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DESIGN OF FIR DIGITAL DIFFERENTIATOR USING DISCRETE HARTLEY TRANSFORM AND BACKWARD DIFFERENCE

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ABSTRACT

In this paper, the design of digital differentiator is presented. First, the interpolation formula of discrete-time sequence is derived by using discrete Hartley transform (DHT). Then, the DHT-based interpolation formula is applied to design FIR digital differentiator by using backward difference formula. The filter coefficients are easily computed because closed-form design is obtained. Finally, design examples are demonstrated to show the proposed method has smaller design error than the conventional window method and maximally flat design using the same parameters.

1. INTRODUCTION

Digital differentiator is very useful tool to determine and estimate the time derivatives of given signals. For example, in radars and sonars, the velocity and acceleration can be computed from the position measurements using differentiators [1]. In image processing, the edge of image can be detected by using differential operation [2]. In biomedical engineering, it is often necessary to obtain the higher-order derivatives of biomedical data [3]. Until now, several methods have been developed to design digital differentiators such as eigenfilter method [4], window method [5] and maximally flat method [6]. The ideal frequency response of linear phase digital differentiator is given by

$$D(\omega) = j\omega e^{-jI\omega} \tag{1}$$

where I is a prescribed integer. The problem is how to design a digital filter such that its frequency response fits $D(\omega)$ as well as possible. On the other hand, the Hartley transform was presented by Hartley for analyzing transmission problem in 1942 [7]. In 1983, Bracewell introduced the discrete Harley transform (DHT) and derived its fast computation algorithm [8][9]. Given the discrete-time sequence $x(0)$, $x(1)$, ..., $x(N\text{-}1)$, the DHT pairs are defined by

$$\begin{aligned} X(k) &= \sum_{m=0}^{N-1} x(m)cas\left(\tfrac{2\pi km}{N}\right) \\ x(m) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)cas\left(\tfrac{2\pi km}{N}\right) \end{aligned} \tag{2}$$

where $cas(\cdot) = \cos(\cdot) + \sin(\cdot)$. So far, DHT has been successfully applied to image processing [10], data interpolation [11], transform-domain adaptive filtering [12], data compression [13] and fractional delay filter design [14]. Moreover, there have been several efficient ways proposed to implement DHT including optical implementation [15], microwave realization [16] and systolic array [17]. In this paper, we will use DHT-based interpolation method to design FIR digital differentiator. The details are described in next section.

2. DESIGN METHOD

In this section, the zero-padding in DHT domain is first applied to interpolate discrete-time signal $x(0)$, $x(1)$, …., $x(N\text{-}1)$. Then, this interpolation method and backward difference formula are used to design FIR digital differentiator.

2.1 DHT Interpolation Method

In the following, we only consider the case of even-length N. Also, we assume that M is an integer multiple of N, say M=NL, where L is the interpolation factor. Given the DHT $X(k)$ in Eq.(2), let us define the zero-padded DHT as

$$X_d(k) = \begin{cases} LX(k) & k \in [0, \frac{N}{2}-1] \\ \frac{L}{2}X(\frac{N}{2}) & k = \frac{N}{2} \\ 0 & k \in [\frac{N}{2}+1, M-\frac{N}{2}-1] \\ \frac{L}{2}X(\frac{N}{2}) & k = M-\frac{N}{2} \\ LX(k-M+N) & k \in [M-\frac{N}{2}+1, M-1] \end{cases} \tag{3}$$

for $M = LN$. The above DHT has zero values at high frequencies. Now, the interpolated sequence $x_d(n)$ is defined as the length-M inverse DHT of $X_d(k)$, that is,

$$x_d(n) = \frac{1}{M}\sum_{k=0}^{M-1} X_d(k)cas\left(\tfrac{2\pi kn}{M}\right) \tag{4}$$

Substituting Eq.(3) into Eq.(4), we get

$$\begin{aligned} x_d(n) = &\frac{1}{N}\left\{X(0) + \sum_{k=1}^{\frac{N}{2}-1} X(k)cas(\tfrac{2\pi kn}{M})\right\} + \frac{1}{N}X(\tfrac{N}{2})\cos(\tfrac{n\pi}{L}) \\ &+ \frac{1}{N}\sum_{k=1}^{\frac{N}{2}-1} X(N-k)cas(\tfrac{-2\pi kn}{M}) \end{aligned} \tag{5}$$

