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ON THE PARAUNITARY AND PERFECT RECONSTRUCTION FILTER BANKS WITH ARBITRARY FILTER LENGTHS

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ABSTRACT

This paper investigates a more general class of M-channel FIR filter banks (FBs) with *arbitrary* equal filter lengths $L = KM + \beta$ $(0 \le \beta < M)$. The motivation for such FBs is to develop more general theories, factorizations and designs to overcome the limitations of traditional works with *unnecessary* length constraint $L = KM$, and to achieve a better trade-off between the design and computational complexities and the filter length. The existence conditions and lattice factorizations are developed for both paraunitary and perfect reconstruction FBs with any number of channels $M \ge 2$. Furthermore, it can be shown that the novel derived lattice factorization is *complete* for any FIR paraunitary FB and for any order-one perfect reconstruction FB. Finally, a design example is presented to validate the proposed lattice structure.

1. INTRODUCTION

The multichannel critically sampled filter bank (FB), a powerful tool in time-frequency signal analysis, has been extensively studied and employed in various signal processing applications [1, 2]. Fig. 1(a) illustrates a typical M-channel FB in the regular form, where $H_k(z)$ and $F_k(z)$ $(0 \le k \le M-1)$ are analysis and synthesis filters, respectively, and the corresponding polyphase form is shown in Fig. 1(b), where $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are the analysis and synthesis polyphase matrices, respectively [1]. After the analysis bank, the low rate subband signals can provide a more efficient and compact representation than the input signal itself, which is often used for signal compression and denoising, etc. In addition, M-band wavelets can be generated by iterating M-channel FBs with regularity. In the past two decades, there has been persistent interest in the study of theory, factorizations and designs for FIR FBs due to their wide applications in signal processing. The fundamental theory and factorization results for M-channel FIR FBs were reported in [3] for paraunitary (PU) FBs and in [4, 5] for more general perfect reconstruction (PR) FBs. In addition, they were studied in [6] under the framework of extended lapped transforms. The relationship between two-channel FBs and wavelets was further investigated in [7]. For multichannel PUFBs, several more efficient lattice structures were reported in [8], [9] and [10]. For general PRFBs, an important subclass of PRFBs, i.e., causal FIR FBs with anticausal FIR inverses (CAFACAFI), was introduced in [5] for the characterization of FIR PRFBs. Recently, [11] and [12] studied such class of PRFBs and PUFBs with regularity in the dyadic-based form, respectively. In addition, image coding based on general nonlinear phase FIR FBs was studied in [11] and [13].

In this paper, we study a more general class of FBs with arbitrary equal filter lengths $L = KM + \beta$ $(0 \le \beta < M, K \in \mathbb{N})$, in contrast to most previous works with the unnecessary length constraint of $L = KM$. The main motivation for this work is to further develop more general theories and designs for FIR FBs to complement previous works. Our work can also yield more flexible choices on designing FBs for some desired applications like low bit rate image coding, whereas traditional works [3]-[13] with unnecessary length constraint of $L = KM$, i.e., $\beta = 0$, greatly limit the possible designs. For example, if the length is constrained to be maximum $2M$ (lapped transforms) due to the constraint of computational complexity, there are only two possible choices in conventional designs in contrast to $M+1$ possible choices by our method. The restriction becomes more severe for larger M.

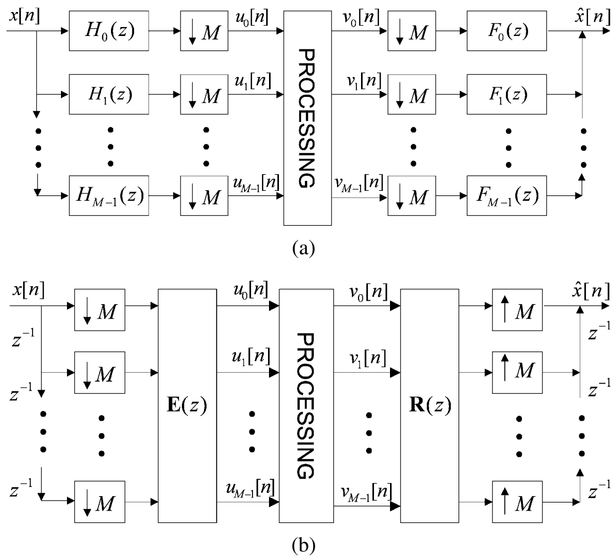


Figure 1: M-channel critically sampled uniform filter bank: (a) Direct form (b) Polyphase form

