FREQUENCY ESTIMATION BASED ON ADJACENT DFT BINS

Michaël Betser, Patrice Collen,

France Telecom R&D
35512 Cesson-Sévigné, France
first.name@francetelecom.com

Gaël Richard.

Telecom Paris
75634 Paris, France
first.name@enst.fr

ABSTRACT

This paper presents a method to derive efficient frequency estimators from the Discrete Fourier Transform (DFT) of the signal. These estimators are very similar to the phase-based Discrete Fourier Spectrum (DFS) interpolators but have the advantage to allow any type of analysis window (and especially non-rectangular windows). As a consequence, it leads to better estimations in the case of a complex tone (cisoid) perturbed by other cisoids. Overall, our best estimator leads to results similar to those of phase vocoder and reassignment estimators but at a lower complexity, since it is based on a single Fast Fourier Transform (FFT) computation.

1. INTRODUCTION

Sinusoidal modeling [1] is a very popular and efficient representation for speech and music signals. It has led to numerous applications such as audio coding, analysis, synthesis and sound transformation. However, to be eligible for such applications, these models require accurate parameter estimations and, in particular, accurate frequency estimation.

Many frequency estimators use the Short Time Fourier Transform (STFT) as a starting point. Such estimators can be classified into three main categories: namely, *time methods* where the time of the STFT varies as in the phase vocoder [2] and the derivative method [1], *window methods* where the window varies as in the spectral reassignment [3], and *frequency methods* where the frequency varies as in the amplitude spectrum interpolation [4] or the phase-based DFS interpolators [5, 6]. A comparison of the state-of-the-art frequency methods can be found in [7, 8]. The latter methods have the advantage to require only one FFT computation since they only use adjacent frequency bins of this single FFT.

This article presents a new frequency estimator that is rather similar to the phase-based DFS interpolator. However, the method used to derive the frequency estimator is original and presents the advantage to allow the use of windowing, which was not the case in [5]. Similarly to previous work, it is supposed in this paper that the studied signal is a complex sinusoid with quasi-constant amplitude and frequency, in the neighborhood of a time t. Such a signal can be written as:

$$x(t+\tau) \triangleq \tilde{A}\, e^{j\beta\tau} + w(t+\tau) \tag{1.1}$$

where $\tilde{A} \triangleq A\, e^{j\alpha}$ is the complex amplitude of the sinusoid, (A is the real amplitude and α the constant phase), and β is the pulsation, both for the time t. τ is the local time in the neighborhood of the time t. This sinusoid can be perturbed by a complex noise w, which is supposed to be zero-mean white Gaussian. The asymptotic properties of the estimator presented in section 4 also holds under weaker noise properties described in [9, 6].

The analysis is based on the Short Time Fourier Transform (STFT) of the partial, defined as:

$$X(t,\omega_k;h) \triangleq \sum_{n=-N/2}^{N/2} x(\tau_n+t)\, h(\tau_n) \exp(-j\,\tau_n \omega_k) \tag{1.2}$$

where N is the size in samples of the window support h, F is the sampling frequency, k is the frequency bin, $\tau_n = n/F$ is the time in seconds of the corresponding sample number (n is an integer). Finally, $\omega_k = \frac{2\pi k F}{N}$ is the pulsation of the bin k. Here N is supposed to be odd. This STFT corresponds to the centered form of the FT, which is sometimes called zero-phased FT, because the phase spectrum response of a symmetric window h has a phase equal to zero in the neighborhood of the frequency zero. This definition is preferred here, because it will simplify the developments presented in this article. If N is even, a centered form of the STFT is also possible, but will be slightly different as the sum will not be on integer values anymore. The STFT of the signal (1.1) can be put under the form:

$$X(t,\omega_k;h) = \tilde{A}\Gamma(\omega_k-\beta;h) + W(t,\omega_k;h) \tag{1.3}$$

where $\Gamma(\omega;h)$ is the discrete time FT, using the definition (1.2), of the window h for the pulsation ω, and W is the STFT of the noise.

In practice, the definition of the FT used is often the linear-phased FT (i.e. the sum in the FT is done from 0 to $N-1$), as for many implementations of the FFT for example. A practical way to zero-phase the FFT is to perform a $(N-1)/2$ sample circular permutation of the windowed signal before computing the FFT [1]. In the remainder of this article, all the FT will be zero-phase.

