

WAVELETS ON THE SPHERE. APPLICATION TO THE DETECTION PROBLEM

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ABSTRACT

A new method is presented for the construction of a natural continuous wavelet transform on the sphere. It incorporates the analysis and synthesis with the same wavelet and the definition of translations and dilations on the sphere through the spherical harmonic coefficients. We construct a couple of wavelets as an extension of the flat *Mexican Hat Wavelet* to the sphere and we apply them to the detection of sources on the sphere. We remark that no projections are used with this methodology.

1. INTRODUCTION

Multiscaling analysis techniques dealing with the analysis/synthesis of nD-images defined on intervals of R^n have been applied in many fields of physics in the last 15 years.

For instance, in the case $n = 1$ one has electronics and audio signals, in the case $n = 2$ one has optical or infrared images whereas for $n = 3$ one deals with fluid dynamics or the large-scale structure of the universe as 3D-images. However, there are data given on other manifolds like the circle S_1 (e. g. scanning along circles the microwave sky) and the sphere S_2 (e. g. geophysics). In this paper, we are interested in data distributed on the sphere. Trivially, for the study of local properties (e. g. detection of objects) one can project on the tangent plane at any point on the sphere to make this type of analysis but when global properties are taken into account the curvature of the sphere can not be neglected.

A first approach to deal with these global properties is to make some global projection of all the points of the sphere. The stereographic projection has been recently used dealing with the continuous wavelet transform. In this case, to get the wavelet coefficient at any point on the sphere, one projects from the opposite point to the local tangent plane. [1] have made a connection to group theory. The translations and dilations in the wavelet have their definition on the plane. Clearly, such a projection does not take into account the topological structure of the sphere. Some applications to cosmology, in particular the study of anisotropies of the cosmic microwave background radiation have been done by some authors ([3],[7],[10]) using the projection of the *Mexican hat wavelet*. A drawback of such projection is the obvious deformation of the pixels and wavelets near the projection pole. We remark that the synthesis can be done in terms of another biorthogonal wavelet [11].

Another approach uses some analyzing wavelet functions that are defined in terms of spherical harmonics [5] with a definition of the dilation operator and conditions on the wavelets in such a way to get a synthesis formula. The draw-

back of such methodology is: the dilations do not satisfy the appropriate flat limit in general. Also some examples of wavelet functions are poorly localized (e. g. Abel-Poisson wavelets).

A different approach assumes from the beginning discrete wavelets incorporating tensor product approaches in polar coordinates, then the two poles are singular points regarding approximation/stability properties ([4],[6]). Another approach is adapted to arbitrary point systems or triangulations on the spheres, then there is no efficient tool as fast wavelet algorithms. In the approach by [8] basis are defined on a quasi-uniform icosahedral triangulation on the sphere allowing for a fast algorithm. However, biorthogonal wavelets are needed and a lifting scheme for the multiresolution is applied avoiding the concepts of translations and dilations and also it is not clear whether the construction leads to a stable $L_2(S_2)$ basis. Haar-type wavelets have been developed using different pixel combinations ([2],[7],[9]). The first case uses the lifting scheme weighting for the area of the pixels whereas in the other two cases an equal area pixelization is used but the Haar-type transform is only applied on regions of the sphere covering only $\frac{1}{12}$ of the total area. Clearly, with any pixelization the symmetry on the sphere is lost.

In this paper we will consider a continuous approach, we will introduce a methodology that incorporates the analysis and synthesis of any function defined on the sphere S_2 using the same circularly-symmetric wavelet and also we will introduce the generalization of the translations/dilations. In this sense we follow Freedman's approach working with spherical harmonics. Examples will be given that have the appropriate flat limit. Finally, the application to the detection of a spot is given, studying the concentration of the wavelet coefficients.