Previous works for the case of $\beta > 0$ have only been reported in some restricted forms of FBs. In [14], the cosine-modulated FBs were studied. Another one is the class of linear phase PUFBs with even channels [15]. For more efficient image and video coding with backward compatible with the existing standards, a class of linear phase PRFBs has also been studied under the framework of lapped transforms via pre/post-filtering structure [16, 17]. However, different from before, we systematically investigate general arbitrary equal-length PU and PR FBs without any phase constraint in this paper. First, some existence conditions on filter length L for such FBs are given, followed by a brief review of the general lattice structure. Next, arbitrary equal-length PUFBs are studied in Sec. 3. The PRFBs are further investigated in Sec. 4. In addition, regular FBs are to be studied in Sec. 5. Finally, one design example is shown in Sec. 6 to validate the proposed design method.

Notations: Bold-faced quantities denote matrices and vectors. $\mathbf{I}_M$ denotes the identity matrix with size M, and $\mathbf{0}$ is a null matrix of appropriate size. $\mathbf{U}^H$, $\mathbf{U}^T$ and $\mathbf{U}^{-1}$ denote the conjugated transpose, the transpose and the inverse of matrix $\mathbf{U}$, respectively. The rank of a matrix $\mathbf{U}$ is denoted by $\rho(\mathbf{U})$. The diag() denotes a diagonal (or block diagonal) matrix with specified diagonal elements (or blocks) in bracket. The polyphase matrix of FIR FBs can be written as $\mathbf{E}(z) = \sum_{i=0}^{K-1} \mathbf{E}_i z^{-i}$, where $\mathbf{E}_{K-1} \neq \mathbf{0}$, and the maximum possible filter length $L = KM$. The tilde operation on a polyphase matrix is defined by $\tilde{\mathbf{E}}(z) = \mathbf{E}^H(1/z^*)$, where $*$ denotes complex conjugation for scalars.

2. EXISTENCE CONDITIONS AND GENERAL LATTICE STRUCTURE

A FB has PR property if $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}_M$. The FB is called PU if $\tilde{\mathbf{E}}(z)\mathbf{E}(z) = \mathbf{I}_M$. Due to the PR or PU constraints, there is a *necessary* condition on the filter length L for such FBs. It can be shown that filter length L cannot be $KM+1$ for both PR and PU (nontrivial) FBs. The proof is based on the rank constraint for the highest-order block $\mathbf{E}_K$ of $\mathbf{E}(z)$ from the PR or PU condition, which is omitted here due to space limitation. Such existence condition is very useful for FB designs. It can narrow down the solution space for FB designer, which is helpful for deriving efficient lattice structures and avoiding impossible design specifications. It also helps to explain why only some solutions exist. For example, there does not exist any two-channel PUFB with odd length [7], which is just a special case $M = 2$ of our result.

Then, we present the general lattice structure for FIR FBs with length L, which is expressed by a factorization of polyphase matrix $\mathbf{E}(z)$ in the following product form,

$$\mathbf{E}(z) = \mathbf{G}_{K-1}(z) \cdots \mathbf{G}_1(z)\mathbf{E}_0(z) \tag{1}$$

where $\mathbf{E}_0(z)$ is a FB with prescribed properties like PU/PR and length $L_0 = N_0 M + \beta$, and each block $\mathbf{G}_i(z)$ with order N_1 can *simultaneously* increase filter length by $N_1 M$ and propagate those desired properties held in $\mathbf{E}_0(z)$. In this way, we can easily construct FBs with desired properties and prescribed length $L = [(K-1)N_1 + N_0]M + \beta$. The lattice (1) has a similar modular structure of linear-predictive lattice filters widely used in speech processing, which means we can easily obtain the given filter length just by varying the number of blocks $\mathbf{G}_i(z)$. In addition, many desired properties like PU/PR and filter length L can be *structurally enforced* into the lattice structure (1), which means the desired properties are always held independent of which specific values are chosen for the lattice parameters. Thus, the FBs can be designed with *exact* desired properties and fast convergence via *unconstrained* optimization routine, in contrast to some other methods only capable of approximation.