2. PROPOSED METHOD

The proposed method combines Fourier Transforms (FT) computed for two different frequencies ω_1 and ω_2. The window h is supposed to be symmetric, real and positive. In this part, the noise is not considered. Let's introduce the following FT ratio:

$$\mathscr{H} \triangleq \frac{X(t,\omega_1;h) - X(t,\omega_2;h)}{X(t,\omega_1;h) + X(t,\omega_2;h)} \tag{2.1}$$

$$= \frac{\Gamma(\omega_1-\beta;h) - \Gamma(\omega_2-\beta;h)}{\Gamma(\omega_1-\beta;h) + \Gamma(\omega_2-\beta;h)} \tag{2.2}$$

Because of its particularity, one can show that this ratio can be understood as a ratio of two FT differing in windows. Let $\Delta\omega \triangleq \frac{\omega_2-\omega_1}{2}$, $\omega_b \triangleq \frac{\omega_1+\omega_2}{2}$ and $\delta \triangleq \omega_b - \beta$, then $\mathscr{H}$ can be written as:

$$\mathscr{H} = j\frac{\Gamma(\delta;h_s)}{\Gamma(\delta;h_c)} \tag{2.3}$$

where h_s and h_c are new analysis windows defined by:

$$h_s(\tau) \triangleq \sin(\Delta\omega\tau)h(\tau), \qquad h_c(\tau) \triangleq \cos(\Delta\omega\tau)h(\tau)$$

If h is even, then h_s is odd and h_c is even.

Remark that $\mathscr{H}$ is necessarily real. In fact, as h_c is symmetric, $\Gamma(\omega;h_c)$ is purely real, and as h_s is anti-symmetric, $\Gamma(\omega;h_s)$ is purely imaginary. It will now be shown that an estimator can be defined from (2.3) and the parity hypothesis on h.

Taylor expansions around the frequency zero will be done. The frequency derivative property of the FT states that:

$$\frac{\partial^i\Gamma}{\partial\omega^i}(\omega;h) = (-j)^i\Gamma(\omega;\tau^i.h) \tag{2.4}$$

Let $\Gamma(h) = \Gamma(0;h)$. As $\Gamma(\tau^i.h_s) = 0$ if i is even, and $\Gamma(\tau^i.h_c) = 0$ if i is odd, the upper part of (2.3) will be expanded to an order 1 and the lower part to an order 0. c_1 and c_2 in $[0,\delta]$ exist such that:

$$\mathscr{H} = \frac{\Gamma(\tau.h_s)\delta - \Gamma(c_1;\tau^3.h_s)\frac{\delta^3}{6}}{\Gamma(h_c) - \Gamma(c_2;\tau^2h_c)\frac{\delta^2}{2}} \tag{2.5}$$

$$= \delta\frac{\Gamma(\tau.h_s)}{\Gamma(h_c)}\frac{\left(1-P\right)}{\left(1-Q\right)} \tag{2.6}$$

where P and Q are the Lagrange remainders,

$$P \triangleq \frac{\Gamma(c_1;\tau^3.h_s)}{\Gamma(\tau.h_s)}\frac{\delta^2}{6}, \qquad Q \triangleq \frac{\Gamma(c_2;\tau^2.h_c)}{\Gamma(h_c)}\frac{\delta^2}{2} \tag{2.7}$$

The values of $\Gamma(\tau^i.h_s)$ and $\Gamma(\tau^i.h_c)$ depend only on the analysis window h and are known in advance. So if the corrective terms P and Q are small compared to 1 (cf section 3), an estimation of the frequency can be obtained as:

$$\hat{\beta} = \omega_b - \Re(\mathscr{H})\frac{\Gamma(h_c)}{\Gamma(\tau.h_s)} \tag{2.8}$$

In practice, $\mathscr{H}$ is never exactly real, this is why the real part ($\Re()$) of $\mathscr{H}$ is taken. When using an FFT, formula (2.8) can be applied to the two most energetic bins of the cisoid: the maximum DFS bin k and the highest DFS bin between $k+1$ and $k-1$. In this case, $\Delta\omega$ is the half frequency resolution of the FFT and ω_b is the middle of the two bins selected.

