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IMPROVED DESIGN OF IIR DIGITAL DIFFERENTIATOR USING RICHARDSON EXTRAPOLATION AND FRACTIONAL DELAY

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ABSTRACT

In this paper, the improved designs of IIR digital Al-Alaoui and Tustin differentiators are presented. First, the fractional delay is used to reduce the errors of differentiators in the high frequency region. Then, the Richardson extrapolation is utilized to generate high-accuracy results while using low-order formulas. Next, conventional Lagrange FIR fractional delay filter is directly applied to implement the designed IIR differentiator. Finally, several numerical examples are illustrated to demonstrate the effectiveness of this improved design approach.

1. INTRODUCTION

The digital differentiator is an important device in the areas of radars, sonars, image processing and biomedical engineering [1]-[3]. The ideal frequency response of digital differentiator is given by

$$D(\omega) = j\omega \qquad (1)$$

The problem is how to design a digital filter such that its frequency response fits $D(\omega)$ as well as possible. The design methods of digital differentiator generally can be classified into two categories. One is finite-impulse-response (FIR) filter approach in which the filter coefficients are determined by eigenfilter method [4], window method [5] and limit computation method [6], the other is infinite-impulse-response (IIR) filter method in which filter coefficients are obtained from the well-known numerical integration rules [7]-[9]. In the literature, two typical transfer functions of IIR digital differentiators are given by

Al-Alaoui: $$H_1(z) = \frac{8}{7}\frac{1 - z^{-1}}{1 + \frac{1}{7}z^{-1}} \qquad (2)$$

Tustin: $$H_2(z) = 2\frac{1 - z^{-1}}{1 + z^{-1}} \qquad (3)$$

So far, these two IIR digital differentiators have been successfully applied to design fractional order differentiators [10]-[12]. Fig.1(a)(b) show the magnitude responses (solid lines) of both differentiators, where dashed lines are ideal responses. It is clear that the design accuracies of these two differentiators are not good enough in high frequency region. To get more accurate result, the fractional delay and a simple algebra procedure called Richardson extrapolation are used to improve the accuracy. The Richardson extrapolation was proposed in 1927 and its historical background can be found in [13]. Until now, this extrapolation procedure has been successfully applied to compute accurate bifurcation values of periodic responses [14], to solve paraxial wave equation [15], and to improve the accuracy of one-sided finite-difference approximations [16] etc. Although there are fractional delay elements involved in the transfer function of the improved differentiator, the Lagrange FIR fractional delay filter in [17] can be directly applied to implement the designed differentiator. Several design examples are finally illustrated to demonstrate the effectiveness of this design approach.

2. AL-ALAOUI DIFFERENTIATOR

In this section, the fractional delay and Richardson extrapolation is first used to reduce the approximation error of IIR Al-Alaoui differentiator. Then, the implementation issue of the improved differentiator is discussed.

2.1 Improved Method

The Al-Alaoui differentiator $H_1(z)$ can be modified into the following form to reduce the approximation error in the high frequency range:

$$A_0(z,\alpha) = \frac{8}{7\alpha}\frac{1 - z^{-\alpha}}{1 + \frac{1}{7}z^{-\alpha}} \qquad (4)$$

where α is a real positive number in $[0,1]$. Because $A_0(z,1)$ is equal to $H_1(z)$, Eq.(4) is a generalization of the Al-Alaoui differentiator. Fig.2(a)(b) show the magnitude responses (solid lines) of $A_0(z,\alpha)$ for $\alpha = 0.5$ and $\alpha = 0.3$, where dashed lines are ideal responses. Compared Fig.1(a) and Fig.2, it is clear that the errors in high frequency region have been reduced. Replacing z by $e^{j\omega}$, the frequency response of this differentiator is computed as

$$A_0(z, e^{j\omega}) = \frac{8}{7\alpha}\frac{1 - e^{-j\omega\alpha}}{1 + \frac{1}{7}e^{-j\omega\alpha}} \qquad (5)$$

