and finite energy correspond to closed and convex sets in L^2 or ℓ^2 and they are widely used in various signal design and restoration problems [1–5]. The benevolence of the method comes from its convenient use and guaranteed convergence. At each step of the iteration, an orthogonal projection P_m is made onto a convex set C_m as:

$$x_m = P_m x = \arg\min \|x - x_m\| \qquad (2)$$

and the iterates defined by the equation:

$$y_{k+1} = P_1 P_2 \cdots P_M y_k \qquad (3)$$

reaches a feasible solution, which is a member of the intersection $C_0 = \bigcap_{m=1}^{M} C_m$. Note that the feasible solution may not be unique. However, the intersection C_0 of the convex sets is also a convex set and at each step of the iterations we get closer to a solution, so that the convergence is guaranteed regardless of the initial iteration, when C_0 is nonempty.

In the next section, we define the convex sets used in the pulse shape design problem and describe the iterative design algorithm.

3. DESIGN CRITERIA

In this paper, constraints are imposed on the autocorrelation function of the pulse-shape. This approach leads to a globally convergent algorithm because all constraints corresponds to closed and convex sets in ℓ^2.

Let $x[n]$ be the pulse shape and $r_x[k] = \sum_n x[n] x^*[n-k]$ be the corresponding autocorrelation function. The set C_1 is defined as the set of autocorrelation functions in ℓ^2 whose Fourier Transform is below a spectral mask $D(w)$:

$$C_1 = \{r_x \mid S_x(w) \leq D(w)\} \qquad (4)$$

where S_x is the power spectrum of the pulse, or equivalently the Fourier transform of $r_x[k]$.

One can easily check that C_1 satisfies the condition given in (1), using linearity property of the DFT and the well-known triangle inequality. This is also a bound on the pulse energy.

Secondly, another convex set is defined by the time-limitedness of the signal by an interval of duration T_p. Thus, the corresponding autocorrelation function is also time-limited. When the pulse signal is nonzero for $[0, T_p]$ the corresponding autocorrelation function is possibly nonzero in the interval $[-T_p, T_p]$ and the convex set C_2 describing the time-limitedness information is defined as

$$C_2 = \left\{r_x \mid r_x[k] = 0,\ |kT_s| > T_p\right\} \qquad (5)$$

where T_s is the sampling period of the underlying continuous signal. It is trivial to check that this set also satisfies the condition in (1).

Finally, we define the third set as the ℓ^2 signals whose autocorrelation samples at integer multiples of a period K (except 0^{th} sample) magnitude-wise sum up to less than a certain bound b. This corresponds to putting a bound on worst case degradation due to intersymbol interference. Formally,

$$C_3 = \left\{ x \in \ell^2 \mid \sum_{k \neq 0} |r_x[k \cdot K]| \leq b,\ b > 0 \right\} \tag{6}$$

where $r_x[k] = \sum_n x[n] \cdot x[n-k]$ is the autocorrelation of the signal. Careful analysis of (6) reveals that C_3 is not convex due to the cross terms of the autocorrelation. We can still use other sets such as

$$C_h = \left\{ h \in \ell^2 \mid \sum_{k \neq 0} h[k \cdot K] \leq b,\ b > 0 \right\} \tag{7}$$

which is indeed convex, and if we can find a correspondence between C_h and C_3, we can achieve a feasible solution for the three convex sets. In order to find a correspondence between C_h and C_3, we define a subset C_s of C_3 such that:

$$C_s = \left\{ \begin{array}{c} x = F^{-1}\left\{ \sqrt{F\{r_x[k]\}} \right\} \cdot e^{-j\frac{2\pi}{N}mn_0}, \\ x \in C_3 \mid r_x[k] \in C_h \end{array} \right\} \tag{8}$$

where n_0 is a nominal time delay for the pulse shaping filter to be realizable [8], N is the length of the discrete Fourier transformation.

An alternative for this set could be the set of minimum phase signals having the same autocorrelation function $r_x[k]$:

$$C_{s'} = \{ x \in C_3 \mid r_x \in C_h \} \tag{9}$$

Consequently, we can define a scheme for finding a pulse shape satisfying the given requirements. We make successive projections onto the three sets defined above iteratively as described in (3).

