IDENTIFYING NON–LINEAR FRACTIONAL CHIRPS USING UNSUPERVISED HILBERT APPROACH

Arnaud Jarrot[†], Patrick Oonincx[‡], Cornel Ioana[†], André Quinquis[†]

[†] E³I² Laboratory (EA 3876) – ENSIETA,
2 Rue François Verny, 29806, Brest, FRANCE
phone: +33(0) 298 348 720 – fax: +33(0) 298 348 750
emails: [jarrotar, ioanaco, quinquis]@ensieta.fr

[‡] Netherlands Defense Academy
1780 CA Den Helder - NETHERLANDS
phone: +31(0) 223 657134 – fax: +31(0) 223 657414
email: p.j.oonincx@kim.nl

ABSTRACT

Non–linear time–frequency structures, naturally present in large number of applications, are difficult to apprehend by means of Cohen's class methods. In order to improve readability, it is possible to generate other class of time–frequency representations using time and/or frequency warping operators. Nevertheless, this requires the knowledge of a non–linear warping function which characterizes the time–frequency content. For this purpose, an unsupervised approach to estimate the warping function is proposed here in the case where time–frequency structures can be represented by chirps with a fractional order. To this end, a Hilbert transform–based technique is applied in order to robustify phases jumps detection. Since those phases jumps define the fractional order in a unique way, the chirp order can be estimated by a bisection method. Results obtained from synthetic data illustrate the attractive outlines of the proposed method.

1. INTRODUCTION

Signals encountered in engineering applications, such as communications, radar or sonar, often involve frequency modulation (FM) which has non–linear variation of the instantaneous frequency law. Typical time–frequency analysis of signals having a non–linear time–frequency structures involves the use of the Cohen's class distributions such as spectrogram and Wigner–Ville distribution [1], or wavelet–based methods [2]. While these methods are natural for signals containing pulses, sinusoids or linear chirps, many other signal classes exist that are not well described in terms of time, frequency or scale.

An alternative is the class of transformations that changes non–linear time–frequency structures into linear ones. This allows the use of more classical characterization tools that are efficient in the case of linear time–frequency structures. The concept of this class of transformations, known as time and/or frequency warping, has been proposed in [3] and is based on the compression of the time and frequency axes by means of a non linear mapping. However, the design of warping operators requires an a-priori knowledge on the non–linearity of the instantaneous frequency law. In the case of unknown signals, the knowledge of the law cannot be easily inferred.

A very useful model for signals estimation is the class of fractional order chirps. This class is based on the representation of the phase of a time–frequency component by means of a non–integer power of time. While the warping transformation of this class of signals is straightforward, an estimation of the chirp order remains difficult, especially for noisy multi–component signals.

To this end, we propose a procedure based on the Hilbert transform of the signal. The Hilbert transform is used to extend the signal in the complex plane for determining its phase. The curvature of the phase is related to the chirp order in a unique way. This relation is translated into a numerical procedure to estimate the chirp order involved in automatic warping methods.

The paper is organized as follows. Section 2 starts with the definition of the signal model considered here. Then, the concept of warping–based time–frequency distribution is presented. In Section 3, properties of the Hilbert transform for chirps are presented. Section 4 deals with the use of the Hilbert transform for identifying chirp orders. The new representation tool based on both Hilbert transform and warping technique is illustrated in Section 5. Section 5 also illustrates the benefits of this approach through results provided for synthetic signals. Finally, concluding remarks are given in Section 6.

2. TIME–FREQUENCY REPRESENTATION FOR FRACTIONAL CHIRP ORDER

2.1 Fractional power chirp model

A classical model for non–linear time–frequency structures is the class of fractional order chirps, defined by

$$s(t) = \exp\left(j\, 2\pi(\omega_0\, t + c\, t^k) \right), \qquad (1)$$

where ω_0 is the start frequency, c is the modulation rate and k stands for the chirp order.