Using Eq.(2) and the following equality

$$\begin{aligned} &cas(\theta_1)cas(\theta_2) + cas(-\theta_1)cas(-\theta_2) \\ &= 2\cos(\theta_1 - \theta_2) \end{aligned} \tag{6}$$

then Eq.(5) can be rewritten as

$$\begin{aligned} x_d(n) &= \frac{1}{N}\sum_{m=0}^{N-1} x(m) + \frac{1}{N}\sum_{m=0}^{N-1} x(m)\sum_{k=1}^{\frac{N}{2}-1} cas(\tfrac{2\pi km}{N})cas(\tfrac{2\pi kn}{M}) \\ &+ \frac{1}{N}\cos(\tfrac{n\pi}{L})\sum_{m=0}^{N-1}(-1)^m x(m) \\ &+ \frac{1}{N}\sum_{m=0}^{N-1} x(m)\sum_{k=1}^{\frac{N}{2}-1} cas(\tfrac{-2\pi km}{N})cas(\tfrac{-2\pi kn}{M}) \\ &= \frac{1}{N}\sum_{m=0}^{N-1} x(m)\left\{\begin{array}{l} 1+(-1)^m\cos(\frac{n\pi}{L}) \\ +2\sum_{k=1}^{\frac{N}{2}-1}\cos\left(\frac{2\pi k(m-\frac{n}{L})}{N}\right)\end{array}\right\} \end{aligned} \tag{7}$$

Clearly, the interpolated value of $x_d(n)$ is just the weighted average of the data $x(m)$ (m=0,1,...,N-1). Moreover, this interpolator will satisfy the following property:

$$x_d(iL) = x(i) \tag{8}$$

that is, the interpolation becomes an identity at the time points of the original length-N signal. Because $x_d(n)$ is the interpolated sequence of $x(n)$ with factor L, we have the following relation:

$$x_d(iL+p) \approx x(i+\tfrac{p}{L}) \tag{9}$$

for $0 \le p \le L-1$ and $0 \le i \le N-1$. When p=0, Eq.(9) reduces to Eq.(8). Combining Eq.(7) and Eq.(9), we have

$$\begin{aligned} x(i+\tfrac{p}{L}) &\approx \frac{1}{N}\sum_{m=0}^{N-1} x(m)\left\{\begin{array}{l} 1+(-1)^m\cos(\frac{(iL+p)\pi}{L}) \\ +2\sum_{k=1}^{\frac{N}{2}-1}\cos\left(\frac{2\pi k(m-\frac{iL+p}{L})}{N}\right)\end{array}\right\} \\ &= \sum_{m=0}^{N-1} x(m)b(m,i+\tfrac{p}{L}) \end{aligned} \tag{10}$$

where interpolation basis is given by

$$b(m,i+\tfrac{p}{L}) = \frac{1}{N}\left\{\begin{array}{l} 1+(-1)^m\cos\left(\pi(i+\frac{p}{L})\right) \\ +2\sum_{k=1}^{\frac{N}{2}-1}\cos\left(\frac{2\pi k(m-i-\frac{p}{L})}{N}\right)\end{array}\right\} \tag{11}$$

Using the identities $\cos(\theta_1-\theta_2)=\cos(\theta_1)\cos(\theta_2)+\sin(\theta_1)\sin(\theta_2)$, $\cos(m\pi)=(-1)^m$ and $\sin(m\pi)=0$, we have

$$\begin{aligned} &(-1)^m\cos(\pi(i+\tfrac{p}{L})) \\ &= \cos(m\pi)\cos(\pi(i+\tfrac{p}{L})) \\ &= \cos\left(\frac{2\pi(i+\frac{p}{L}-m)\frac{N}{2}}{N}\right) \end{aligned} \tag{12}$$

Substituting Eq.(12) into Eq.(11), the basis can be rewritten as

$$b(m,i+\tfrac{p}{L}) = \frac{1}{N}\sum_{k=0}^{\frac{N}{2}}\beta_k\cos\left(\frac{2\pi(i+\frac{p}{L}-m)k}{N}\right) \tag{13}$$

where

$$\beta_k = \begin{cases} 1 & k=0,\frac{N}{2} \\ 2 & k=1,2,\cdots,\frac{N}{2}-1 \end{cases} \tag{14}$$

One remark is now made. Let $t = i+\frac{p}{L}$, then the value of t can be any real number in [0,N) if interpolation factor L approaches infinity. Substituting $t = i+\frac{p}{L}$ into Eq.(10), we get

$$x(t) \approx \sum_{m=0}^{N-1} x(m)b(m,t) \tag{15}$$

This means that the continuous-time signal $x(t)$ can be approximately reconstructed from its samples $x(0)$, $x(1)$, ..., $x(N-1)$ in the range [0,N) by using continuous-time interpolation basis $b(m,t)$.