Algorithm:

1. Initialization: compute $\Gamma(\tau.h_s)$ and $\Gamma(h_c)$ for the window h used in the FFT.
2. Compute the zero-phased FFTs for a time t
3. Select the maximum bin $k = \arg\max_i |X(t,\omega_i;h)|$ and the second maximum $k' = \arg\max_{i\in\{k+1,k-1\}} |X(t,\omega_i;h)|$
4. Compute the ratio $\mathscr{H} = \frac{X(t,\omega_k;h)-X(t,\omega_{k'};h)}{X(t,\omega_k;h)+X(t,\omega_{k'};h)}$
5. Compute the estimated frequency: $\hat{\beta} = \omega_b - \Re(\mathscr{H})\frac{\Gamma(h_c)}{\Gamma(\tau.h_s)}$, where $\omega_b = (\omega_k + \omega_{k'})/2$

	Han	Ham	Rec	Bla	Gau
b_1	1.5e-4	2.1e-2	3.9e-6	5.4e-3	2.4e-2
b_2	1.3e-1	1.4e-1	2.5e-1	1.0e-1	1.3e-1
Error bound(Hz)	2.6e-3	3.8e-1	8.3e-5	9.4e-2	4.3e-1

Table 1: Bound values for different analysis windows.

3. ERROR BOUND

In this section the performances of the algorithm without noise are studied. Using equation (2.6) and the definition of the estimator (2.8), the error between β and the estimation $\hat{\beta}$ can be rewritten as:

$$\beta - \hat{\beta} = \frac{(Q-P)}{(1-Q)}\delta \tag{3.1}$$

Since β is inside $[\omega_1,\omega_2]$, $|\delta|$ is lower than the half frequency resolution of the DFT, $R = \pi F/N$.

We will first try to bound P and Q. The FT of a real symmetric and positive window reaches its maximum in $\omega = 0$ and is decreasing for $\omega \in [0;R]$. Therefore P and Q are positive, and tight bounds on P and Q are:

$$P \leq \frac{\Gamma(\tau^3.h_s)}{\Gamma(\tau.h_s)}\frac{R^2}{6}, \qquad Q \leq \frac{\Gamma(\tau^2.h_c)}{\Gamma(h_c)}\frac{R^2}{2} \tag{3.2}$$

If no additional hypotheses on the window h are made, $|P| \leq \pi^2/24$ and $|Q| \leq \pi^2/8$. It means that Q could be equal to 1. Let's now suppose that h verifies the following property:

$$\sum_{|n|\leq N/2} h(n) \geq 2 \sum_{N/4\leq|n|\leq N/2} h(n) \tag{3.3}$$

For all the usual windows, the energy of the center is superior to the energy of the edges. Consequently all the usual windows verify (3.3). With this hypothesis, the bound on Q becomes: $|Q| \leq 5\pi^2/64 < 1$, which proves that $1/(1-Q)$ is $O(1)$ [1]. As P is also $O(1)$, and δ is $O(N^{-1})$, then, from (3.1), the estimate error $\beta - \hat{\beta}$ is $O(N^{-1})$, for all the windows verifying property (3.3). For some windows, the corrective terms P and Q will be of the same order, leading to smaller order of error: it has been shown that for the rectangular window, the estimator is $O(N^{-2})$ [6].

In order to find the bound on $P-Q$, the Lagrange remainders P and Q will be rewritten as :

$$P = \sum_{i=1}^{\infty}(-1)^i\frac{\Gamma(\tau^{2i+1}.h_s)}{\Gamma(\tau.h_s)}.\frac{\delta^{2i}}{(2i+1)!} \tag{3.4}$$

$$Q = \sum_{i=1}^{\infty}(-1)^i\frac{\Gamma(\tau^{2i}.h_c)}{\Gamma(h_c)}.\frac{\delta^{2i}}{(2i)!} \tag{3.5}$$

[1] Let n be an integer variable which tends to infinity, let $g(n)$ be a positive function and $f(n)$ any function. Then $f = O(g)$ means that $|f| \leq A.g$ for some constant A and all values of n. [10]

$$E\{(\beta-\hat{\beta})^2\} = \delta^2\frac{(Q-P)^2}{(1-Q)^2} + \frac{2\sigma^2}{A^2\Gamma(\delta;h_c)^2}\left[\delta^2\Gamma({h_c}^2)\frac{(1-P)^2}{(1-Q)^2} + \frac{\Gamma(h_c)^2\Gamma({h_s}^2)}{\Gamma(\tau.h_s)^2}\right] + O(N^{-3.5}\ln(N)^{1.5}) \quad (4.4)$$

$$E\{(\beta-\hat{\beta})^2\} \lesssim \frac{R^2{b_1}^2}{(1-b_2)^2} + \frac{2\sigma^2}{A^2\Gamma(R;h_c)^2}\left[\frac{R^2\Gamma({h_c}^2)}{(1-b_1)^2} + \frac{\Gamma(h_c)^2\Gamma({h_s}^2)}{\Gamma(\tau.h_s)^2}\right] \quad (4.5)$$