2. PROPERTIES OF THE WAVELET

We will consider a circularly-symmetric filter defined on the sphere S_2

$$\Psi(\vec{n} \cdot \vec{\gamma}; R), \quad (1)$$

where \vec{n} is a fixed direction. $\vec{\gamma}$ is another fixed, but arbitrary direction, therefore $\vec{n} \cdot \vec{\gamma}$ will represent a rotation on the sphere with respect to the direction \vec{n} defined by the angle θ ($\cos(\theta) \equiv \vec{n} \cdot \vec{\gamma}$). $R > 0$ will represent a dilation, which will be defined later on through the spherical harmonics.

We assume the following properties of the filter:

(i) the analysis of any function $f(\vec{n})$ will be done with the wavelets $\Psi(\vec{n} \cdot \vec{\gamma}; R)$,

(ii) the synthesis of any function $f(\vec{n})$ will be done with the wavelets coefficients and the wavelets $\Psi(\vec{n} \cdot \vec{\gamma}; R)$,

(iii) it will incorporate the definition of translation and dilation on the sphere.

We remark that no assumption about compensation of the filter (i. e. $\int d\Omega(\vec{n}) \Psi(\vec{n} \cdot \vec{\gamma}; R) = 0$) and projection from R^2 to S_2 is imposed.

3. ANALYSIS WITH THE FILTER Ψ

We define the wavelet coefficients associated to the translation $\vec{\gamma}$ and dilation R for the function $f(\vec{n})$ defined on S_2

$$w(R, \vec{\gamma}) = \int d\Omega(\vec{n}) f(\vec{n}) \Psi(\vec{n} \cdot \vec{\gamma}; R). \quad (2)$$

Let us assume the standard decomposition of $f(\vec{n})$ in spherical harmonics $Y_{lm}(\vec{n})$

$$f(\vec{n}) = \sum_{lm} f_{lm} Y_{lm}(\vec{n}), \quad f_{lm} = \int d\Omega(\vec{n}) f(\vec{n}) Y_{lm}^*(\vec{n}). \quad (3)$$

By introducing Eq.(3) into Eq.(2) and taking into account that $Y_{lm}(\vec{n})$ is an orthonormal base of S_2 , we obtain

$$w(R, \vec{\gamma}) = \sum_{lm} \left(\frac{4\pi}{2l+1} \right) f_{lm} \Psi_l(R) Y_{lm}(\vec{\gamma}), \quad (4)$$

where the Legendre coefficients associated to the circularly-symmetric filter Ψ are given by

$$\begin{aligned} \Psi(\vec{n} \cdot \vec{\gamma}; R) &= \sum_l \Psi_l(R) P_l(\vec{n} \cdot \vec{\gamma}), \\ \Psi_l(R) &= \left(l + \frac{1}{2} \right) \int_{-1}^1 dy P_l(y) \Psi(y; R). \end{aligned} \quad (5)$$

4. SYNTHESIS WITH THE FILTER Ψ

Now, let us show that in order to have a reconstruction equation, i. e. $f(\vec{n})$ as a functional integral of the wavelet coefficients and the wavelet base Ψ one can impose the condition

$$\Psi_l(R) \equiv \left(\frac{2l+1}{4\pi} \right) \psi(lR), \quad (6)$$

i. e. $\Psi_l(R)$ depends on the product lR and $\psi(l)$ satisfies the admissibility condition

$$C_\psi \equiv \int_0^\infty \frac{dl}{l} \psi^2(l) < \infty, \quad (7)$$

where l runs in the interval $[0, \infty)$. We remark that the analogous condition to have a reconstruction on the plane by substituting $l \rightarrow q$, q being the wave number in Fourier space. Therefore, the filter Ψ -given by Eq. (5)- can be rewritten as

$$\Psi(\vec{n} \cdot \vec{\gamma}; R) = \sum_l \Psi_l(R) P_l(\vec{n} \cdot \vec{\gamma}) = \sum_{lm} \psi(lR) Y_{lm}^*(\vec{n}) Y_{lm}(\vec{\gamma}). \quad (8)$$

Firstly, we remark that the previous equation defines a dilation on the sphere in terms of dilation of the number l and a translation on the sphere in terms of a rotation through the spherical harmonics $Y_{lm}(\vec{\gamma})$. We think that such a definition is the most natural on the sphere and generalizes the one associated to dilations and translations in the plane R^2 via Fourier space.