2.2 Design of FIR Digital Differentiator

Given the signal $s(n)$, the backward difference formula to estimate its derivative $s'(n-I)$ is given by

$$s'(n-I) \approx \frac{s(n-I)-s(n-I-d)}{d} \tag{16}$$

The smaller d is, the better estimation $s'(n-I)$ is. Because d is a very smaller positive real number, the unknown signal $s(n-I-d)$ needs to be estimated from the sampled signals $s(n)$, $s(n-1)$, ..., $s(n-N+1)$. The zero-padding DHT interpolation formula in Eq.(10) can be used to achieve this estimation purpose by choosing

$$x(m) = s(n-(N-1)+m) \qquad 0 \le m \le N-1 \tag{17}$$

Substituting Eq.(17) into Eq.(10), we get

$$\begin{aligned} &s(n-(N-1)+i+\tfrac{p}{L}) \\ &\approx \sum_{m=0}^{N-1} s(n-(N-1)+m)b(m,i+\tfrac{p}{L}) \end{aligned} \tag{18}$$

Replacing $i+\frac{p}{L}$ by $N-1-I-d$, the above equation can be rewritten as

$$\begin{aligned} &s(n-I-d) \\ &\approx \sum_{m=0}^{N-1} s(n-(N-1)+m)b(m,N-1-I-d) \end{aligned} \tag{19}$$

Let $m = N-1-r$, the expression becomes

$$s(n-I-d) \approx \sum_{r=0}^{N-1} g(r,d)s(n-r) \tag{20}$$

where $g(r,d)$ is given by

$$\begin{aligned} g(r,d) &= b(N-1-r,N-1-I-d) \\ &= \frac{1}{N}\sum_{k=0}^{\frac{N}{2}}\beta_k\cos\left(\frac{2\pi(r-I-d)k}{N}\right) \end{aligned} \tag{21}$$

Substituting Eq.(20) into Eq.(16), we obtain

$$\begin{aligned} s'(n-I) &\approx \frac{s(n-I)-\sum_{r=0}^{N-1} g(r,d)s(n-r)}{d} \\ &\approx \sum_{r=0}^{N-1} h(r,d)s(n-r) \end{aligned} \tag{22}$$

where filter coefficients $h(r,d)$ are

$$h(r,d)=\begin{cases}\frac{-g(r,d)}{d} & r\neq I\\ \frac{1-g(I,d)}{d} & r=I\end{cases} \quad (23)$$

Taking the Fourier transform at both sides in Eq.(22), we get

$$(j\omega e^{-j\omega I})S(\omega)\approx\left(\sum_{r=0}^{N-1}h(r,d)e^{-j\omega r}\right)S(\omega) \quad (24)$$

where $S(\omega)$ is the Fourier transform of the signal $s(n)$. Cancelling $S(\omega)$ at both sides, we have

$$\begin{aligned}j\omega e^{-j\omega I} &\approx \sum_{r=0}^{N-1}h(r,d)e^{-j\omega r}\\ &= H(z)\Big|_{z=e^{j\omega}}\end{aligned} \quad (25)$$

where FIR filter $H(z)$ is given by

$$H(z)=\sum_{r=0}^{N-1}h(r,d)z^{-r} \quad (26)$$

The filter coefficients $h(r,d)$ are easily computed because Eq.(23) is a closed-form formula. Fig.1(a)(b) shows the magnitude response and group delay (solid lines) of the proposed DHT-based FIR digital differentiator for N=80, I=40, and d=0.1. The dashed lines are ideal responses. Obviously, the specification is fitted well.