A bound on $P-Q$ is therefore the infinite sum:

$$|P-Q| \le \sum_{i=1}^{\infty}\frac{R^{2i}}{(2i+1)!}|\frac{\Gamma(\tau^{2i+1}.h_s)}{\Gamma(\tau.h_s)} - (2n+1)\frac{\Gamma(\tau^{2i}.h_c)}{\Gamma(h_c)}| \quad (3.6)$$

Alembert's rule shows that this bound is a convergent series, and as R is usually small, this series converges fast. The first terms give a good approximation of this bound.

Let's note b_1 and b_2 the bounds on $P-Q$ (3.6) and Q (3.2) respectively. Values of these bounds for different typical windows are given in table 1. As $b_2 < 1$, one can conclude that a bound on the error is:

$$|\beta-\hat{\beta}| \le \frac{b_1}{1-b_2}|\delta| \quad (3.7)$$

Theorem 1 *If the window h is real, symmetric, positive, and verifies the property* (3.3)*, then the error of the estimator* (2.8) *without considering the noise influence is at least* $O(N^{-1})$ *and is bounded by* $\frac{b_1}{1-b_2}R$*, where* b_1 *and* b_2 *are the bounds in equations* (3.6) *and* (3.2) *respectively, and R is the half Fourier resolution.*

In the last line of table 1, the bounds are given in Hz for $F = 16000$ and $N = 512$, and for various analysis windows. The small values of the bound $b1$ show that the two error terms P and Q compensate each other, especially for the rectangular window. This is why the first order Taylor expansion of equation (2.5), which seems a bit rough at first, can nevertheless give good results, depending on the window used. It can be noted that superior order Taylor expansions of (2.3) can lead to more precise estimators, but at the cost of an increase in complexity. In this case the frequency estimation is now one of the roots of a polynomial which has the same order as the expansion order.

4. STATISTICAL PROPERTIES OF THE ESTIMATOR

The noise influence on the performance of the estimator is discussed in this section. The analysis will be very similar to the one given in [6], as they consider almost the same estimator but only in the case of a rectangular window. It also follows the strategy adopted by Quinn in [9, 10].

Let $\mathscr{X} \triangleq j\,\frac{\Gamma(\delta;h_s)}{\Gamma(\delta;h_c)}$ and $d \triangleq \tilde{A}\Gamma(\delta;h_c)$. From equation (2.3), if no noise is present, there is identity between $\mathscr{X}$ and $\mathscr{H}$. From the definition of $\mathscr{H}$ in equation (2.1), and using equation (1.3), the ratio $\mathscr{H}$ has the form:

$$\mathscr{H} = \frac{\mathscr{X} + \frac{(W_1-W_2)}{d}}{1+\frac{(W_1+W_2)}{d}} \quad (4.1)$$

where W_1 and W_2 are the STFT of the noise for the frequencies ω_1 and ω_2 respectively.

In [11], it is proved that if w is white Gaussian, then $W(t,\omega;h)$ is $O(\sqrt{N\ln(N)})$ almost surely. This property is still true for more general assumptions on the noise, which are described in [11, 9]. If the function used to construct the discrete window h is continuous, positive, and normalized, i.e. ≤ 1 on the interval of definition, then $\sum_n h_n$ will be $O(N)$ and $1/(\sum_n h_n)$ will be $O(N^{-1})$. This property will be useful to determine the function orders. As $\Gamma(\delta;h_c)$ is $O(N)$, then $(W_1+W_2)/d$ is $O(N^{-1/2}\ln(N)^{1/2})$.

$$\mathscr{H} = \big(\mathscr{X} + \frac{(W_1-W_2)}{d}\big)\big(1 - \frac{(W_1+W_2)}{d} + \frac{(W_1+W_2)^2}{d^2} + O(N^{-1.5}\ln(N)^{1.5})\big) \quad (4.2)$$

The expansion has been done to an order 2, because order 1 terms will be canceled when considering the expectation.