Using the power series expansion of exponential function:

$$\begin{aligned} e^{-j\omega\alpha} &= 1 + \sum_{k=1}^{\infty}\frac{(-j\omega\alpha)^k}{k!} \\ &= 1 + \sum_{k=1}^{\infty} g_k \alpha^k \end{aligned} \qquad (6)$$

where $g_k = \frac{(-j\omega)^k}{k!}$, then Eq.(5) can be rewritten as

$$A_0(z, e^{j\omega}) = \frac{-\sum_{k=1}^{\infty} g_k \alpha^{k-1}}{1 + \frac{1}{8}\sum_{k=1}^{\infty} g_k \alpha^k} \quad (7)$$

Using long-division, the Eq.(7) becomes the following polynomial form of α :

$$A_0(e^{j\omega}, \alpha) = -g_1 + (\tfrac{g_1^2}{8} - g_2)\alpha + (\tfrac{g_1 g_2}{4} - g_3 - \tfrac{g_1^3}{64})\alpha^2 + \cdots \quad (8)$$

Substituting $g_1 = -j\omega$ into the first term in Eq.(8), we have

$$A_0(e^{j\omega}, \alpha) = j\omega + \sum_{k=1}^{\infty} a_k \alpha^k = D(\omega) + O(\alpha) \quad (9)$$

where $a_1 = \frac{g_1^2}{8} - g_2$, $a_2 = \frac{g_1 g_2}{4} - g_3 - \frac{g_1^3}{64}$, …, and notation $O(\alpha)$ denotes the error term decays as fast as α . If the parameter α approaches to zero, we obtain the following result:

$$\lim_{\alpha \to 0} A_0(e^{j\omega}, \alpha) = D(\omega) \quad (10)$$

That is, $A_0(z,\alpha)$ approaches to ideal differentiator if parameter α tends to zero. However, this convergence speed is not good enough. In order to improve the speed, the Richardson extrapolation is used to generate high-accuracy results from low-order formulas. To achieve this purpose, two expressions from Eq.(9) are given by

$$A_0(e^{j\omega}, \alpha) = D(\omega) + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + \cdots \quad (11)$$

$$A_0(e^{j\omega}, 2\alpha) = D(\omega) + 2a_1\alpha + 4a_2\alpha^2 + 8a_3\alpha^3 + \cdots \quad (12)$$

If we multiply Eq.(11) by 2 and subtract Eq.(12) from this product, then the terms involving a_1 cancel and the result is

$$2A_0(e^{j\omega}, \alpha) - A_0(e^{j\omega}, 2\alpha) = D(\omega) + 0 - 2a_2\alpha^2 + \cdots \quad (13)$$

After some manipulation, the above equation can be rewritten as

$$\begin{aligned} A_1(e^{j\omega}, \alpha) &= 2A_0(e^{j\omega}, \alpha) - A_0(e^{j\omega}, 2\alpha) \\ &= D(\omega) - 2a_2\alpha^2 - 6a_3\alpha^3 + \cdots \\ &= D(\omega) + \sum_{k=2}^{\infty} b_k \alpha^k \\ &= D(\omega) + O(\alpha^2) \end{aligned} \quad (14)$$

where coefficients b_k are given by

$$b_k = (2 - 2^k)a_k \quad (15)$$

Clearly, the order of error term of $A_1(e^{j\omega}, \alpha)$ is $O(\alpha^2)$ which produces faster convergence speed than $A_0(e^{j\omega}, \alpha)$ when α approaches to zero. Replacing $e^{j\omega}$ by z and using Eq.(4), then Eq.(14) reduce to the following form

$$\begin{aligned} A_1(z, \alpha) &= 2A_0(z, \alpha) - A_0(z, 2\alpha) \\ &= 2\left(\frac{8}{7\alpha}\frac{1 - z^{-\alpha}}{1 + \frac{1}{7}z^{-\alpha}}\right) - \left(\frac{8}{14\alpha}\frac{1 - z^{-2\alpha}}{1 + \frac{1}{7}z^{-2\alpha}}\right) \\ &= \frac{4}{\alpha}\frac{21 - 29z^{-\alpha} + 11z^{-2\alpha} - 3z^{-3\alpha}}{49 + 7z^{-\alpha} + 7z^{-2\alpha} + z^{-3\alpha}} \end{aligned} \quad (16)$$

