SPECTRAL ANALYSIS TECHNIQUES OF STATIONARY POINT PROCESSES USED FOR THE ESTIMATION OF CROSS-CORRELATION: APPLICATION TO THE STUDY OF A NEUROPHYSIOLOGICAL SYSTEM

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ABSTRACT

In this work we apply spectral analysis techniques of bivariate stationary point processes for the estimation of the cross-correlation (CC). This is used for the study of a component of the neurophysiological system called muscle spindle. We are interested in the effect of different stimuli to the function of the muscle spindle by recording the response from the Ia sensory axon transferred to the spinal cord. The estimate of the cross-correlation, a measure of the association between the input and the output of the muscle spindle, is obtained using the periodogram statistic which is a function of the frequency domain. It is shown that the asymptotic distribution of the estimator is Normal. Thus approximate 95% pointwise confidence intervals for the estimates of the modified CC can be computed which show if the behaviour of the system is excitatory and/or inhibitory.

1. INTRODUCTION

In the study of a number of neurophysiological problems, the theory of stationary point processes plays an important role [3], [7], [10]. Interesting parameters of the stationary point processes can be defined in the time domain describing important characteristics, like the mean rate of occurrence of events, the association or not between two point processes and many others.

In this paper we examine the degree of association between two point processes. Many parameters have been proposed for this purpose [3], [10]. A measure of association which is of interest is the cross-correlation (CC). The estimate of the CC is obtained using spectral analysis techniques and its asymptotic distribution is shown to be Normal with variance depending on the true value of the estimate. The use of variance-stabilizing transformations permits the construction of approximate pointwise confidence intervals which show if there is association between two stationary point processes.

The theoretical results developed in this paper are applied to the study of an elementary component of the neuromuscular system called muscle spindle. This system plays a critical role in the initiation movement and in the maintenance of posture. In particular two cases are examined which affect the function of the muscle spindle:

I. innervation of the muscle spindle coming from a gamma motoneuron (gamma stimulation)

II. gamma stimulation modified by the presence of an alpha motoneuron.

2. BIVARIATE POINT PROCESSES

2.1 Definition of the bivariate point process

Let \( N(t) = \{N_1(t), N_2(t)\}, \ 0 < t < \infty \) be a bivariate stochastic point process, where \( N_a(t), \ a = 1, 2 \) denotes the number of type \( a \) events occurred in the time interval \((0, t]\).

Every individual component \( N_a(t) \) is a random, non-negative, integer-valued variable, with increments given by \( dN_a(t) = N_a(t + dt) - N_a(t) \).

We assume that the stochastic point process satisfies the following conditions:
- It is stationary. This means that the probabilistic properties of the process do not change with time \( t \).
- It is orderly. This means that the points of the process are isolated and thus the point process can be considered as a binary time series.
- It is strong mixing. This means that the increments of the point process well-separated in time are approximately independent.

More details about stationary point processes and their properties discussed above can be found in [6].

2.2 Definition of certain parameters

In this section we define certain parameters of point processes in the time and in the frequency domain.

2.2.1 Time domain

The mean intensity of type \( a \) events, \( a = 1, 2 \), is defined as:

\[
p_a = \lim_{h \to 0} \text{Pr}\{ \text{type } a \text{ event occurs in } (t, t+h) \} / h.
\]

This parameter describes the average number of points per unit time of the component \( N_a(t) \). The second-order product density function between type \( a \) events and type \( b \) events (\( a, b = 1, 2 \)), is defined by:
\[ p_{ab}(u) = \lim_{h_j \to 0} \Pr\{ \text{type a event in } (t+u,t+u+h_j] \text{ and type b event in } (t,t+h_j]\} / (h_j). \]

The above expression may be interpreted as the probability of having a type a event in the time interval \((t+u,t+u+h_j]\) and a type b event in the time interval \((t,t+h_j]\). The parameters \(p_a\) and \(p_{ab}(u)\) do not depend on \(t\) because of stationarity.

The condition of orderliness implies that the product densities functions are connected with the moments of the process:

\[ E\{dN_a(t)\} = p_a, \]
\[ E\{dN_a(t+u)N_b(t)\} = p_{ab}(u)dtdu, a \neq b. \]

From the definition of the strong mixing condition we have that the increments \(dN_a(t+u)\) and \(dN_b(t)\) become independent as \(u\) becomes large. Therefore,

\[ p_{ab}(u) \to p_a p_b, \quad a \neq b. \]

This leads to the definition of the second-order cumulant density \(q_{ab}(u)\) as

\[ q_{ab}(u) = p_{ab}(u) - p_a p_b, \quad a \neq b, \]

which is continuous at the origin and tends to zero as \(u \to \infty\). More details about these functions can be found in [2].