Next, we present some existing structures for blocks $\mathbf{G}_i(z)$ widely used before [5]. For general (nontrivial) PR FBs, $\mathbf{G}_i(z) = \mathbf{I}_M - \mathbf{U}_i\mathbf{V}_i^H + z^{-1}\mathbf{U}_i\mathbf{V}_i^H$ with $\mathbf{V}_i^H\mathbf{U}_i$ to be an upper-triangular matrix, where $\mathbf{U}_i$ and $\mathbf{V}_i$ are $M \times r_i$ matrices ($1 \leq r_i < M$). For the equal filter length L between analysis and synthesis filters, we need to further impose an additional condition $\mathbf{V}_i^H\mathbf{U}_i = \mathbf{I}_{r_i}$ [5, 9]. For the PUFBs, we just need to further constrain $\mathbf{V}_i = \mathbf{U}_i$. Note that the above dyadic-form based block $\mathbf{G}_i(z)$ can also be rewritten as $\mathbf{W}_i(z) = \mathbf{W}_i\mathbf{\Lambda}_i(z)$, where $\mathbf{W}_i$ is an $M \times M$ invertible matrix for PRFBs or a unitary matrix for PUFBs, and $\mathbf{\Lambda}_i(z)$ is a delay matrix in the form of $\mathrm{diag}(\mathbf{I}_{M-r_i}, z^{-1}\mathbf{I}_{r_i})$. Furthermore, such $\mathbf{W}_i(z)$ can be further simplified [9, 10, 13] by successively removing redundant parameters in the product of $\mathbf{W}_i(z)$ with the remaining lattice structure.

However, the *difference* between our work on the lattice factorization (1) and conventional ones lies in the starting block $\mathbf{E}_0(z)$. Contrary to [3]-[13], for the general case of $\beta \neq 0$, $\mathbf{E}_0(z)$ cannot be made order zero, i.e., a constant matrix, which is treated as a trivial case because it would impose multiple of $(M-\beta)$ zero filter coefficients at fixed positions. Thus, $\mathbf{E}_0(z)$ has at least order one, i.e., $N_0 \geq 1$.

3. PARAUNITARY FILTER BANKS

Consider an M-channel PUFB with filter length $L = KM + \beta$. Since the general PU propagating block $\mathbf{G}_i(z)$ in Sec. 2 has order one, i.e., $N_1 = 1$, the minimal order of initial block $\mathbf{E}_0(z)$ can be made one, i.e., $N_0 = 1$ and length $M + \beta$. Thus, $\mathbf{E}_0(z)$ can be written in the form $\mathbf{E}_0(z) = [\mathbf{E}_{00} + z^{-1}\mathbf{E}_{01}, \mathbf{E}_{02}]$, where $\mathbf{E}_{00}$ and $\mathbf{E}_{01}$ are both $M \times \beta$ matrices, and $\mathbf{E}_{02}$ is an $M \times (M-\beta)$ matrix. Then, with the PU condition $\tilde{\mathbf{E}}_0(z)\mathbf{E}_0(z) = \mathbf{I}_M$, the following matrix equations can be established.

$$\mathbf{E}_{00}^H\mathbf{E}_{00} + \mathbf{E}_{01}^H\mathbf{E}_{01} = \mathbf{I}_\beta \tag{2}$$

$$\mathbf{E}_{02}^H\mathbf{E}_{02} = \mathbf{I}_{M-\beta} \tag{3}$$

$$\mathbf{E}_{02}^H\mathbf{E}_{00} = \mathbf{0}_{(M-\beta)\times\beta} = \mathbf{E}_{02}^H\mathbf{E}_{01}, \qquad \mathbf{E}_{00}^H\mathbf{E}_{01} = \mathbf{0}_\beta \tag{4}$$

With these equations, a set of rank conditions on the matrices $\mathbf{E}_{00}$, $\mathbf{E}_{01}$ and $\mathbf{E}_{02}$ can be derived, which is the key to our lattice factorization.