This model is particularly useful to describe strictly increasing or decreasing time–frequency components with a small number of parameters. Typical examples where this model is useful include underwater acoustic propagation in constant celerity profile channels where modal components can be represented by a fractional order, or natural animal sounds such as bat echo–location chirps.

In the case of unknown signals, a general problem is to find the best pair (ω_0, k) for which the model described in (1) best matches the unknown signal. In truth, the unknown information principally reduces to the estimation of the fractional order k since the start frequency ω_0 can be easily estimated by classical stationary spectral analysis. For this reason, we focus on the case where the starting frequency is equal to 0.

2.2 Time–warping principle

Let $s(t) \in L^2(\mathbb{R})$ be a signal with a non–linear instantaneous frequency law and $\mathcal{W} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the warping oper-

ator whose effect on $s(t)$ is given by [1]

$$(\mathcal{W}s)(t) = \sqrt{|\dot{w}(t)|}\, s(w(t)). \qquad (2)$$

The time–warping function $w(t) \in \mathcal{C}^2(\mathbb{R})$ is assumed to be a smooth, one–to–one function, and $\dot{w}(t)$ its derivative with respect to t. Generally, the warping function is chosen to transform the non–stationary time–frequency content into an equivalent stationary one. For a signal $s(t)$ given by

$$s(t) = \exp(j2\pi c t^k),$$

the associated time–warping function is defined as the inverse of modulation function t^k. Applying the warping operator leads to

$$
\begin{aligned}
(\mathcal{W}s)(t) &= \sqrt{|\,t^{1/k-1}/k|}\, \exp\left(j2\pi c (t^{1/k})^k\right) \\
&= \sqrt[2k]{t^{1-k}}/\sqrt{k}\, \exp(j2\pi c t), \quad t,k \in \mathbb{R}^+
\end{aligned}
$$

which is a signal that exhibits a constant instantaneous frequency law.

Generally, the extension of the class of warping operators in the case of discrete–time sequences is not straightforward. Please refer to [4] for details about the discrete–time implementation of warping operators.

2.3 Warping–based signal representation

Matching signals that have non–linear time–frequency structures requires a joint distribution with different instantaneous frequency and group delay localization properties.

A well–known technique [3] is the unitary equivalence. This concept allows to design distributions that match almost any one–to–one group delay or instantaneous frequency characteristics.

This transformation is based on the transformation of the time axis by means of a non–linear mapping, in order to obtain an equivalent signal with linear instantaneous frequency. Next step consists in the estimation of a time–frequency distribution of the Cohen-class which is the optimal class for signals with linear instantaneous frequency. For a particular warping–function, the time–frequency distribution exhibits reduced intraference terms, thanks to the effect non linear mapping. Final step, consists to decompress time and frequency axes in the time–frequency distribution.

This can be illustrated by considering the signal defined by

$$s(t) = \exp(j2\pi\, 0.38\, t^{1.3}),$$

Defining the warping function as

$$w(t) = t^{\frac{1}{1.3}},$$

allows to obtain an equivalent signal with highly reduced frequency support. In such case, new time–frequency coordinates are related to the standard ones by [3]

$$\tilde{t} = w(t), \quad \tilde{f} = \dot{w}^{-1}(w(t))\, f.$$

This example is illustrated in Fig. 1. As can be seen on the Wigner–Ville distribution, the warping operator allows to reduce the frequency support of the warped signal. This effect gives a time–frequency representation for which interference terms are highly reduced which is not the case for the original signal.
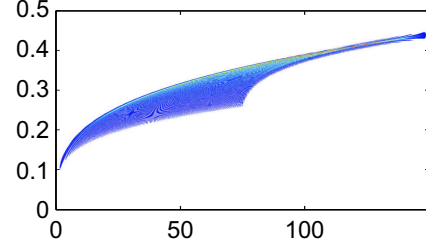
(a) Wigner–Ville distribution of the original signal supported by (t, f).
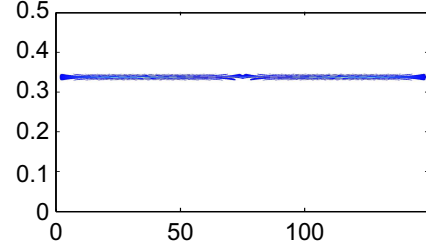
(b) Wigner–Ville distribution of the time–warped signal supported by $(\tilde{t}, \tilde{f})$.
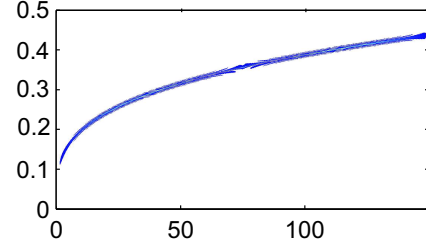
(c) Wigner–Ville distribution of the time–warped signal supported by (t, f).