### 2.2.2 Frequency domain

If the first two moments of the process \(N(t)\) exist we define the cross-spectral density function \(f_{ab}(\lambda)\), as the Fourier transform of \(q_{ab}(u)\):

\[ f_{ab}(\lambda) = (2\pi)^{-1} \int q_{ab}(u)e^{-i\lambda u}du, \]

with \(-\infty < \lambda < \infty\) and \(a \neq b\). The inverse transform of the above relation gives

\[ q_{ab}(u) = \int_{-\infty}^{\infty} e^{i\lambda u} f_{ab}(\lambda)d\lambda, \quad -\infty < u < \infty. \] (1)

### 3. MEASURING ASSOCIATION

Cross-correlation is proposed for measuring the association between the increments of the components of a bivariate point process. We begin by

\[ \text{corr} \{dN_a(t+u),dN_b(t)\} = q_{ab}(u) / \sqrt{p_a p_b} \sqrt{dtdu}. \]

The above relation suggests the quantity

\[ \rho_{ab}(u) = q_{ab}(u) / \sqrt{p_a p_b}, \]

as a measure of the degree of association at lag \(u\) between the type a events and type b events [3]. This statistic is not bounded but it still can be used to examine association [7]. If there is no association then \(\rho_{ab}(u) = 0\). If \(\rho_{ab}(u) > 0\) the association between \(N_a\) and \(N_b\) is positive, consequently the effect of the process \(N_b\) on the process \(N_a\) is excitatory, whereas if \(\rho_{ab}(u) < 0\) the effect is inhibitory.

### 4. ESTIMATION OF THE CROSS CORRELATION

In order to obtain an estimate of the CC we consider the periodogram, a function in the frequency domain, which leads to the estimation of the time domain parameters described in the previous section.

Let \(N(t) = \{N_1(t),N_2(t)\}\) be observed for \(0 < t \leq T\).

The estimates of the mean intensities \(p_a, a, b = 1, 2\) are given by

\[ \tilde{p}_a = \frac{N_a(T)}{T}. \]

The finite Fourier-Stieltjes transform of the increments is a fundamental tool for our statistical analysis, defined by

\[ d_s(\lambda) = \int_0^T e^{-i\lambda u}dN_a(u). \]

Sometimes it is better to use the modified finite Fourier – Stieltjes transform, given by

\[ d_s^T(\lambda) = \int_0^T e^{-i\lambda u}dN_a(u) - \tilde{p}_a \Delta^T(\lambda), \]

where \(\Delta^T(\lambda) = \int_0^T e^{-i\lambda u}du\). Taking into account that

\[ \Delta^T(\lambda) = \begin{cases} T, & \text{if } \lambda = 0, \\ 0, & \text{if } \lambda = \pm \frac{2\pi j}{T}, j = \pm 1, \pm 2, \ldots. \end{cases} \]

we get \(d_s^T(\lambda) = d_s(\lambda)\), for \(\lambda = \pm \frac{2\pi j}{T}, j = \pm 1, \pm 2, \ldots.\) The cross-spectral density function is then estimated from the modified periodogram statistic defined by

\[ I_{ab}^T(\lambda) = \frac{1}{2\pi T} d_s^T(\lambda) d_s^T(\lambda), \] (2)

where \(d_s^T(\lambda)\) is the conjugate function of \(d_s(\lambda)\). In practice we approximate \(d_s^T(\lambda)\), by the following sum:

\[ d_s^T(\lambda) = \sum_{j=0}^{T-1} e^{-i\lambda j} [N_a(t+j) - \tilde{p}_a j], N_a(0) = 0. \]

Combining relations (1) and (2) we can estimate the second-order cumulant density function as follows:

\[ \tilde{q}_{ab}(u) = \frac{2\pi}{T} \sum_j W_j(\lambda) I_{ab}^T(\lambda) e^{i\lambda u}, \] (3)

where \(W_j(\lambda) = W(b_j)\) is a convergence factor [8] with bandwidth \(b\) and \(\lambda = \frac{2\pi j}{T}, j = 0, \pm 1, \pm 2, \ldots\).

Finally, the estimate of the CC is given by

\[ \tilde{\rho}_{ab}(u) = \frac{\tilde{q}_{ab}(u)}{\sqrt{\tilde{p}_a \tilde{p}_b}}. \]
5. Asymptotic Distributions of the Estimated Parameters

In this section we examine the asymptotic distributions of \( \hat{q}_{ab}(u) \) and \( \hat{p}_{ab}(u) \).