So far, we have already seen how to use parameters α and 2α to remove the error term involving α . To see how α^2 is removed, two expressions from Eq.(14) are given by

$$A_1(e^{j\omega}, \alpha) = D(\omega) + b_2\alpha^2 + b_3\alpha^3 + \cdots \quad (17)$$

$$A_1(e^{j\omega}, 2\alpha) = D(\omega) + 4b_2\alpha^2 + 8b_3\alpha^3 + \cdots \quad (18)$$

If we multiply Eq.(17) by 4 and subtract Eq.(18) from this product, then the terms involving b_2 cancel and the result is given by

$$4A_1(e^{j\omega}, \alpha) - A_1(e^{j\omega}, 2\alpha) = 3D(\omega) + 0 - 4b_3\alpha^3 + \cdots \quad (19)$$

After some algebra, the above equation can be rewritten as

$$\begin{aligned} A_2(e^{j\omega}, \alpha) &= \frac{4A_1(e^{j\omega}, \alpha) - A_1(e^{j\omega}, 2\alpha)}{3} \\ &= D(\omega) - \tfrac{4}{3}b_3\alpha^3 - 4b_4\alpha^4 + \cdots \\ &= D(\omega) + O(\alpha^3) \end{aligned} \quad (20)$$

Clearly, the order of error term of $A_2(e^{j\omega}, \alpha)$ is $O(\alpha^3)$. Replacing $e^{j\omega}$ by z and using Eq.(16), then Eq.(20) becomes the form

$$A_2(z, \alpha) = \frac{4A_1(z, \alpha) - A_1(z, 2\alpha)}{3} = \frac{2}{3\alpha}\frac{\begin{pmatrix} 7203 - 11515z^{-\alpha} + 6762z^{-2\alpha} - 2618z^{-3\alpha} \\ +1456z^{-4\alpha} - 1840z^{-5\alpha} + 854z^{-6\alpha} - 390z^{-7\alpha} \\ +109z^{-8\alpha} - 21z^{-9\alpha} \end{pmatrix}}{\begin{pmatrix} 2401 + 343z^{-\alpha} + 686z^{-2\alpha} + 98z^{-3\alpha} + 392z^{-4\alpha} \\ +56z^{-5\alpha} + 98z^{-6\alpha} + 14z^{-7\alpha} + 7z^{-8\alpha} + z^{-9\alpha} \end{pmatrix}} \quad (21)$$

Based on the above results, the general recursive formula for Richardson's improvement process is stated below:

$$A_k(z, \alpha) = \frac{2^k A_{k-1}(z, \alpha) - A_{k-1}(z, 2\alpha)}{2^k - 1} \quad (22)$$

Then, the frequency response has the form

$$A_k(e^{j\omega}, \alpha) = D(\omega) + O(\alpha^{k+1}) \quad (23)$$

To evaluate the performance of differentiator $A_k(z,\alpha)$, the error function is defined by

$$E(\alpha) = 10\log_{10}\left(\int_0^{\pi} \left|D(\omega) - A_k(e^{j\omega}, \alpha)\right|^2 d\omega\right) \quad (24)$$

Fig.3 shows the error curves $E(\alpha)$ for k=0,1,2. It is clear that the error can be reduced by increasing k or decreasing α . Moreover, Fig.4 depicts the frequency response error $20\log_{10}(|D(\omega) - A_k(e^{j\omega}, \alpha)|)$ for $\alpha = 0.1$ and k=0,1,2. Clearly, the error at entire frequency range is reduced when k increases. Although the implementation of digital differentiator $A_k(z,\alpha)$ involves fractional delay, this problem is easily solved by Lagrange FIR fractional delay filter. The details are described in next subsection.