Lemma 1. *For the class of PUFBs and its starting block $\mathbf{E}_0(z)$ stated above, the matrix $\mathbf{E}_{02}$ has full column rank $M-\beta$ i.e., $\rho(\mathbf{E}_{02}) = M-\beta$, and $\rho(\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}) = \beta$.*

Proof. From (3), it can be seen that matrix $\mathbf{E}_{02}$ has full rank, thus $\rho(\mathbf{E}_{02}) = M-\beta$. According to (2), we can obtain that $\beta = \rho(\mathbf{E}_{00}^H\mathbf{E}_{00} + \mathbf{E}_{01}^H\mathbf{E}_{01}) \leq \rho(\mathbf{E}_{00}^H\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}^H\mathbf{E}_{01}) = \rho(\mathbf{E}_{00}) + \rho(\mathbf{E}_{01})$, thus we know $\rho(\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}) \geq \beta$. Define matrix $\mathbf{F} = [\mathbf{E}_{00}, \mathbf{E}_{01}, \mathbf{E}_{02}]$, then from (4), we can see easily that $\mathbf{F}^H\mathbf{F} = \mathrm{diag}(\mathbf{E}_{00}^H\mathbf{E}_{00}, \mathbf{E}_{01}^H\mathbf{E}_{01}, \mathbf{I}_{M-\beta})$. Due to such diagonal structure, it can be seen $\rho(\mathbf{F}^H\mathbf{F}) = \rho(\mathbf{E}_{00}^H\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}^H\mathbf{E}_{01}) + \rho(\mathbf{I}_{M-\beta}) = \rho(\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}) + M - \beta = \rho(\mathbf{F}) \leq \min\{M, M+\beta\} = M$, which implies $\rho(\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}) \leq \beta$. Finally, we obtain $\rho(\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}) = \beta$ by combining the above two inequalities. □

These rank constraints derived from the PU conditions establish the *necessary* constraints on those matrices $\mathbf{E}_{0i}$. Without loss of generality, we can assume the rank of matrix $\mathbf{E}_{01}$ to be α with $0 \leq \alpha \leq \beta$ and thus $\rho(\mathbf{E}_{00}) = \beta - \alpha$ due to Theorem 1 (Note that the case of $\alpha = 0$ or $\alpha = \beta$ would lead to the trivial case of the starting block $\mathbf{E}_0(z)$ with β zero filter coefficients at either left boundary or right boundary, which would result in the PUFB $\mathbf{E}(z)$ with β zero coefficients at the boundary. Thus, we only consider $1 \leq \alpha \leq \beta - 1$ in the following). For the lattice factorization of $\mathbf{E}_0(z)$, we propose parameterized forms for $\mathbf{E}_{00}$ and $\mathbf{E}_{01}$ according to the above derived necessary constraints as follows,

$$\mathbf{E}_{00} = \mathbf{U}_{00}\mathbf{A}_0, \;\; \mathbf{E}_{01} = \mathbf{U}_{01}\mathbf{A}_1, \;\; \mathbf{E}_{02} = \mathbf{U}_{02} \qquad (5)$$

where matrices $\mathbf{U}_{00}$, $\mathbf{U}_{01}$ and $\mathbf{U}_{02}$ have sizes of $M \times (\beta - \alpha)$, $M \times \alpha$ and $M \times (M - \beta)$, respectively, and matrices $\mathbf{A}_0$ and $\mathbf{A}_1$ have sizes of $(\beta - \alpha) \times \beta$ and $\alpha \times \beta$, respectively. According to our assumption on the rank of $\mathbf{E}_{01}$ and the constraints stated in Lemma 1, the matrices $\mathbf{U}_{0i}$ should be of full column rank and matrices $\mathbf{A}_i$ should be of full row rank. This can be seen from the reduced singular value decomposition (SVD) [22] in which the columns of $\mathbf{U}_{0i}$ are the left singular vectors and the rows of $\mathbf{A}_i$ are the right singular vectors scaled by the singular values.