The projection operators can be defined as follows, respectively:

$$P_1 x[n] = F^{-1}\{X_m[k]\} \tag{10}$$

$$X_m[k] = \left\{ \begin{array}{c} X[k],\ |X[k]| \leq D[k] \\ D[k] \cdot e^{j\Phi[k]},\ o.w \end{array} \right. \tag{11}$$

where $\Phi[k]$ is the phase of $X[k]$, and

$$P_2 x[n] = \left\{ \begin{array}{c} x[n],\ nT_s \in (0, T_p) \\ 0,\ o.w. \end{array} \right. \tag{12}$$

where T_s is the sampling period, and

$$P_3 x[n] = F^{-1}\left\{ \sqrt{F\{r_x^h[k]\}} \cdot e^{-j\frac{2\pi}{N}mn_0} \right\} \tag{13}$$

where $r_x^h[k] = P_h r_x[k]$, and finally the projection T onto C_h can be defined as:

$$Tz[n] = \left\{ \begin{array}{c} \frac{z[n]}{b} \cdot \sum_{k \neq 0} |z[k \cdot K]|,\ \sum_{k \neq 0} |z[k \cdot K]| > b,\ n = k \cdot K \\ z[n],\ o.w. \end{array} \right. \tag{14}$$

If one would like to project onto $C_{s'}$ instead, we can define the associated projection as [9]:

$$T'x[n] = F^{-1}\{\exp[H(\ln F\{x[n]\})]\} \tag{15}$$

$$Hy[m] = \left\{ \begin{array}{c} 0,\ m < 0 \\ \frac{y[0]}{2},\ m = 0 \\ y[m],\ m > 0 \end{array} \right. \tag{16}$$

It is worth also noting that we work with real signals, and taking real parts of the iterations corresponds to projecting onto convex sets of real signals, which we could denote by P_4.

4. EXAMPLE DESIGN

In this section, we present some exemplary design approaches through our method. In order to achieve a feasible solution quickly, we start from an initial root raised-cosine signal with roll-off factor $\alpha = 1$. In fact, this would not have been necessary if all the projections we defined in the previous section were made onto convex sets. We can still get to a feasible solution starting from a random signal; although we take this heuristic approach, which by no means is a part of the convention. It is even necessary to note that in many iterative solutions consisting of projections onto non-convex sets, it may be better to start with a random signal, since behaving otherwise may consistently lead to non-convergent results, due to the deterministic nature of the projection operators. In our case, however, we are aware of a signal (root raised-cosine) which is somewhat close to satisfying our requirements; and we simply use that fact by making the root raised-cosine signal our starting point.

First we identify the values that result in the worst case degradation for the k^{th} bit as:

$$I_k(j) = \left\{ \begin{array}{c} 1,\ r_u(|j-k|T) > 0 \\ 0,\ o.w. \end{array} \right.,\ j \neq k \tag{17}$$

This is simply because the intersymbol interference (ISI) term should be the negative of the matched filter output at zero lag, for the worst case degradation to occur.

Then we can define the worst case ISI for a unit energy pulse shape $u(t)$ as:

$$ISI = \sum_{k \neq 0} |r_u(kT)| \tag{18}$$

for which the degradation in signal-to-noise ratio (SNR) is:

$$d = -20\log_{10}(1 - ISI) \tag{19}$$

Note that $d' = -20\log_{10}(1 + ISI)$ is not the worst case degradation since $d' < d$, $ISI > 0$. Placing a constraint on the worst case degradation $d < 0.25$ dB directly puts a bound on the ISI as:

$$-20\log_{10}(1 - ISI) < 0.25 \Longrightarrow ISI < 1 - 10^{-\frac{0.25}{20}} \tag{20}$$

which constitutes the b value in (14). Henceforth, we apply the proposed iterative scheme with two other constraints given by the spectral mask in Fig. 1 as D in equation (4), the set C_2 given in equation (5) and (8), as we have a finite duration of 40 μs for the example in consideration.

To achieve the outcome of successive projections onto the sets we defined in the previous section, we stop the iterations immediately as we reach a feasible solution. The pulse shape given below in Fig. 1 yields a symbol rate of 218 kHz, causing a worst case degradation less than 0.25 dB.

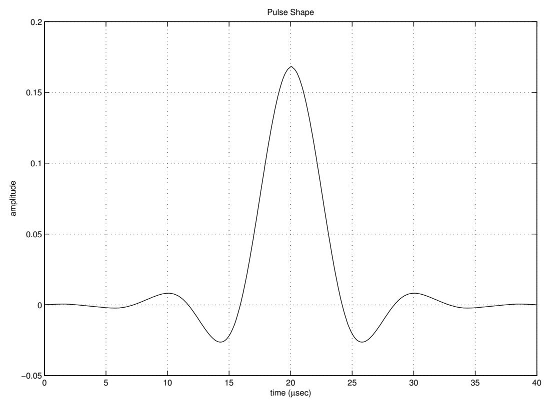

Figure 1: Pulse shape designed via proposed method

Fig. 2 illustrates the matched filter output at the receiver, and the power spectrum of the designed pulse. The mask is nowhere exceeded by the pulse spectrum, as expected.