**Assumption 5.1** The stationary bivariate point process \( N(t) = \{N_1(t), N_2(t)\} \), \( 0 < t < \infty \), satisfies the relation

\[
\sum_{i=1}^{\infty} \left(1 + u_i\right) q_{ab} \delta(u_i - u) du_i < \infty,
\]

for \( j = 1,2,\ldots,\ell - 1; a_1,\ldots,a_\ell = 1,2 \) and \( \ell = 2,3,4,\ldots \).

This assumption implies that well-separated points of the process are nearly independent and that spectral densities of all orders are bounded and uniformly continuous.

**Lemma 5.1** Let \( N(t) = \{N_1(t), N_2(t)\} \), \( 0 < t < \infty \), be a stationary bivariate point process. We also assume that the process is orderly and satisfies the Assumption 5.1. Then if \( b_1T \to \infty \) as \( T \to \infty \) and \( b_1 \to 0 \), the estimate \( \hat{q}_{ab}(u) \) is asymptotically Normal with mean \( q_{ab}(u) \) and variance given by

\[
\frac{\hat{p}_{ab}(u)}{2nb_1T} \int W^2(\lambda) d\lambda.
\]

**Proof**

Firstly, we prove that \( \hat{q}_{ab}(u) \) is asymptotically an unbiased estimate of \( q_{ab}(u) \). The mean value in relation (3) gives

\[
E[\hat{q}_{ab}(u)] = \frac{2\pi}{T} \sum_q W_q(\lambda_q)e^{iu\lambda_q}.
\]

According to [2] and the Appendix,

\[
E[\hat{q}_{ab}(u)] = \int q_{ab}(v) W_q(v-u) dv + O(b_1^{-1}T^{-1}) + O(T^{-1}),
\]

where \( W_q(v) = b_1^{-1} W(b_1^{-1}v) \) is the Fourier transform of the convergence factor \( W_q \). If we set \( z = b_1^{-1}(v-u) \), then

\[
E[\hat{q}_{ab}(u)] = q_{ab}(u) + O(b_1^{-1}T^{-1}) + O(T^{-1}).
\]

Next we prove the formula for the asymptotic variance. Applying the properties of the cumulants [5],

\[
\text{Var}[\hat{q}_{ab}(u)] = \text{cov}[\hat{q}_{ab}(u), \hat{q}_{ab}(u)]
\]

is approximately

\[
\left(\frac{2\pi}{T}\right)^2 \sum_{j=1}^{\infty} \sum_{\lambda_j} W_j(\lambda_j) W_j(-\lambda_j) e^{iu\lambda_j} \text{cov}\left[I_{ab}^2(\lambda_j), I_{ab}^2(-\lambda_j)\right]
\]

\[
= \left(\frac{2\pi}{T}\right)^2 \sum_{j=1}^{\infty} \sum_{\lambda_j} W_j(\lambda_j) e^{iu\lambda_j} \int \delta(\lambda_j - \lambda_k) f_{ab}(\lambda_k) f_{ab}(\lambda_k) d\lambda_k
\]

\[
+ \left(\frac{2\pi}{T}\right)^2 \sum_{j=1}^{\infty} \sum_{\lambda_j} W_j(\lambda_j) e^{iu\lambda_j} f_{ab}(\lambda_j, -\lambda_j, -\lambda_j),
\]

where \( \delta(\lambda) \) is the Kronecker delta function. According to the form of the fourth-order spectral density [2] and the relation

\[
\lim_{T \to \infty} \frac{f_{ab}(\lambda)}{\pi} = \begin{cases} \frac{p_a}{2\pi} , & \text{if } a = b, \\ 0 , & \text{if } a \neq b \end{cases}
\]

we finally get

\[
\lim_{T \to \infty} \frac{b_1TVar[\hat{q}_{ab}(u)]}{2} = \frac{p_a(u)}{2\pi} \int W^2(\lambda) d\lambda.
\]

That is,

\[
\text{Var}[\hat{q}_{ab}(u)] = \frac{p_a(u)}{2\pi} \int W^2(\lambda) d\lambda.
\]

The estimate \( \hat{q}_{ab}(u) \) follows asymptotically a Normal distribution because the normalized cumulant of order \( R \) tends to zero according to the assumptions of the Lemma, for \( R > 2 \) [9]. Let,

\[
\text{cum}[\hat{q}_{ab}(u), \ldots, \hat{q}_{ab}(u)]
\]

be the cumulant of order \( R \) of \( \hat{q}_{ab}(u) \). Then,

\[
(b_1T)^2 \text{cum}[\hat{q}_{ab}(u), \ldots, \hat{q}_{ab}(u)] = O\left[\left(b_1T^{-1}T^{-1}\right)^{R-1}\right],
\]

which tends to zero, for \( R > 2 \), as \( b_1T \to \infty \).