Figure 1: Comparison of the Wigner–Ville distributions of the original signal and the time–warped signal.

3. THE HILBERT TRANSFORM FOR CHIRPS

We consider the Hilbert transform for signals $s(t)$ in $L^2(\mathbb{R})$, given by

$$\mathcal{H}s(t) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{s(u)}{t-u}\, du, \qquad (3)$$

using a Cauchy's Principal Value (PV) integral. An other way of writing this transform is by means of a convolution product

$$\mathcal{H}s = (s * PV(1/t)) \qquad (4)$$

yielding straightforwardly its Fourier transform

$$\mathcal{F}(\mathcal{H}s)(\omega) = -j\,\text{sign}(\omega)\,\hat{s}(\omega), \qquad (5)$$

with $\hat{s}$ the Fourier transform of s. For computing the Hilbert transform of distributions, with known Fourier transforms like trigonometric functions, Equation (5) is the most suitable starting point.

The Hilbert transform can be added to $s(t)$ as its imaginary part in the complex plane to generate a complex-valued signal $z(t) = s(t) + j\mathcal{H}s(t)$. We observe that $\hat{z}(\omega) = 0$, $\omega < 0$, yielding that $z(t)$ is analytical as a result from [5]. Moreover, $z(t)$ is the analytical continuation of $s(t)$ in the complex plane

and can be represented as

$$z(t) = a(t) \exp(j\varphi(t)), \qquad (6)$$

with $a(t)$ its time-varying amplitude and $\varphi(t)$ its time-varying phase.

As an example we compute $\varphi(t)$ of $f(t) = \sin(2\pi t)$ the most straightforward way to this is given by (5). Observe that its Fourier transform is given by

$$\hat{s}(\omega) = j\pi\, \delta(\omega + 2\pi) - j\pi\, \delta(\omega - 2\pi).$$

Substitution in (5) yields

$$\mathcal{F}(\mathcal{H}s)(\omega) = -\pi\, (\delta(\omega + 2\pi) + \delta(\omega - 2\pi)),$$

i.e., the Fourier transform of $-\cos(2\pi t)$. This results in the analytical signal

$$z(t) = \sin(2\pi t) - j\cos(2\pi t) = \exp(j(2\pi t - \pi/2)).$$

We note that for $s(t) = \sin(2\pi t)$ we get $\varphi(t) = 2\pi t - \pi/2$ and $\omega(t) = 2\pi$, as one may have expected already.

In this paper our aim is to compute $\varphi(t)$ of some chirp $s(t) = \sin(t^k)$ of arbitrary order $k > 1$. Once this phase is known, in particular the chirp order k is known. This order can be used to construct effective warping operators for arbitrary order chirps. The problem we have to deal with is that there is generally no analytical expression for the Hilbert transform of high order chirps. However, using complex analysis one is able to compute the transform in the case $k = 2$. Leaving the analysis to [6] it is shown that the Fourier transform of $s(t) = \sin(t^2)$ in the sense of distributions can be expressed as $\mathcal{F}s(t) = \sqrt{\pi/2}\,(\cos(\omega^2/4) - \sin(\omega^2/4))$. Using (5) one can derive [6]