3. DESIGN EXAMPLES AND DISCUSSION

In this section, numerical examples are used to demonstrate the effectiveness of the proposed DHT-based method. To evaluate performance, the normalized root mean squares (NRMS) error is used and defined by

$$E=\left(\frac{\int_0^{\alpha\pi}\left|H(e^{j\omega})-D(\omega)\right|^2d\omega}{\int_0^{\alpha\pi}\left|D(\omega)\right|^2d\omega}\right)^{\frac{1}{2}}\times 100\ \% \quad (27)$$

The error is computed on the frequency band $[0,\alpha\pi]$. Obviously, the smaller the NRMS error E is, the better the performance of design method is.

Example 1: In this example, we will study the relation between NRMS error E and parameter d. When the parameters $N=80$, $I=40$ and $\alpha=0.95$ are chosen, Fig.2 shows the NRMS error curve. It is clear that the error E increases linearly when parameter d increases. Thus, the optimal choice of d is zero, that is, the best filter coefficients are given by

$$h_o(r)=\lim_{d\to 0}h(r,d) \quad (28)$$

From Eq.(21) and (23), we have

$$\begin{aligned}h_o(I) &= \lim_{d\to 0}\frac{1-g(I,d)}{d}\\ &= \lim_{d\to 0}\frac{1-\frac{1}{N}\sum_{k=0}^{\frac{N}{2}}\beta_k\cos\left(\frac{2\pi dk}{N}\right)}{d}\end{aligned} \quad (29)$$

The calculation of the above limit is the case of zero by zero. Thus, using L'Hopital rule in [18], the $h_o(I)$ can be computed as

$$\begin{aligned}h_o(I) &= \lim_{d\to 0}\frac{\frac{1}{N}\sum_{k=0}^{\frac{N}{2}}\beta_k\left(\frac{2\pi k}{N}\right)\sin\left(\frac{2\pi dk}{N}\right)}{1}\\ &= 0\end{aligned} \quad (30)$$

When $r\neq I$, from Eq.(21) and (23), we have

$$\begin{aligned}h_o(r) &= \lim_{d\to 0}\frac{-g(r,d)}{d}\\ &= \lim_{d\to 0}\frac{-\frac{1}{N}\sum_{k=0}^{\frac{N}{2}}\beta_k\cos\left(\frac{2\pi(r-I)k}{N}-\frac{2\pi kd}{N}\right)}{d}\\ &= T_1+T_2\end{aligned} \quad (31)$$

where

$$\begin{aligned}T_1 &= \lim_{d\to 0}\frac{-\frac{1}{N}\sum_{k=0}^{\frac{N}{2}}\beta_k\cos\left(\frac{2\pi(r-I)k}{N}\right)\cos\left(\frac{2\pi kd}{N}\right)}{d}\\ T_2 &= \lim_{d\to 0}\frac{-\frac{1}{N}\sum_{k=0}^{\frac{N}{2}}\beta_k\sin\left(\frac{2\pi(r-I)k}{N}\right)\sin\left(\frac{2\pi kd}{N}\right)}{d}\end{aligned} \quad (32)$$

It can be shown that the following equality is valid:

$$\sum_{k=0}^{\frac{N}{2}}\beta_k\cos\left(\frac{2\pi(r-I)k}{N}\right)=0 \quad (33)$$

, so the calculation of the limit T_1 in Eq.(32) is the case of zero by zero. Thus, using L'Hopital rule, we have

$$\begin{aligned}T_1 &= \lim_{d\to 0}\frac{\frac{1}{N}\sum_{k=0}^{\frac{N}{2}}\beta_k\left(\frac{2\pi k}{N}\right)\cos\left(\frac{2\pi(r-I)k}{N}\right)\sin\left(\frac{2\pi kd}{N}\right)}{1}\\ &= 0\end{aligned} \quad (34)$$

Moreover, using L'Hopital rule, the limit T_2 in Eq.(32) is computed by

$$\begin{aligned}T_2 &= \lim_{d\to 0}\frac{-\frac{1}{N}\sum_{k=0}^{\frac{N}{2}}\beta_k\left(\frac{2\pi k}{N}\right)\sin\left(\frac{2\pi(r-I)k}{N}\right)\cos\left(\frac{2\pi kd}{N}\right)}{1}\\ &= -\frac{2\pi}{N^2}\sum_{k=0}^{\frac{N}{2}}k\beta_k\sin\left(\frac{2\pi(r-I)k}{N}\right)\end{aligned} \quad (35)$$