The next theorem is an important result for our work.

**Theorem 5.1** We assume that the bivariate point process \( N(t) = \{N_1(t), N_2(t)\} \), \( 0 < t < \infty \) satisfies the assumptions of the Lemma 5.1. Then the estimate \( \hat{p}_{ab}(u) \) follows asymptotically a Normal distribution with mean \( p_{ab}(u) \) and variance given by

\[
\frac{\hat{p}_{ab}(u)}{2\pi b_1T p_b} \int W^2(\lambda) d\lambda.
\]

**Proof**

For the estimates of the mean intensities, \( \hat{p}_{ab}(u) \), \( a = 1,2 \), it is known [4] that the distribution is asymptotically Normal with mean \( p_a \) and variance of order \( O(T^{-1}) \). Then, using the square root transformation we can show that \( \sqrt{p_a} \) tends in probability to \( \sqrt{p_a} \). Therefore it follows from [12] that \( \hat{p}_{ab}(u) \) tends in distribution to Normal with mean \( p_{ab}(u) \) and variance \( \frac{\hat{p}_{ab}(u)}{2\pi b_1T p_b} \int W^2(\lambda) d\lambda \), which gives the required result.

The variance of \( \hat{p}_{ab}(u) \) is depending on \( p_{ab}(u) \), because

\[
\hat{p}_{ab}(u) = q_{ab}(u) + p_ap_b - \sqrt{p_ap_b},
\]

In order to have a constant variance we apply a variance-stabilizing transformation as it is shown in the following corollary.
Corollary 5.1 We assume that the bivariate point process
\( \mathcal{N}(t) = (N_1(t), N_2(t)) \), \( 0 < t < \infty \) satisfies the assumptions
of the Theorem 5.1. Then the modified estimate
\( \hat{s}_{ab}(u) = \sqrt{\hat{p}_{ab}(u)} + \sqrt{p_b} \) follows asymptotically a Normal
distribution with mean \( \sqrt{p_{ab}(u)} + \sqrt{p_b} \) and variance given by
\[
\frac{1}{8p_b T \sqrt{p_p p_b}} \int W^2(\lambda) d\lambda.
\]

Proof
Let \( g \) be the appropriate transformation for the stabilization
of the variance. Then \( g \) will be given by the solution of the
equation [12]:
\[
g = c \left( \frac{\text{Var} \left( \hat{p}_{ab}(u) \right) }{\hat{p}_{ab}(u)} \right)^{1/2}.
\]
where the constant \( c \) is the asymptotic standard division of
the transformed statistic \( g \left( \hat{p}_{ab}(u) \right) \). The choice of
\[
c = \frac{\int W^2(\lambda) d\lambda}{2\sqrt{2p_b T \sqrt{p_p p_b}}} \text{ completes the proof.} \]

6. EXAMPLES
In this section we present two illustrative examples from the
field of Neurophysiology. We examine the behaviour of the
muscle spindle under the influence of different stimuli as
they described in the introduction.

The muscle spindle is an element of the neuromuscular
system located within the skeletal muscles. It is innervated
from gamma and alpha motoneurons by sequences of action
potentials (also known as spike trains). The response of the
muscle spindle is transmitted to the spinal cord by the Ia
afferent axon. More details about the structure and the
function of the muscle spindle can be found in Boyd [1].

In our experiments the tenuissimus muscle in
anesthetized cats was used, and the responses of single
sensory axons in dorsal root filaments were recorded. Gamma and alpha motoneuron axons isolated in ventral root
filaments were stimulated by sequences of pulses at twice
threshold. The samples recorded from the gamma and alpha
motoneurons are approximately Poisson. A description of the characteristics of these samples is given in [11].

We proceed now to discuss the way of estimating the
CC. In the estimation of the cumulant density, a Parzen
convergence factor is used, with \( p_\gamma = 0.6 \). The approximate
95% pointwise confidence interval of the estimate \( \hat{s}_{ab}(u) \) is obtained by
\[
\hat{s}_{ab}(u) \pm 1.96 \sqrt{\text{Var} \left( \hat{s}_{ab}(u) \right)}.
\]

The results presented in Figures 1 and 2 are discussed below.