By the parameterized form (5) for matrices $\mathbf{E}_{0i}$, we can establish the lattice factorization of $\mathbf{E}_0(z)$ as follows,

$$\begin{aligned}
\mathbf{E}_0(z) &= [\mathbf{U}_{00}\mathbf{A}_0 + z^{-1}\mathbf{U}_{01}\mathbf{A}_1, \mathbf{U}_{02}] \\
&= [\mathbf{U}_{00}, z^{-1}\mathbf{U}_{01}, \mathbf{U}_{02}] \begin{bmatrix} \mathbf{A}_0 & \mathbf{0} \\ \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{M-\beta} \end{bmatrix} \\
&= [\mathbf{U}_{00}, \mathbf{U}_{01}, \mathbf{U}_{02}] \, \mathrm{diag}(\mathbf{I}_{\beta-\alpha}, z^{-1}\mathbf{I}_{\alpha}, \mathbf{I}_{M-\beta})\mathbf{\Gamma} \\
&= \mathbf{E}_0\mathbf{\Lambda}_0(z)\mathbf{\Gamma} \qquad (6)
\end{aligned}$$

where $\mathbf{E}_0 = [\mathbf{U}_{00}, \mathbf{U}_{01}, \mathbf{U}_{02}]$ is a square matrix of size M and $\mathbf{\Gamma} = \mathrm{diag}(\mathbf{T}_\beta, \mathbf{I}_{M-\beta})$ with square matrix $\mathbf{T}_\beta = [\mathbf{A}_0^T, \mathbf{A}_1^T]^T$ of size β. It can be shown easily that such factorization can guarantee the PU property of FBs as long as matrices $\mathbf{E}_0$ and $\mathbf{T}_\beta$ are unitary. Thus, the submatrices $\mathbf{U}_{0i}$ of square matrix $\mathbf{E}_0$ must be of full column rank as in our parameterized form (5). Similar argument is also applicable to matrices $\mathbf{A}_i$.

Combining (1) and (6), we can obtain the overall forward factorization for $\mathbf{E}(z)$ of PUFBs as follows.

$$\mathbf{E}(z) = \mathbf{G}_{K-1}(z) \cdots \mathbf{G}_1(z)\mathbf{E}_0\mathbf{\Lambda}_0(z)\mathbf{\Gamma} \qquad (7)$$

It can be seen easily that the traditional factorization of PUFBs with length constraint $L = KM$, i.e., $\beta = 0$, is only a special case of our proposed lattice factorization. Since the matrices $\mathbf{\Lambda}_0(z)$ and $\mathbf{\Gamma}$ will disappear when β is zero, which leads the factorization (7) to degenerate into the same as before [1, 8, 9, 10]. In addition, a special case of $\beta = 1$ deserves more attention. In this case, the matrix $\mathbf{T}_\beta$ becomes a scalar and we can only have $0 \le \alpha \le 1$. However, either $\alpha = 0$ or $\alpha = 1$ would lead to the zero filter coefficient at the boundary of $\mathbf{E}_0(z)$. Moreover, these zeros will be held at the boundary at every stage of lattice structure (1). Thus, any PUFB with length $KM + 1$ would always have zero filter coefficient at the boundary, which is actually a PUFB with shorter length KM (with possible shifting) since the proposed lattice structure is complete. This also explains the statement on the existence conditions for filter lengths in Sec. 2.

It is clear that the proposed lattice structure of (7) can generate M-channel PUFBs with arbitrary lengths $L = KM + \beta$. Furthermore, a stronger result is that the converse, i.e., *completeness* of the proposed lattice factorization, is also true, which is stated in the following theorem.

Theorem 1. *Any M-channel FIR PUFB can be factorized into the lattice (7) with appropriate choice of K and* β.

Due to the space limitation, the detailed proof is omitted here, which relies on a modified order reduction, i.e., *length reduction*, and the rank constraints shown in Lemma 1. Theorem 1 reveals the theoretical significance of the proposed lattice factorization (7), i.e., it can completely cover the whole space of FIR PUFBs. To our knowledge, this is the first complete lattice factorization for the FIR PUFB with *any* length, reported in the literature.

Degrees of freedom: Any FIR PUFB is characterized by the lattice structure (7). For the case of the filter length $L = KM + \beta$, there are $K - 1$ PU propagating blocks $\mathbf{G}_i(z)$ and two free unitary matrices $\mathbf{E}_0$ and $\mathbf{T}_\beta$. Recall that the simplified order-one building blocks derived in [10] and [13] are equivalent to $\mathbf{G}_i(z)$ in Sec. 2. Each simplified block has $2r_i(M - r_i)$ free parameters for the complex coefficient case and $r_i(M - r_i)$ for the real coefficient case. The two general square unitary matrices $\mathbf{E}_0$ and $\mathbf{T}_\beta$ need $M^2 + \beta^2$ and $M(M-1)/2 + \beta(\beta - 1)/2$ parameters for the complex and real coefficient cases, respectively. Thus, the degrees of freedom are given by $D_L = M^2 + \beta^2 + \sum_{i=1}^{K-1} 2r_i(M - r_i)$ for the complex case and $D_L = M(M-1)/2 + \beta(\beta-1)/2 + \sum_{i=1}^{K-1} r_i(M - r_i)$ for the case of real-valued coefficients. Note that there still exist redundant parameters in the above characterization, which will be addressed in future.