Secondly, we can write the following equation

$$\begin{aligned} \int \frac{dR}{R} \int d\Omega(\vec{\gamma}) w(R, \vec{\gamma}) \Psi(\vec{n} \cdot \vec{\gamma}; R) = \\ \sum_{lm} f_{lm} Y_{lm}(\vec{n}) \left[\int \frac{dR}{R} \left(\frac{4\pi}{2l+1} \right)^2 \Psi_l^2(R) \right], \end{aligned} \quad (9)$$

where we have taken the harmonic expansions for $w(R, \vec{\gamma})$ and $\Psi(\vec{n} \cdot \vec{\gamma}; R)$. If one wants to have this equation proportional to $\sum_{lm} f_{lm} Y_{lm}(\vec{n}) = f(\vec{n})$, i.e. to be able to reconstruct $f(\vec{n})$, then it is obvious that necessarily

$$\int \frac{dR}{R} \left(\frac{4\pi}{2l+1} \right)^2 \Psi_l^2(R) = C_\psi, \quad (10)$$

where $C_\psi \neq 0$ must be a constant. A particular solution to the previous equation is given by Eq. 6 and the admissibility condition. In this case, the synthesis equation can be written as

$$f(\vec{n}) = \frac{1}{C_\psi} \int \frac{dR}{R} \int d\Omega(\vec{\gamma}) w(R, \vec{\gamma}) \Psi(\vec{n} \cdot \vec{\gamma}; R). \quad (11)$$

and the Equation (4) can be rewritten as

$$w(R, \vec{\gamma}) = \sum_{lm} f_{lm} \psi(lR) Y_{lm}(\vec{\gamma}). \quad (12)$$

These equations are the analysis/synthesis counterparts on S_2 of the corresponding ones on R^2 .

5. PROPERTIES OF THE FILTER Ψ

Let us focus on some of the properties of the filter:

Ψ is a compensated filter

Taking into account Eq. (7), $C_\psi < \infty$ implies that $\psi(l) \rightarrow l^\varepsilon$, $\varepsilon > 0$ as $l \rightarrow 0$, i. e. $\psi(l=0) = 0$. Now, taking into account Eq. (8), one obtains

$$\int d\Omega(\vec{n}) \Psi(\vec{n} \cdot \vec{\gamma}; R) = \psi(0) = 0, \quad (13)$$

the filter is compensated (hereinafter wavelet).

Energy of the wavelet

Taking into account Eq. (8) and the orthonormal property of the spherical harmonics, one obtains

$$\int d\Omega(\vec{n}) \Psi^2(\vec{n} \cdot \vec{\gamma}; R) = \sum_l \frac{2l+1}{4\pi} \psi^2(lR). \quad (14)$$

Energy of the function $f(\vec{n})$

Taking into account the standard properties of the spherical harmonics

$$\|f\|^2 \equiv \int d\Omega(\vec{n}) f^2(\vec{n}) = \sum_{lm} f_{lm}^2, \quad f_{lm}^2 \equiv f_{lm} f_{lm}^*. \quad (15)$$

We can also prove the following equivalence

$$\|f\|^2 = \frac{1}{C_\psi} \int \frac{dR}{R} \int d\Omega(\vec{n}) w^2(R, \vec{n}). \quad (16)$$