Figure 1 presents the estimate \( \hat{s}_{ab}(u) \) when the muscle
spindle is affected by a gamma motoneuron. The numbers of events for the gamma motoneuron and the response of the
muscle spindle recorded in the time interval \( T = 15872 \) ms
are \( N_1(T) = 1010 \) and \( N_2(T) = 538 \) respectively. The solid
line corresponds to the estimate of the modified CC, \( \hat{s}_{ab}(u) \),
while the dotted lines represent the 95% approximate
pointwise confidence limits. If the lower confidence limit is
greater than \( \sqrt{\hat{p}_b \hat{p}_b} \) (value of \( \hat{s}_{ab}(u) \) when \( \hat{p}_{ab}(u) = 0 \)),
the behaviour of the muscle spindle is excitatory, namely the
system is almost certain to fire. On the contrary, if the upper
confidence limit is lower than \( \sqrt{\hat{p}_b \hat{p}_b} \), the behaviour of the
muscle spindle is inhibitory and the probability of the system
to fire is small. In this case the effect of the gamma
motoneuron on the muscle spindle occurs mainly in the
interval between 10 and 40 ms. The upper confidence limit is
very small and the peak is substantially smaller. Afterwards the behaviour of the muscle spindle
becomes completely random.

Figure 2 presents the estimate \( \hat{s}_{ab}(u) \) when the muscle
spindle is affected by a gamma motoneuron and at the same
time a simultaneous effect by an alpha motoneuron is
present. The numbers of events for the gamma motoneuron and the response of the muscle spindle recorded in the time
interval \( T = 11360 \) ms are \( N_1(T) = 691 \) and \( N_2(T) = 358 \)
respectively. In this case the muscle spindle has similar
behaviour with the Figure 1, but it is clear that the presence
of the alpha motoneuron reduces the effect of the gamma
motoneuron. Therefore, it is almost certain to fire in the time
period 7-40 ms but the peak is substantially smaller.
Afterwards the behaviour of the system tends slowly towards
a completely random situation.
In the present paper we present a new approach for the computation of the modified CC for stationary point processes. This approach is based on the frequency domain techniques and provides an alternative method to find the asymptotic properties of the modified CC. Consequently, the construction of approximate pointwise confidence intervals is feasible which present a better way of indicating the degree of association between two point processes. The theoretical results are applied to the study of the muscle spindle when is affected by a gamma stimulation and there is: a) no other effect and b) a simultaneous effect by an alpha motoneuron. The presence of the alpha motoneuron reduces significantly the effect of the gamma stimulation in the region between 7-25 ms.

The issue of the association between stationary point processes discussed in this paper can be extended in order to measure higher order association between three stationary point processes. This consists the aim of future work.

APPENDIX

If a function \( g \) has finite total variation \( V \), on the interval \([a,b]\), then

\[
\int_a^b g(x) \, dx - \frac{b-a}{n} \sum_{k=1}^{n} g \left( \frac{a+k}{n} (b-a) \right) \leq \frac{(b-a)V}{n}.
\]

Proof


REFERENCES


Figure 2. \( \tilde{s}_{ab}(u) \) between the stimulus of the gamma motoneuron and the response of the muscle spindle, when an alpha motoneuron is present.

The statistical treatment of the data sets has been carried out with R (version 2.4.0), the well known environment for statistical computing [13]. The built in function for the Fourier transform is used for both the input and output processes and gives fast results, with the following R commands.

\[
daT \leftarrow \text{fft}(dNa-pa) \
dbT \leftarrow \text{fft}(dNb-pb)
\]

Then, the periodogram statistic, \( I_{ab}(\lambda) \), which is used as the base of the computation of CC, is calculated rather easily avoiding loops and using vectors, as it is advisable for the R language.

\[
I_{ab} \leftarrow \text{daT}^* \text{Conj}(\text{dbT})/(2\pi*\text{Time})
\]

The excellent graphical capabilities of R produce the Figures 1 and 2. A code has been written consisting of about forty lines in which the modified estimates of CC are computed and is available on [http://utopia.duth.gr/~rigas](http://utopia.duth.gr/~rigas).

7. CONCLUSIONS

In the present paper we present a new approach for the computation of the modified CC for stationary point processes. This approach is based on the frequency domain techniques and provides an alternative method to find the asymptotic properties of the modified CC. Consequently, the construction of approximate pointwise confidence intervals is feasible which present a better way of indicating the degree of association between two point processes. The theoretical results are applied to the study of the muscle spindle when is affected by a gamma stimulation and there is: a) no other effect and b) a simultaneous effect by an alpha motoneuron. The presence of the alpha motoneuron reduces significantly the effect of the gamma stimulation in the region between 7-25 ms.

The issue of the association between stationary point processes discussed in this paper can be extended in order to measure higher order association between three stationary point processes. This consists the aim of future work.