$$\begin{aligned}
\mathcal{H}s(t) &= -\cos(t^2) \cdot (C(t\sqrt{2/\pi}) + S(t\sqrt{2/\pi})) \\
&\quad + \sin(t^2) \cdot (C(t\sqrt{2/\pi}) - S(t\sqrt{2/\pi})) \\
&= -\cos(t^2) + \varepsilon_1(t),
\end{aligned}$$

with $C(t)$ and $S(t)$ the Fresnel cosine and sine integral functions respectively, see [7]. Since $C(t), S(t) \to 1/2$ for $t \to \infty$, we have derived

$$\mathcal{H}\sin(t^2) = -\cos(t^2) + \varepsilon_1(t), \ \varepsilon_1(t) \to 0. \qquad (7)$$

To show the behaviour of $\varepsilon_1(t)$ we depicted $-\cos(t^2)$ and $\mathcal{H}\sin(t^2)$ (red) together in Fig. 2. Obviously, $\varepsilon_1(t)$ has a fast
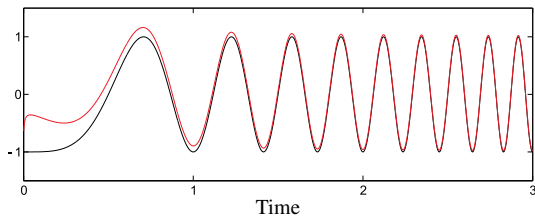

Figure 2: The signal $-\cos(t^2)$ (black) as an approximation of $\mathcal{H}\sin(t^2)$ (red).

decay towards zero making $-\cos(t^2)$ a rather good approximation already for small values of t. The result for quadratic

chirps can be extended to arbitrary order chirps yielding the following proposition.

Proposition 4.1
For $s(t) = \sin(t^k), k > 1$ its Hilbert transform is given by

$$\mathcal{H}\sin(t^k) = -\cos(t^k) + \varepsilon_2(t), \qquad (8)$$

with $\varepsilon_2(t) \to 0$ for $t \to \infty$.

For the rather technical proof we refer to [6]. Here we give a sketch of the proof based on approximating $\sin(t^k)$ by means of a series of localised sine waves. Consider the time intervals $I_{k,n} = (\sqrt[k]{2\pi n}, \sqrt[k]{2\pi(n+1)})$, with $n = 0, 1, \ldots$. The length of such interval is denoted by $T_{k,n}$. For each interval $I_{k,n}$ we write $\sin(t^k) = \sin(2\pi(t - \sqrt[k]{2\pi n})/T_{k,n}) + \varepsilon_n(t)$. For large n it can be shown that the residual $\varepsilon_n(t)$ tends to zero. To illustrate, for the particular case $k = 3$ the decay of ε_n (by means of its power) has been depicted in Fig. 3. Combining all $I_{k,n}$ precisely and taking the energy conservation by means of the Hilbert transform into account we arrive at (8).
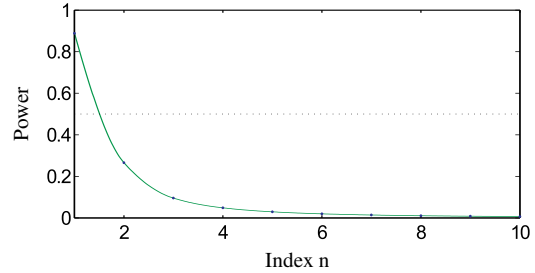

Figure 3: The power of ε_n (green) as a function of n versus the power of $\sin(2\pi(t - \sqrt[k]{2\pi n})/T_{k,n})$ (dotted).

4. CONSTRUCTING WARPING OPERATORS FOR CHIRPS

For a chirp $\sin(2\pi t^k)$ we would like to use the warping operator

$$\mathcal{W}_k s(t) = \sqrt[2k]{t^{1-k}} \cdot s(\sqrt[k]{t})/\sqrt{k}, \quad t > 0,$$

resulting in the warped chirp $\dfrac{\sqrt[2k]{t^{1-k}}}{\sqrt{k}} \cdot \sin(2\pi t)$. To warp a given (measured) chirp of arbitrary order k in this way, we have to know the value of k. In this section we discuss on how to find k for a given $s(t) = \sin(c(t - t_0)^k + \phi_0)$, using the results of the previous section. The fact that not for all $k > 1$ a closed expression for $\mathcal{H}s$ can be found will not effect the method's practical use. In practice $s(t)$ will be given by a finite number of data samples with finite time duration. $\mathcal{H}s$ will then be computed using (5) and the fast Fourier transform.