What interests us is the expectation of the squared error between the true frequency and the estimated frequency, which corresponds to the Mean Squared Error (MSE) in section 5:

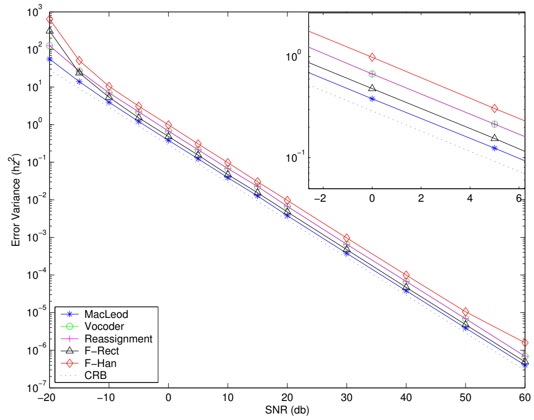
$$E\{(\beta-\hat{\beta})^2\} = E\left\{\Big(\frac{\Gamma(h_c)}{\Gamma(\tau.h_s)}\Re(\mathscr{H}) - \delta\Big)^2\right\} \quad (4.3)$$

Substituting equation (4.2) inside eq (4.3) leads us to an asymptotic development of the MSE. After some simplifications, one can found that this development is given by equation (4.4). If N is large enough, a variance estimate, or at least a tight bound on the variance, could be computed for each value of δ. We have chosen to present only the worst estimation case bound which is given by equation (4.5). This bound has been computed for different typical windows, represented in figure 1. The bound is composed of two terms: one corresponding to the deterministic error, and another to the error caused by the noise. If we consider this bound as a function of the SNR: $\sigma^2/A^2 = 10^{(-SNR/10)}$, the deterministic error will be constant, and the noise error will be a linear function of the SNR in a log scale. Therefore, when the deterministic error is dominant - i.e. for high SNRs - the estimator error will be constant, and when the noise error becomes dominant, for low SNRs, the error will be linear (in log scale). This explains the shape of the curves, in two parts, of the figure 1.

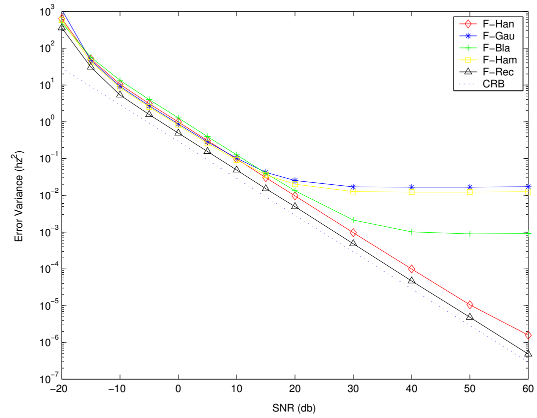
Theorem 2 *If the function used to construct the window h is real, continuous, positive, normalized, symmetric and verifies the property* (3.3)*, then the estimator defined in* (2.8) *is asymptotically unbiased and, when N is large enough, a worst case bound on the variance is given by equation* (4.5).

5. PERFORMANCE COMPARISON

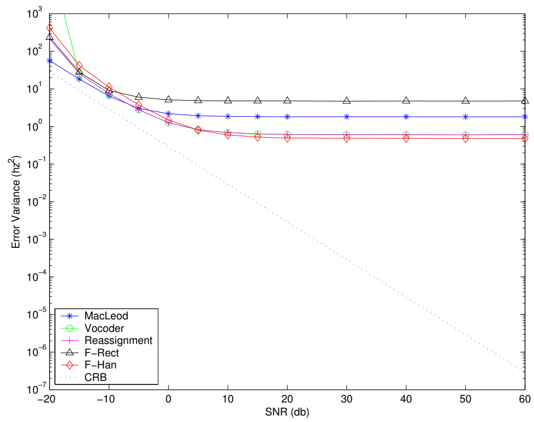
The purpose of this section is to compare the behavior of the new estimators to the classical ones. As studies comparing