2.2 Implementation Issue

From Eq.(4), (16) and (21), the transfer function $A_k(z,\alpha)$ can be rewritten as a unified form below:

$$A_k(z,\alpha) = \frac{\sum_{m=0}^{3^k} r_m^{(k)} z^{-m\alpha}}{\sum_{m=0}^{3^k} s_m^{(k)} z^{-m\alpha}} \quad (25)$$

To realize this differentiator, the numerator and denominator are both multiplied by integer delay z^{-I} to get

$$A_k(z,\alpha) = \frac{\sum_{m=0}^{3^k} r_m^{(k)} z^{-(I+m\alpha)}}{\sum_{m=0}^{3^k} s_m^{(k)} z^{-(I+m\alpha)}} \quad (26)$$

In [17], the Lagrange interpolation method has been used to design an FIR filter for approximating fractional delay $z^{-(I+p)}$. In this method, the transfer function of FIR fractional delay filter is given by

$$z^{-(I+p)} \approx \sum_{n=0}^{L} h_n(p) z^{-n} \quad (27)$$

where filter coefficients $h_n(p)$ have the explicit form as

$$h_n(p) = \prod_{l=0,l\neq n}^{L} \frac{I+p-l}{n-l} \quad (28)$$

Substituting Eq.(27) into the numerator and denominator in Eq.(26), the $A_k(z,\alpha)$ reduces to a transfer function containing only integer delays below:

$$A_k(z,\alpha) \approx \frac{\sum_{n=0}^{L}\sum_{m=0}^{3^k} r_m^{(k)} h_n(m\alpha) z^{-n}}{\sum_{n=0}^{L}\sum_{m=0}^{3^k} s_m^{(k)} h_n(m\alpha) z^{-n}} \quad (29)$$

Fig.5 shows the frequency response error curves $20\log_{10}(|D(\omega)-A_k(e^{j\omega},\alpha)|)$ of the differentiator in Eq.(29) for $L=50$, $I=30$ and $\alpha=0.1$. Clearly, the error curves look similar to those in Fig.4 except there are some distortions in the high frequency region.

3. TUSTIN DIFFERENTIATOR

In this section, the fractional delay and Richardson extrapolation will be used to reduce the error of the IIR Tustin differentiator. To achieve this purpose, the Tustin differentiator $H_2(z)$ in Eq.(3) is modified into the following form:

$$B_0(z,\alpha) = \frac{2}{\alpha}\frac{1-z^{-\alpha}}{1+z^{-\alpha}} \quad (30)$$

where α is a real positive number in $[0,1]$. Because $B_0(z,1)$ is equal to $H_2(z)$, Eq.(30) is a generalization of the Tustin differentiator. Fig.6(a)(b) show the magnitude responses (solid lines) of $B_0(z,\alpha)$ for $\alpha=0.2$ and $\alpha=0.1$, where dashed lines are ideal responses. Compared Fig.1(b) and Fig.6, it is clear that the errors in high frequency region have been reduced. Replacing z by $e^{j\omega}$, the frequency response of this differentiator is computed as

$$\begin{aligned} B_0(e^{j\omega},\alpha) &= \frac{2}{\alpha}\frac{1-e^{-j\omega\alpha}}{1+e^{-j\omega\alpha}} \\ &= \frac{2}{\alpha} j\tan\left(\frac{\omega\alpha}{2}\right) \end{aligned} \quad (31)$$