For simplicity we assume $c = 2\pi$. Without proof we will mention the results for $c \neq 2\pi$ at the end of this section. The algorithm to find the chirp order k is based on computing the phase $\varphi(t)$ using the Hilbert transform of $s(t)$. Based on Proposition 4.1 and elementary properties of the Hilbert transform we get the analytical continuation of $s(t)$

$$\begin{aligned}
z(t) &\sim \sin(2\pi(t - t_0)^k + \phi_0) - j\cos(2\pi(t - t_0)^k + \phi_0) \\
&= \exp(j(2\pi(t - t_0)^k + \phi_0 - \pi/2)),
\end{aligned}$$

resulting in the phase $\varphi(t) = 2\pi(t - t_0)^k + \phi_0 - \pi/2$.

Since in general we do not know the time and phase offset of measured data we cannot derive k from substituting one given pair $(t_1, \varphi(t_1))$ in the expression for the phase and then compute

$$k = \frac{\log((\varphi(t_1) - \phi_0 + \pi/2)/2\pi)}{\log(t_1 - t_0)}.$$

However, three different positions $(x_i, \varphi(x_i))$ on a curve x^k already fully determine the order k. This is due to the increasing locally unique curvature of x^k for any $k > 1$. The method we use here is based on collecting all t_i at which $\varphi(t)$ jumps from $+\pi$ to $-\pi$. Taking $x_i = t_i - t_0$ we get for all i the equations

$$\begin{align} (x_i + \delta_{i,i+1})^k &= x_i^k + 1, & (9) \\ (x_i + \delta_{i,i+2})^k &= x_i^k + 2, & (10) \end{align}$$

with $\delta_{i,m} = t_m - t_i$. From these two equations we have to solve k (and x_i). A way to find a numerical solution for this set of equations is by means of a kind of bisection method [8].

Take $k = k_0 > 1$ (first guess) and put $\alpha = 1, \beta = 0$. Now, solve x_1 from Equation (9) and x_2 from Equation (10), e.g. by using computer algebra. Next, we update the estimation for the order k following

- $x_2 > x_1$ and $\beta = 0$ $\rightarrow$ take $\alpha := k$ and next $k := 2k_0$.
- $x_2 > x_1$ and $\beta \neq 0$ $\rightarrow$ take $\alpha := k$ and next $k := (\alpha + \beta)/2$.
- $x_2 < x_1$ $\rightarrow$ take $\beta := k$ and next $k := (\alpha + \beta)/2$.
- The procedure is repeated until either $x_1 = x_2$ or a desired accuracy for k is achieved.

After applying this bisection method to a set of three consecutive t_i we get an estimate for the order k. Applying the method on other sets of measured t_i yields similar estimates. Taking the mean of all obtained estimates for k finally yields a good estimation for k to be used for the warping operator.

Finally we like to mention without proof, that in the case of dilated chirps ($c \neq 2\pi$) Equations (9) and (10) should be replaced by

$$\begin{align} 2x_i^k &= (x_i + \delta_{i,i+1})^k + (x_i + \delta_{i,i-1})^k, \\ 2x_i^k &= (x_i + \delta_{i,i+2})^k + (x_i + \delta_{i,i-2})^k. \end{align}$$

5. EXAMPLES AND RESULTS

We first illustrate the proposed method by means of the example signal $s(t) = \sin(2\pi t^{1.3})$, as illustrated in Fig. 4.a. Using its Hilbert transform the phase $\varphi(t)$ is computed. Fig. 4.b shows $\varphi(t)$ and the first five samples t_i (see arrows) used for the bisection method. Using this modified bisection approach we find $k = 1.24$ yielding the warped signal $0.9t^{-0.097}\sin(2\pi t^{1.05})$, as depicted in Fig. 4.c. For the bisection approach all consecutive phase jumps $0 < t_i < 10$ have been used.