6. AN EXAMPLE

As an example of the previous ideas we will consider the generalization to the sphere of the *Mexican Hat wavelet* (MHW). We focus on the MHW because it is a widely used tool in Astronomy, well suited for the detection of pointlike objects such as extragalactic sources ([3],[7],[10]), but the same ideas can be applied to other wavelet families as well. Two natural generalizations of the mother Mexican Hat wavelet on the plane are possible

$$\psi_1(l) \propto l^2 e^{-\frac{1}{2}l^2}, \quad \psi_2(l) \propto l(l+1)e^{-\frac{1}{2}l(l+1)}, \quad (17)$$

where we have taken a unit width to define the mother wavelet. For $l \gg 1$ both functions approach the MHW on the plane. We have represented the harmonic coefficients $\psi_1(lR)$, $\psi_2(lR)$ for different values of $R = 0.2 \times 2^j$, $j = -2, -1, 0, 1, 2$, as well as the profile of the wavelets on real space for the same cases, in Figure 1. The differences between wavelets ψ_1 and ψ_2 is shown in Figure 2.

Next, in order to study the concentration in wavelet space we will consider a spot with spherical symmetry placed on the north pole, i. e. it is defined by a function $f(\theta)$. In this case, we get for the wavelet coefficients

$$w(R, \theta) = \sum_l w_l(R) P_l(\cos \theta), \quad w_l(R) \equiv \int f_l \psi_l(lR),$$

$$f_l \equiv \left(l + \frac{1}{2}\right) \int_0^\pi d\theta \sin \theta P_l(\cos \theta) f(\theta). \quad (18)$$

Let us now consider a very simple spot defined by a top hat $f(\theta) = 1_{[0, \theta_0]}$. A simple way to test how the wavelets allow us to concentrate the information in a few number of coefficients is to measure somehow the width of the curve $w(R, \theta)$ of wavelet coefficients for the spot. An intuitive way to do this is to define the “energy” as the integral under the squared curve $w^2(R, \theta)$ and to see which is the radius θ_e that contains a given fraction of the total energy. The smaller θ_e is, the more concentrated the coefficients are.

We have performed numerical simulations with a simple toy model to see which of the two generalizations of the MHW, ψ_1 or ψ_2 , concentrates more the coefficients. We have placed a top hat spot of size $\theta_0 = 0.2$ rad in the North Pole and we have filtered it with the wavelets ψ_1 and ψ_2 using different scales ranging from $R = 0.1$ to $R = 1.6$ rad. The results are shown in the upper panel of Figure 3. We have plotted the radius $\theta_e(R)$ such that 68% of the energy is inside the circle of radius $\theta_e(R)$ as a function of the dilation scale R . As can be seen, the wavelet ψ_1 concentrates more the coefficients, that is, produces smaller values of θ_e .

Now we repeat the same process but using a Gaussian spot instead of a top hat. The Gaussian spot is given by $f(\theta) = \exp(-\theta^2/2\theta_0^2)$. This case is interesting since most of the detectors that operate in microwave Astronomy experiments have approximately Gaussian response. The lower panel of Figure 3 show the results, that are very similar to the top hat case. Again, ψ_1 concentrates more the coefficients. We have tested the results for different levels of concentration and we have found that up to 75% the wavelet ψ_1 concentrates more the coefficients. Above 75% this is still true for large values of the dilation R , while for small R ψ_2 concentrates more the coefficients. This is due to the fact that ψ_1 produces more small oscillations in the tail of the curve $w(R, \theta)$ for small R .

7. CONCLUSIONS

We have developed a constructive wavelet approach on the sphere without any projection from the plane. It is a continuous transform that allows the analysis and synthesis of any function defined on the sphere and incorporates the concepts of translation and dilation as generalizations of the elementary ones defined on the plane. It is a compensated filter that conserves the energy of any function. We have considered some natural generalizations of the plane Mexican hat wavelet and we have applied them to the detection of a big spot. The conclusion is that one of the wavelets (ψ_1) concentrates more the information than the other one.