Using the power series expansion of tangent function:

$$\tan(\theta) = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)}{(2k)!} |\beta_{2k}| \theta^{2k-1} \quad (32)$$

where β_{2k} is the Bernoulli number [18], then the frequency response can be rewritten as

$$\begin{aligned} B_0(e^{j\omega},\alpha) &= j\left[\sum_{k=1}^{\infty}\left(\frac{2^{2k}(2^{2k}-1)|\beta_{2k}|}{(2k)!}\frac{\omega^{2k-1}}{2^{2k-2}}\right)\alpha^{2k-2}\right] \\ &= j\omega + \sum_{m=1}^{\infty} c_m \alpha^{2m} \\ &= D(\omega) + O(\alpha^2) \end{aligned} \quad (33)$$

where notation $O(\alpha^2)$ denotes the error term decays as fast as α^2. If parameter α approaches to zero, we have the following result

$$\lim_{\alpha\to 0} B_0(e^{j\omega},\alpha) = D(\omega) \quad (34)$$

That is, the filter $B_0(z,\alpha)$ approaches to ideal integrator if parameter α tends to zero. However, this convergence speed is not good enough. To order to improve the speed, the Richardson extrapolation is used to generate high-accuracy results from low-order formula. To achieve this purpose, two expressions from Eq.(33) are given by

$$B_0(e^{j\omega},\alpha) = D(\omega) + c_1\alpha^2 + c_2\alpha^4 + c_3\alpha^6 + \cdots \quad (35)$$

$$B_0(e^{j\omega},2\alpha) = D(\omega) + 4c_1\alpha^2 + 16c_2\alpha^4 + \cdots \quad (36)$$

If we multiply Eq.(35) by 4 and subtract Eq.(36) from this product, then the terms involving c_1 cancel and the result is

$$4B_0(e^{j\omega},\alpha) - B_0(e^{j\omega},2\alpha) = 3D(\omega) - 12c_2\alpha^4 + \cdots \quad (37)$$

After some manipulation, the above equation can be rewritten as

$$\begin{aligned} B_1(e^{j\omega},\alpha) &= \frac{4B_0(e^{j\omega},\alpha) - B_0(e^{j\omega},2\alpha)}{3} \\ &= D(\omega) + \sum_{k=2}^{\infty} d_k \alpha^{2k} \\ &= D(\omega) + O(\alpha^4) \end{aligned} \quad (38)$$

where coefficients $d_k = (4-2^{2k})c_k/3$. Obviously, the order of error term of $B_1(e^{j\omega},\alpha)$ is $O(\alpha^4)$ which produces faster convergence speed than $B_0(e^{j\omega},\alpha)$ when α approaches to zero. Replacing $e^{j\omega}$ by z and using Eq.(30), then Eq.(38) reduce to the following form

$$\begin{aligned} B_1(z,\alpha) &= \frac{4B_0(z,\alpha)-B_0(z,2\alpha)}{3} \\ &= \frac{1}{3\alpha}\frac{7-9z^{-\alpha}+9z^{-2\alpha}-7z^{-3\alpha}}{1+z^{-\alpha}+z^{-2\alpha}+z^{-3\alpha}} \end{aligned} \tag{39}$$

Now, let us see how α^4 is removed. Two expressions from Eq.(38) are given by

$$B_1(e^{j\omega},\alpha) = D(\omega)+d_2\alpha^4+d_3\alpha^6+\cdots \tag{40}$$

$$B_1(e^{j\omega},2\alpha) = D(\omega)+16d_2\alpha^4+64d_3\alpha^6+\cdots \tag{41}$$

If we multiply Eq.(40) by 16 and subtract Eq.(41) from this product, then the terms involving d_2 cancel and the result is

$$16B_1(e^{j\omega},\alpha)-B_1(e^{j\omega},2\alpha)=15D(\omega)-48d_3\alpha^6+\cdots \tag{42}$$