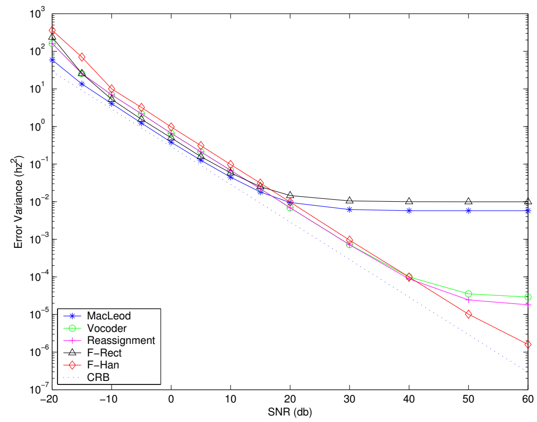
(a) Single cisoid comparison



(b) Window comparison



(c) Two cisoids with 100Hz separation



(d) Two cisoids with 1000Hz separation

Figure 2: Performance comparison

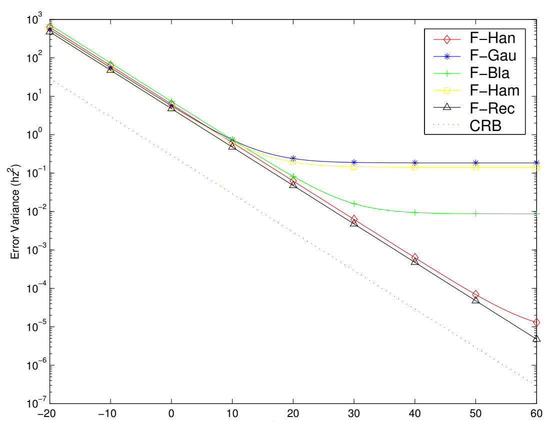


Figure 1: Theoretical window comparison

the classical frequency estimation methods have been done more than once [7], only the algorithms giving the best results will be considered, namely the classical phase vocoder ('Vocoder'), the reassignment method ('Reassignment'), and another interesting method, Macleod's 3 samples interpolator ('Macleod'). All the methods are summarized in table 2. The new method is named 'F' followed by the first three letters of the window used.

In order to achieve a frequency estimation, peak detection is needed, but as our purpose is to compare the frequency estimators, it will be assumed in all experiments that the correct maximum bins are known. The second maximum bin is still supposed unknown.The classical Cramer-Rao Bound (CRB) framework is used to compare the estimators for different Signal to Noise Ratios (SNR) [5]. The CRB is represented by a dashed line.

The experiments are presented for $F = 16000$ and $N = 512$. The error between the true and estimated values is based on an average of the possible causes of error: on noise, using K independent realizations, on frequency values, using randomly picked frequency in regularly spaced interval (10Hz size) over all the spectrum, and on the initial phase and amplitude, using random values between $[0, 2\pi[$ and $[0.1, 0.9]$ respectively. The noise variance is computed from the current amplitude value. In the experiments 2(c) and 2(d), the second sinusoid has the same amplitude as the first one.

Figure 2(b) shows the raw performances of the F estimator on a single cisoid, for different analysis windows. As the SNR increases, the noise becomes negligible, and the inherent bias due to approximation (2.6) appears, as explained in section 4. The theoretical curves in figure 1 have the same shape and the same performance relations as the experimental curve. They have also approximately the same magnitude order as the experimental MSE. The shift between the theoretical curves and the experimental MSE is explained by the fact that the bound (4.5) corresponds to the worst estimation

Name	Estimation	Window
Vocoder	$\hat{\beta} = F.(\angle X(t+1/F,\omega;h) - \angle X(t,\omega;h))$	Han
Reassignment	$\hat{\beta} = \omega + \Im(\frac{X(t,\omega;h')}{X(t,\omega;\tau.h)})$	Han
Macleod	$\hat{\beta} = \omega + \Delta\omega\frac{(\sqrt{1+8\delta^2}-1)}{4\delta}$, where $\delta = \frac{\Re((X(t,\omega-\Delta\omega;h)-X(t,\omega+\Delta\omega;h))X^*(t,\omega;h))}{\Re((2X(t,\omega;h)+X(t,\omega-\Delta\omega;h)+X(t,\omega+\Delta\omega;h))X^*(t,\omega;h))}$	Rec

Table 2: Summary of the different methods compared.

case.