4. PERFECT RECONSTRUCTION BANKS

In this section, we further extend the above lattice factorization to general M-channel PRFBs with arbitrary equal filter lengths $L = KM + \beta$. Since the general PR propagating block $\mathbf{G}_i(z)$ has order one as seen from Sec. 2, the initial block $\mathbf{E}_0(z)$ can still be made order-one and length $M + \beta$. Thus, $\mathbf{E}_0(z)$ can also be written in the same structural form as in the PUFB, i.e., $\mathbf{E}_0(z) = [\mathbf{E}_{00} + z^{-1}\mathbf{E}_{01}, \mathbf{E}_{02}]$, where matrices $\mathbf{E}_{0i}$ have the same sizes as those in Sec. 3. Due to the class of PRFBs with CAFACAFI, we know that the initial block $\mathbf{R}_0(z)$ of the synthesis bank can also be made anticausal with order-one and length $M + \beta$ since the PR propagating block $\mathbf{G}_i^{-1}(z)$ for the synthesis bank is order-one anticausal. Thus, we can write $\mathbf{R}_0(z)$ in the following form $\mathbf{R}_0(z) = [\mathbf{R}_{00}^T + z\mathbf{R}_{01}^T, \mathbf{R}_{02}^T]^T$, where $\mathbf{R}_{00}$ and $\mathbf{R}_{01}$ are both $\beta \times M$ matrices, and $\mathbf{R}_{01}$ is an $(M - \beta) \times M$ matrix.

Similar to the case of PUFBs, we can employ the PR conditions to obtain a set of matrix equations (omitted due to space limitation) which consequently lead to the necessary constraints on the ranks of matrices $\mathbf{E}_{0i}$ and $\mathbf{R}_{0i}$, stated formally in the following Lemma.

Lemma 2. *For the class of PRFBs and the starting blocks* $\mathbf{E}_0(z)$ *and* $\mathbf{R}_0(z)$ *stated above, the matrix* $\mathbf{E}_{02}$ *has full column rank* $M - \beta$ *i.e.,* $\rho(\mathbf{E}_{02}) = M - \beta$*, and* $\rho(\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}) = \beta$*. Moreover,* $\rho(\mathbf{R}_{0i}) = \rho(\mathbf{E}_{0i})$ *for* $0 \le i \le 2$.

The detailed proof is omitted here due to the space limitation. Although the statement looks similar to the case of PUFBs, its proof cannot be extended straightforwardly from the proof of Lemma 1 since we no longer have an important rank equality $\rho(\mathbf{A}^H\mathbf{A}) = \rho(\mathbf{A})$ [22] for any matrix $\mathbf{A}$. Moreover, the proof of Lemma 2 brought a *novel perspective* of the PR conditions of critically sampled PRFBs. Different from the case of PUFBs, the PR condition $\mathbf{R}_0(z)\mathbf{E}_0(z) = \mathbf{I}_M$ alone is not enough to characterize those rank constraints on $\mathbf{E}_{0i}$ and $\mathbf{R}_{0i}$. Another PR condition $\mathbf{E}_0(z)\mathbf{R}_0(z) = \mathbf{I}_M$ which is usually overlooked, is actually needed in the proof of Lemma 2 in order to show that the matrices $\mathbf{R}_{0i}$ are the *reflexive generalized inverses* [18] of matrices $\mathbf{E}_{0i}$ for $i = 0, 1$

and consequently establishes the rank conditions stated in Lemma 2.