After some algebra, the above equation can be rewritten as

$$\begin{aligned} B_2(e^{j\omega},\alpha) &= \frac{16B_1(e^{j\omega},\alpha)-B_1(e^{j\omega},2\alpha)}{15} \\ &= D(\omega)-\tfrac{48}{15}d_3\alpha^6+\cdots \\ &= D(\omega)+O(\alpha^6) \end{aligned} \tag{43}$$

Clearly, the order of error term of $B_2(e^{j\omega},\alpha)$ is $O(\alpha^6)$. Based on the above results, the general recursive formula for Richardson's improvement process is stated below:

$$B_k(e^{j\omega},\alpha) = \frac{4^k B_{k-1}(e^{j\omega},\alpha)-B_{k-1}(e^{j\omega},2\alpha)}{4^k-1} \tag{44}$$

Then, the frequency response has the form

$$B_k(e^{j\omega},\alpha) = D(\omega)+O(\alpha^{2k+2}) \tag{45}$$

Moreover, from Eq.(30) and (39), the transfer function $B_k(z,\alpha)$ can be rewritten as a unified form below:

$$B_k(z,\alpha) = \frac{\sum_{m=0}^{3^k} u_m^{(k)} z^{-m\alpha}}{\sum_{m=0}^{3^k} v_m^{(k)} z^{-m\alpha}} \tag{46}$$

If the numerator and denominator are both multiplied by integer delay z^{-I} and substituting Eq.(27) into fractional delay parts, the $B_k(z,\alpha)$ reduces to a transfer function containing only integer delays so it can be realized.

4. CONCLUSIONS

In this paper, the improved designs of IIR digital Al-Alaoui and Tustin differentiators have been presented. First, the fractional delay is used to reduce the errors of differentiators in the high frequency region. Then, the Richardson extrapolation is utilized to generate high-accuracy results while using low-order formulas. Next, conventional Lagrange FIR fractional delay filter is directly applied to implement the designed IIR differentiator. Finally, several numerical examples are illustrated to demonstrate the effectiveness of this improved design approach.

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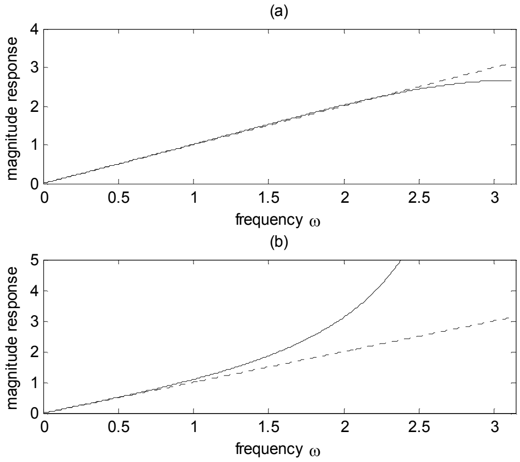

Fig.1 The magnitude responses (solid lines) of differentiators. (a) Al-Alaoui (b) Tustin. The dashed lines are ideal responses.

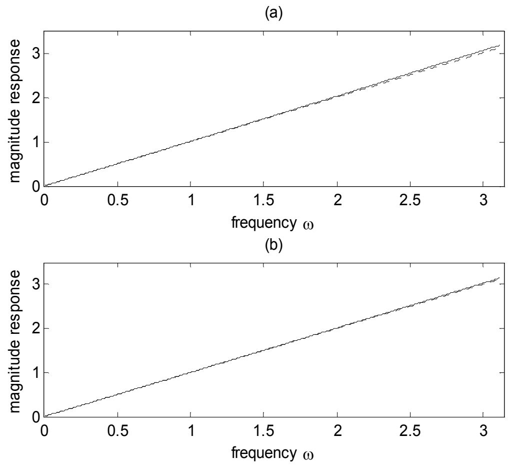

Fig.2 The magnitude responses (solid lines) of $A_0(z,\alpha)$. (a) $\alpha = 0.5$ (b) $\alpha = 0.3$. The dashed lines are ideal responses.