The F estimator is compared to the classical methods described in table 2. For clarity, only two different versions of the F estimator have been retained: F-Rec and F-Han. In the case of a single cisoid estimation, all methods give similar values and perform quite well as all the MSE are contained within 1db of the CRB. F-Han, because of its inherent bias, does not perform as well as the other estimators. The best results are obtained with Macleod's estimator, but F-Rec is very close. These two methods use the rectangular window, and the way Macleod derived his in [5] makes them very similar. If one takes a closer look to the formula of Macleod's estimator (table 2), one can see that it is the solution of an order 2 polynomial. As it has been mentioned in the previous section, the error made with approximation (2.6) may be reduced by using higher Taylor expansion orders, which is what is done in Macleod's estimator. The increase in performance is nonetheless quite small. For low SNR (-20db), the performances of F-Rec and F-Han drop faster than for other estimators. This is caused by the asymmetry of the method: the estimation is best done when using the maximum bin and the second highest bin, but for this noise level the second highest bin is hard to find. A solution, as for Quinn's estimator, is to compute the estimation for both bins around the maximum, and to use a test to determine which estimation is best [5]. But this error appears only in a failure area where all estimators perform badly.

Figures 2(c) and 2(d) represent the errors for an estimation perturbed by a second cisoid which is, respectively, at 100Hz and 1000Hz from the first cisoid. The methods using rectangular windows now give worse results than the others, except when the perturbation due to the second sinusoid becomes smaller than the noise perturbation. The other methods perform better because they use a Hann window which has a better side lobe attenuation. The F-Han method appears to be a good compromise between side lobe attenuation and single cisoid precision, comparable to the phase-vocoder and the reassignment.

6. CONCLUSION

This paper has presented a new frequency estimator based on frequency variations of the FFT. This estimator is very similar to the estimation called DFS interpolator using the phase, but the method presented allows the use of non-rectangular windows, which was not possible before. The advantage of using non-rectangular windows is to keep good performances even if the estimation is perturbed with close cisoids, which was the main default of the DFS interpolator using the phase. The results using a Hann window are similar to the performances of the classical phase vocoder and the reassignment method. In future work, we believe that using superior order expansion and testing different ratio filters $\mathscr{H}$ will lead to estimators performing better than the classical ones in all the scenarii.

REFERENCES

[1] Sylvain Marchand, *Sound models for computer music (analysis, transformation, synthesis)*, Ph.D. thesis, University of Bordeaux, 2000.

[2] Eric Moulines and Jean Laroche, "Non-parametric techniques for pitch-scale and time-scale modification of speech," *Speech Communication*, 1995.

[3] François Auger and Patrick Flandrin, "Improving the readability of time-frequency and time-scale representation by the reassignment method," *IEEE Transaction on Signal Processing*, vol. 43, no. 5, May 1995.

[4] Mototsugu Abe and Julius O. Smith III, "Design criteria for simple sinusoidal parameter estimation based on quadratic interpolation of fft magnitude peaks," *Audio Engineering Society 117th Convention*, 2004.

[5] M. Macleod, "Fast nearly ml estimation of the parameters of real or complex single tones or resolved multiple tones," *IEEE Transaction on Signal Processing*, vol. 46, no. 1, pp. 141–148, Jan 1998.

[6] Elias Aboutanios and Bernard Mulgrew, "Iterative frequency estimation by interpolation on fourier coefficients," *IEEE Transactions on Signal Processing*, vol. 53, no. 4, pp. 1237–1241, Apr 2005.

[7] Konrad Hofbauer, "Estimating frequency and amplitude of sinusoids in harmonic signals," Tech. Rep., Graz University of Technology, Apr 2004.

[8] Stephen Hainsworth and Malcolm Macleod, "On sinusoidal parameter estimation," *Proc. of the 6th Int. Conf. on Digital Audio Effects (DAFx)*, 2003.

[9] Barry G. Quinn, "Estimating frequency by interpolation using fourier coefficients," *IEEE Transactions on Signal Processing*, vol. 42, no. 5, pp. 1264–1268, May 1994.

[10] B. G. Quinn and E. J. Hannan, *The estimation and tracking of frequency*, Cambridge Univ. Press, 2001.

[11] Z.-G. Chen H.-Z. An and E. J. Hannan, "The maximum of the periodogram," *J. Multivariate Anal.*, vol. 13, pp. 383–400, 1983.