Substituting Eq.(34) and (35) into Eq.(31), it yields

$$h_o(r)=-\frac{2\pi}{N^2}\sum_{k=0}^{\frac{N}{2}}k\beta_k\sin\left(\frac{2\pi(r-I)k}{N}\right) \quad (36)$$

Combining the results in Eq.(30) and (36), the filter coefficients $h_o(r)$ are

$$h_o(r)=\begin{cases}0 & r=I\\ -\frac{2\pi}{N^2}\sum_{k=0}^{\frac{N}{2}}k\beta_k\sin\left(\frac{2\pi(r-I)k}{N}\right) & r\neq I\end{cases} \quad (37)$$

Once the optimal d has been chosen, the other problem is how to select optimal integer delay I for a given filter length N. When the parameters $N=80$, $\alpha=0.95$ and $h_o(r)$ are chosen, Fig.3 shows the NRMS error curve versus I. It is clear that the minimum error E occurs at

$I = \frac{N}{2}$. Thus, the optimal choice of I is $\frac{N}{2}$ after N is specified.

Example 2: In this example, we compare the DHT-based design with the conventional window method. Taking the inverse discrete-time Fourier transform of $D(\omega)$ in Eq.(1), the ideal impulse response is given by [19]

$$h_{id}(r) = \begin{cases} \dfrac{\cos[\pi(r-I)]}{r-I} & r \neq I \\ 0 & r = I \end{cases} \tag{38}$$

Then, the filter coefficients $h(r)$ are computed by $h(r) = w(r)h_{id}(r)$ for the conventional window method. In this example, the following Hamming window is used:

$$w(r) = 0.54 - 0.46\cos\left(\frac{2\pi r}{N-1}\right) \tag{39}$$

Fig.4(a)(b) show the magnitude response and error of the differentiator designed by conventional Hamming window method for N=80 and I=40. The NRMS error in this case is 0.196% for $\alpha = 0.95$. Fig.5(a)(b) show the magnitude response and error of the window DHT-based digital differentiator whose filter coefficients are computed by $h(r) = w(r)h_o(r)$. The parameters are also chosen as N=80 and I=40. The NRMS error in this case is 0.13% for $\alpha = 0.95$. Based on the above results, it is clear that the window DHT method has smaller design error than conventional window method.

Example 3: In this example, we compare the DHT-based design with the conventional maximally-flat method [6][20]. If $N = 2I$ is chosen, the filter coefficients are given by

$$h(r) = \begin{cases} 0 & r = I \\ \dfrac{(-1)^{r+I+1}(I!)^2}{(I-r)(N-r)!\,r!} & r \neq I \end{cases} \tag{40}$$

When the design parameters are chosen as N=80 and I=40, Fig.6(a)(b) show the magnitude response and error of the differentiator designed by maximally flat method. The NRMS error in this case is 13.83% for $\alpha = 0.95$. The dashed line in Fig.6(b) is the error curve of DHT method with N=80, I=40 and $h_o(r)$. From this result, it is clear that DHT method provides better result than maximally flat method in the high frequency range. However, maximally flat approach has smaller error in the low frequency region because it is a maximally flat design at $\omega = 0$.

4. CONCLUSIONS

In this paper, the discrete Hartley transform (DHT) interpolation method and backward difference have been used to design FIR digital differentiator. Design examples are demonstrated to show the proposed method has smaller design errors than the conventional window method and maximally flat design. In the future, the following three topics may be studied:

(1) Only DHT is studied here. Thus, it is interesting to use discrete cosine transform (DCT) or discrete Fourier transform (DFT) to design digital differentiator.
(2) Only backward difference is investigated here, so it is interesting to apply central difference or forward difference to design digital differentiator.
(3) Only first order differentiator is designed here, so the proposed DHT method may be extended to design high order differentiators.

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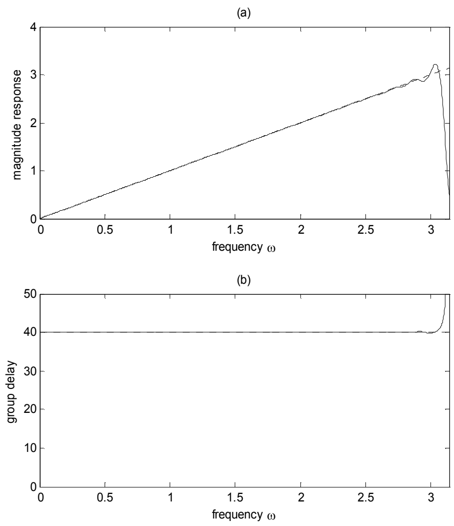

Fig.1 The magnitude response and group delay (solid lines) of the proposed DHT-based digital differentiator for N=80, I=40, and d=0.1. The dashed line is ideal response.