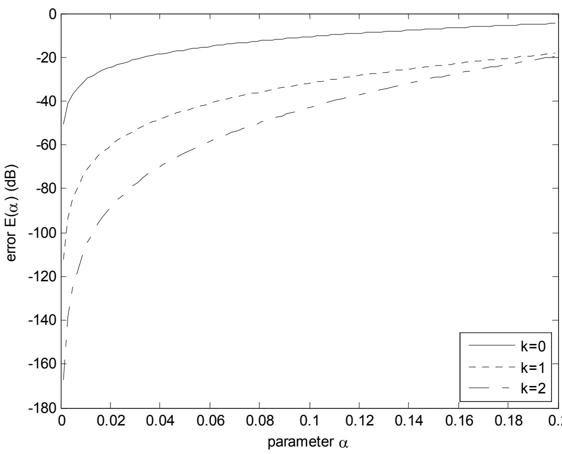

Fig.3 The error curves $E(\alpha)$ for k=0,1,2.

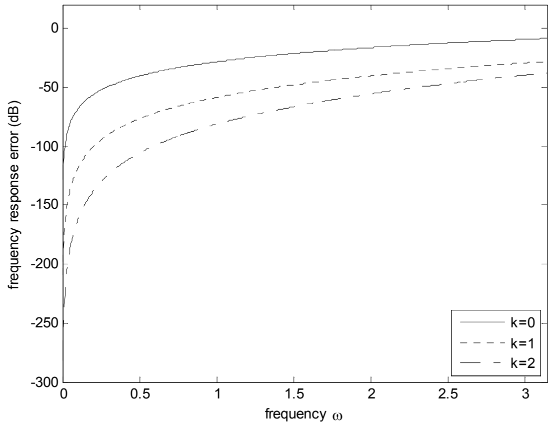

Fig.4 The frequency response error $20\log_{10}(|D(\omega)-A_k(e^{j\omega},\alpha)|)$ for $\alpha = 0.1$ and k=0,1,2.

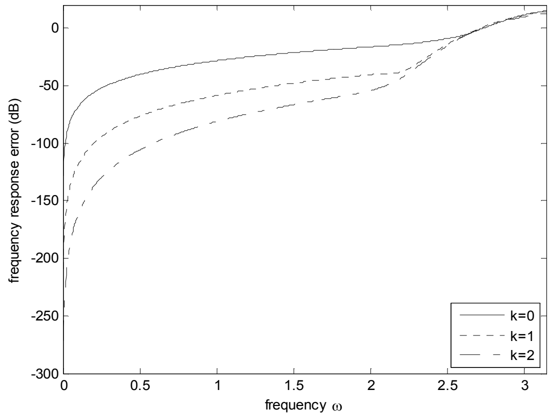

Fig.5 The frequency response error curves $20\log_{10}(|D(\omega)-A_k(e^{j\omega},\alpha)|)$ of the differentiator in Eq.(29) for $L = 50$, $I = 30$ and $\alpha = 0.1$.

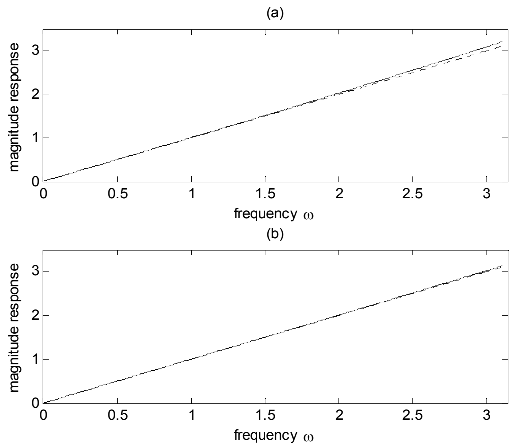

Fig.6 The magnitude responses (solid lines) of $B_0(z,\alpha)$. (a) $\alpha = 0.2$ (b) $\alpha = 0.1$. The dashed lines are ideal responses.