8. ACKNOWLEDGEMENTS

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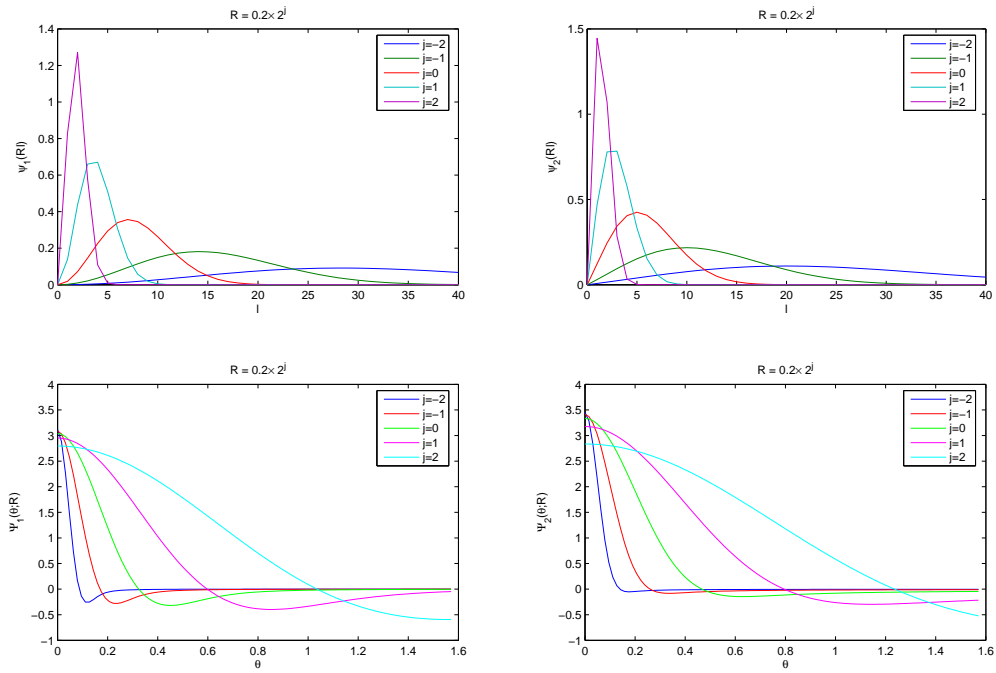


Figure 1: Harmonic coefficients of the wavelets ψ_1 (top left) and ψ_2 (top right). In the bottom panels, the dilated wavelets ψ_1 (left) and ψ_2 (right) are shown on real space. For all the cases, $R = 0.2 \times 2^j$