Note that for this particular example the Hilbert transform is not really needed to obtain phase information. Determining the zero–crossings or extrema of the given signal could have yielded the input data t_i for the bisection method as well. However, in practice for noisy data this method fails,

(a) Chirp signal of fractional order $k = 1.3$

(b) Phase $\varphi(t)$ obtained via analytical continuation

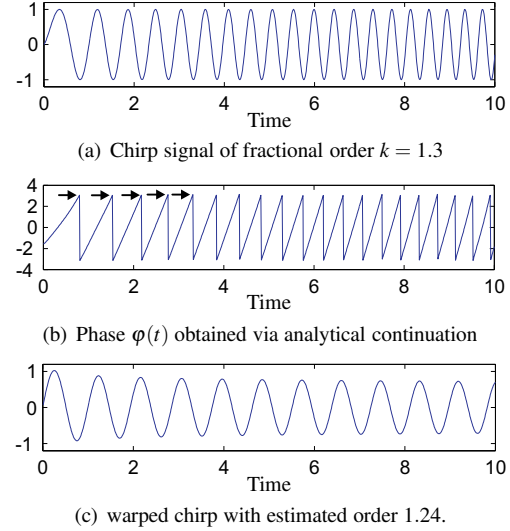(c) warped chirp with estimated order 1.24.

Figure 4: Example of the fractional order estimation method for a chirp signal of fractional order $k = 1.3$. The first five arrows indicates the samples used for the bisection method.

while the Hilbert transform approach seems to be more robust.

Providing a good estimation of the fractional order, the warping operator concentrates the energy of the original signal around a nearly constant instantaneous frequency. This effect can be seen in Fig. 5 where the Wigner–Ville distribution and the frequency marginal of the original signal and the warped signal are depicted. Compared to the frequency spread of the original signal, the warped signal exhibits a high concentration in frequency which can be seen both in the Wigner–Ville distribution and in the frequency marginal.

(a) Original chirp with fractional order $k = 1.3$. Right: Wigner–Ville distribution. Left: frequency marginal.

(b) Warped chirp with estimated fractional order $k = 1.24$. Right: Wigner–Ville distribution. Left: frequency marginal.
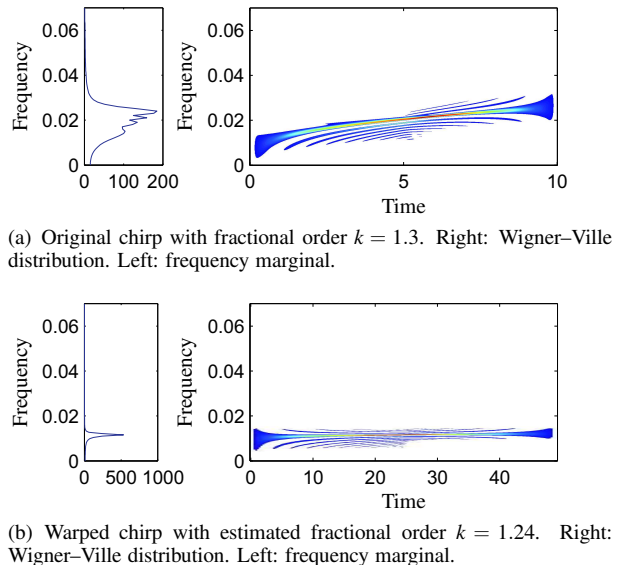
Figure 5: Comparison of the energy concentration between: a) the original signal, b) warped signal with estimated fractional order.