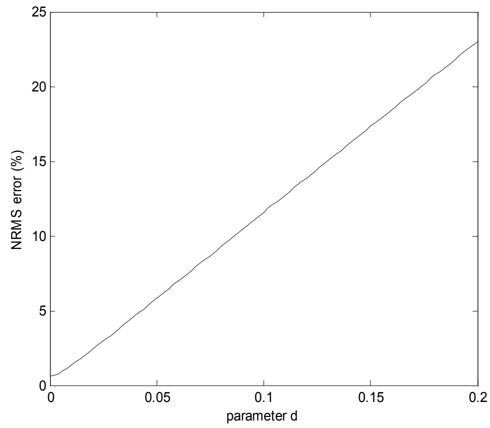

Fig.2 The NRMS error curve versus parameter d.

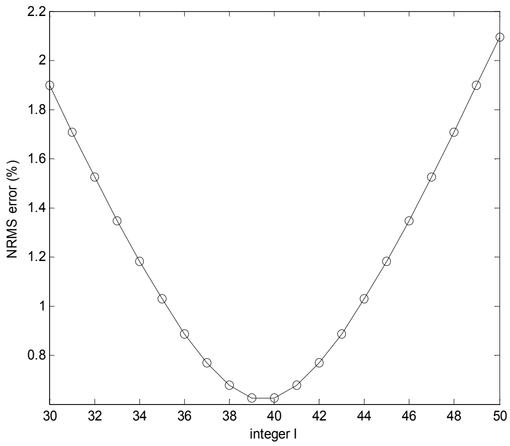

Fig.3 The NRMS error curve versus integer I.

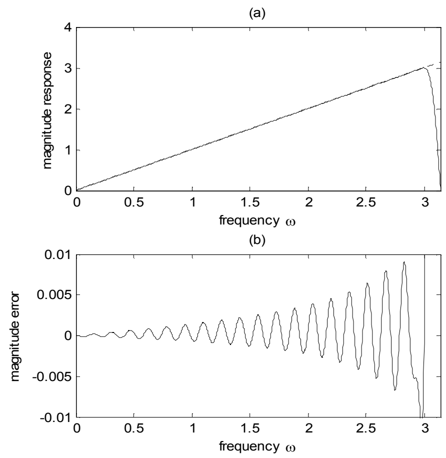

Fig.4 The design results of conventional window method. (a) Magnitude response. (b) Magnitude error.

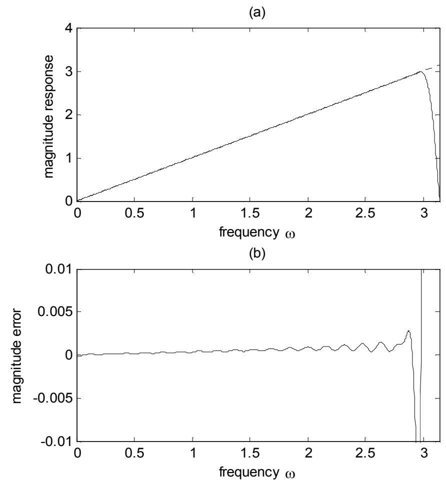

Fig.5 The design results of the window DHT-based method. (a) Magnitude response. (b) Magnitude error.

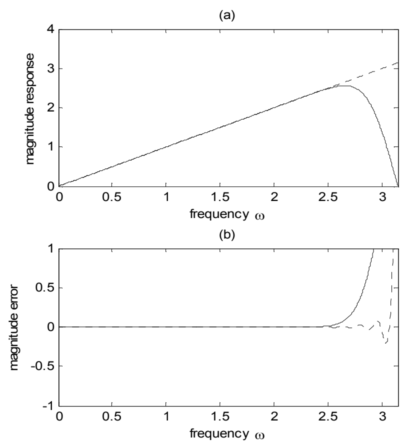

Fig.6 The design results of maximally flat method. (a) Magnitude response. The dashed line is ideal response. (b) Magnitude error. The dashed line is the error of DHT method.