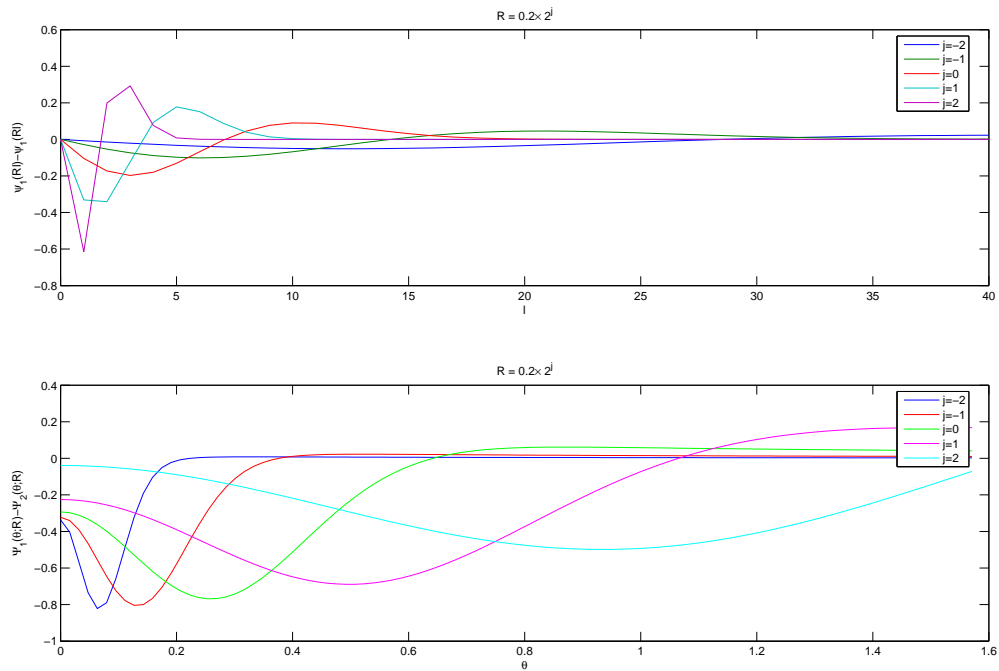


Figure 2: Difference $\psi_1 - \psi_2$ on harmonic (top) and real space (bottom) for the same cases as in Figure 1

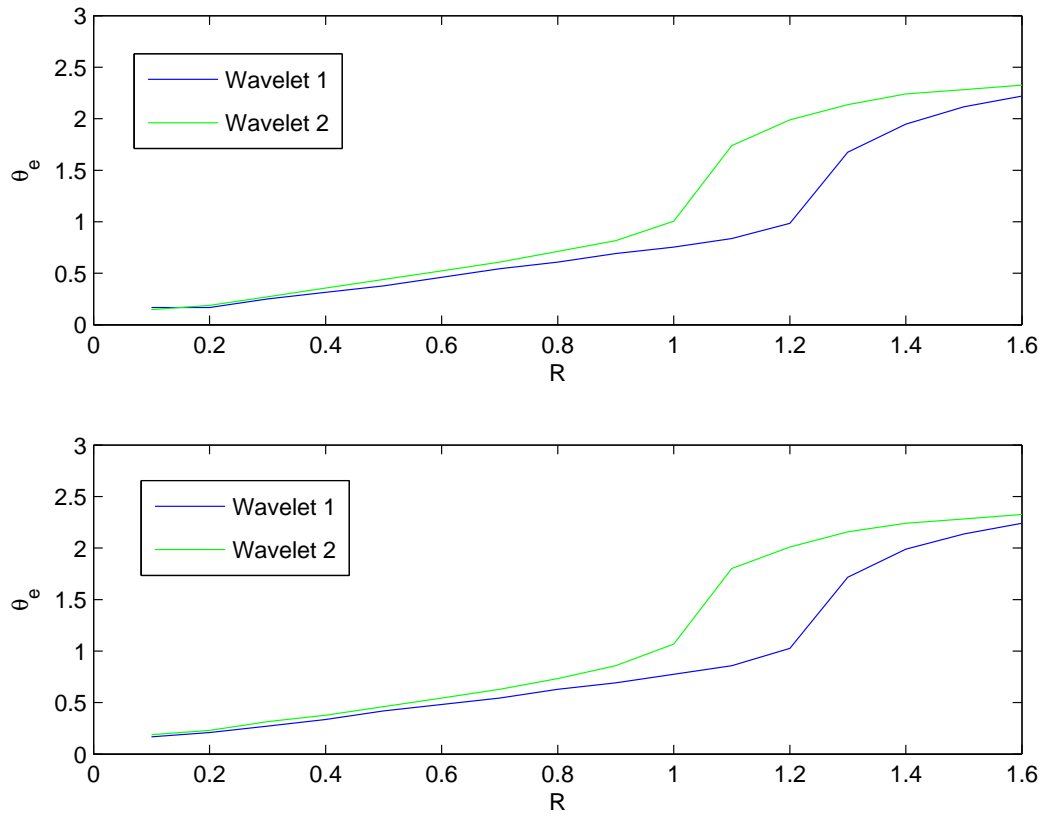


Figure 3: Width of the wavelet coefficient curves $w(R, \theta)$ for the wavelets ψ_1 and ψ_2 applied to a simple top hat spot of size $\theta_0 = 0.2$ rad placed on the north pole and different values of R (top panel) and for a Gaussian spot of size $\theta_0 = 0.2$ rad placed on the north pole and different values of R (lower panel). The width of the curve is defined as the radius θ_e that contains 68% of the energy of the curve.