As a second example signal we added white noise to a chirp, i.e., $s(t) = 1.6\sin(2\pi t^{2.6}) + n(t)$ resulting in a SNR=10 dB, as illustrated in Fig. 5.a. As in the previous example we used the Hilbert transform to derive its phase $\varphi(t)$. Fig. 4.b shows $\varphi(t)$ and some samples t_i. Obviously, the pattern of this graph is very similar to the one in Fig. 4, although the influence of the noise is also visible. Using the bisection approach we find $k = 2.42$ yielding the warped signal

$$s(t) = 0.64 t^{-0.295} \sin(2\pi t^{1.07}) + n(t^{1/2.42}),$$

as depicted in Fig. 5.c. For estimating the fractional order we have used all consecutive phase jumps between $t = 1$ and $t = 3$.

(a) Noisy chirp signal of fractional order $k = 2.6$

(b) Phase $\varphi(t)$ obtained via analytical continuation

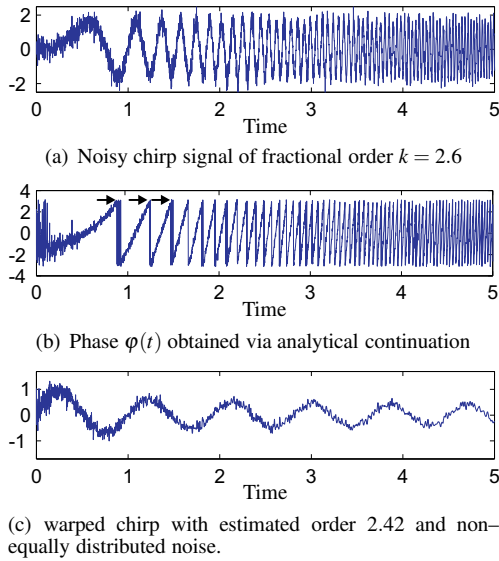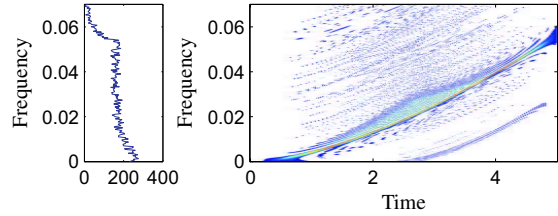(c) warped chirp with estimated order 2.42 and non–equally distributed noise.

Figure 6: Example of the fractional order estimation method for a noisy chirp signal of fractional order $k = 2.6$. The first three arrows indicates the samples used for the bisection method.

Here again, the warping operator constructed from the estimated fractional order, gives a warped signal with the energy concentrated around a nearly constant instantaneous frequency as can be seen the Fig. 7. In Fig. 7.c., the particular noise distribution on the Wigner–Ville distribution is due to the effect of the warping operator on the noise that yields to a non–equally distributed noise.

6. CONCLUSION

In this paper we have proposed a new method for estimating the fractional order of a chirp signal. This method is based on the Hilbert transform technique which is aimed to extend the signal in the complex plane. The phase evaluated in this plane provides, via a numerical estimation procedure, the order of the unknown chirp. Once this order is estimated, we have focused on the design of the warping operator which linearizes the time-frequency content of the signal. Both the Hilbert transform–based chirp order estimation and warping concepts together form an automatic warping procedure. The results proved the efficiency of the method for synthetic data. In the future, we intend to extend this automatic warping procedure for more complex signals (radar, sonar or speech). On

(a) Original noisy chirp with fractional order $k = 1.6$. Right: Wigner–Ville distribution. Left: frequency marginal.
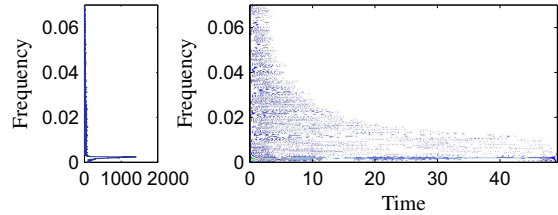
(b) Warped chirp with estimated fractional order $k = 1.42$. Right: Wigner–Ville distribution. Left: frequency marginal.

Figure 7: Comparison of the energy concentration between: a) the original signal, b) warped signal with estimated fractional order.

the other hand, the performance analysis will be done rigorously.

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