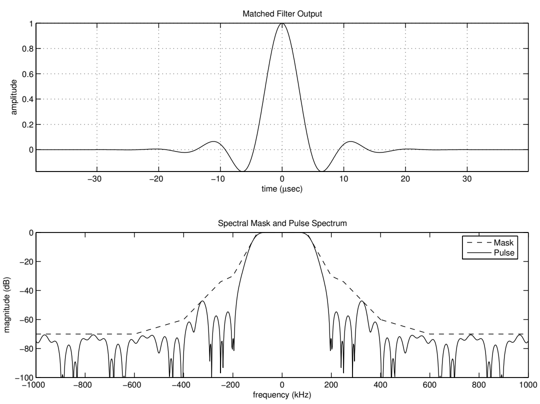

Figure 2: a) Matched filter output, b) Spectral mask and pulse spectrum

In our second design approach, we take the minimal phase root and therefore the corresponding the projection operator onto the set $C_{s'}$ in equations (9,15).

Since minimum phase signals are causal, we observe the projections onto $C_{s'}$ yield signals with little energy before index value 0. The energy spillover is due to the lowpass effect caused by the application of the spectral mask D in (4). Therefore, we need to pick a time delay n_0 as in the previous design, so that most of the energy stays inside the limited duration of the time-domain signal. With trial and error, we observed that a few microseconds were sufficient for this purpose. The initial iteration was chosen to be random. Below is the pulse shape in Fig. 3 and the matched filter output, spectral mask and power spectrum of the pulse in Fig. 4. In order to improve the speed of convergence, we specified tighter bounds in the projection onto the spectral mask set. In this case the worst case degradation in SNR turned out to be 1.75 dB. We observe the tradeoff between speed of convergence (projection with a tighter spectral mask) and the worst case degradation in SNR due to ISI.

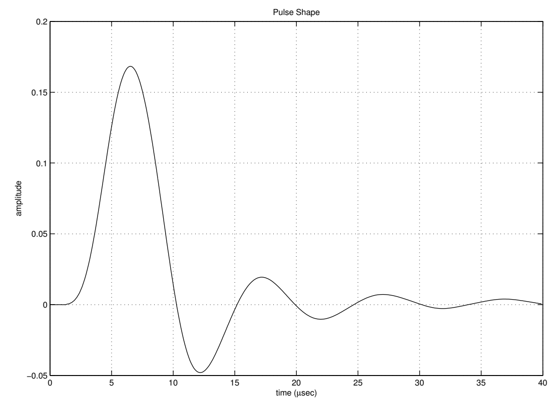

Figure 3: Minimum phase pulse shape

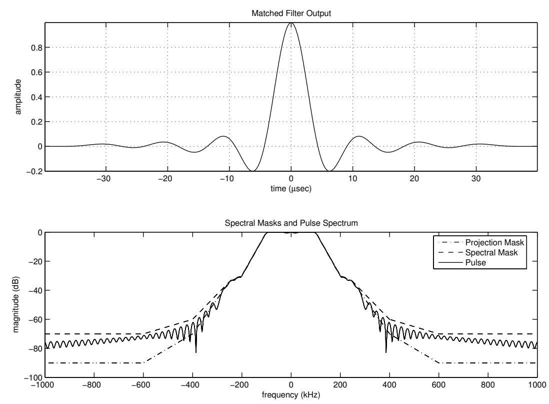

Figure 4: a) Matched filter output, b) Spectral mask used in projections (dashdot), spectral constraint mask (dash), power spectrum of the pulse (solid)

5. CONCLUSIONS

In this paper, we present a method for designing pulse shapes that obey certain constraints defined in time and frequency domains. Other constraints that can be represented as convex sets can be included in the procedure, as well. The method assures the convergence in case all the constraint sets are convex. We develop a method to associate non-convex constraint sets with their convex subsets to overcome the problem of convergence in non-convex sets. We present design examples to illustrate the procedure.

In our examples iterations converged in reasonable numbers of cycles, satisfying all of the requirements. When the constraints are defined to be too tight, the algorithm oscillates between the projections on the constraint sets. In this case, one should restart the procedure with looser constraints. Also, defining the constraints a little tighter than necessary improves the speed of convergence, with a compromise between finding the minimum mean square distance solution, a higher degradation in SNR occurs as a result.

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