Without loss of generality, we can assume the rank of the matrix $\mathbf{E}_{01}$ to be α with $0 \le \alpha \le \beta$ and thus $\rho(\mathbf{E}_{00}) = \beta - \alpha$. Similarly, we know that $\rho(\mathbf{R}_{01}) = \alpha$ and $\rho(\mathbf{R}_{00}) = \beta - \alpha$ due to Lemma 2. The parameterized forms of $\mathbf{E}_{0i}$ shown in (5) and the corresponding lattice factorization of $\mathbf{E}_0(z)$ shown in (6) are still applicable to PRFBs here, since (5) and (6) only exploit the rank structures of matrices $\mathbf{E}_{0i}$ and the PRFBs have the same rank structures as those of PUFBs, as shown in Lemma 1 and 2. Therefore, the overall lattice factorization of a PRFB $\mathbf{E}(z)$ with length $L = KM + \beta$ has the same structural form as (7). However, different from PUFBs, the factorization (6) can ensure the PR property if and only if the lattice parameters $\mathbf{E}_0$ and $\mathbf{T}_\beta$ are just nonsingular matrices, in contrast to more restrictive unitary matrices for PUFBs. Compared to the conventional factorizations for PRFBs with filter lengths $L = KM$ [4, 5, 11, 13], the proposed one is more general since it can degenerate into the traditional one by setting $\beta = 0$. Furthermore, we present the completeness of the proposed lattice factorization for any order-one PRFBs without length constraint, i.e., $L \le 2M$, in the following theorem without proof, which generalizes the previous factorization results (cf. [19, Table I]).

Theorem 2. *For the class of PRFBs stated above, the proposed lattice factorization of (7) is complete for any $L \le 2M$.*

In addition, as in the case of PUFBs, the matrix $\mathbf{T}_\beta$ becomes a scalar when $\beta = 1$. Consequently, the only possible choices both $\alpha = 0$ and $\alpha = 1$ would lead to the zero filter coefficient at the boundary of $\mathbf{E}_0(z)$. Thus, there does not exist any nontrivial PRFB with length $M+1$ according to the completeness of our lattice factorization stated above.

Degrees of freedom: The lattice structure of (7) can generate PRFBs with length $L = KM + \beta$. It needs $K-1$ PR building blocks $\mathbf{G}_i(z)$ and two free invertible matrices $\mathbf{E}_0$ and $\mathbf{T}_\beta$. Recall that the simplified order-one block $\mathbf{G}_i(z)$ [13] has $4r_i(M - r_i)$ free parameters for the complex coefficient case and $2r_i(M - r_i)$ for the real coefficient case. The two square invertible matrices $\mathbf{E}_0$ and $\mathbf{T}_\beta$ need $2(M^2 + \beta^2)$ parameters for the complex case and $M^2 + \beta^2$ parameters for the real case. Thus, the degrees of freedom for the whole lattice are given by $D_L = 2(M^2 + \beta^2) + \sum_{i=1}^{K-1} 4r_i(M - r_i)$ for the complex case and $D_L = M^2 + \beta^2 + \sum_{i=1}^{K-1} 2r_i(M - r_i)$ for the real case. Note that there are still some redundant parameters in the above characterization, which will be our future work.

5. REGULAR FILTER BANKS

Another desired property of FB is the regularity which is closely related to the performance of image coding based on FBs. In this paper, regularity of FBs is referred to as the number of multiple zeros at the aliasing frequencies [20, 21]. By virtue of our derived lattice factorization, we can structurally impose the regularity by further structural constraints on lattice parameters. In the following, we only present lattice structure for one degree of regularity due to the space limitation. Such FBs have no DC leakage, which is desirable for avoiding the artificial checkerboard effect in image coding. In addition, it also serves as the necessary condition [20, 21] for M-band wavelets generated by iterating M-channel FBs. The necessary conditions in terms of polyphase matrices are,

$$\mathbf{E}(z^M)\mathbf{d}(z)|_{z=1} = c\mathbf{e}_0, \;\; \tilde{\mathbf{R}}(z^M)\mathbf{J}_M\mathbf{d}(z)|_{z=1} = d\mathbf{e}_0 \qquad (8)$$

where delay chain vector $\mathbf{d}(z) = [1, z^{-1}, \cdots, z^{-(M-1)}]^T$, Euclidean basis vector $\mathbf{e}_0 = [1, 0, \cdots, 0]^T$ and c, d are nonzero constants with $cd = M$.

For general PRFBs factorized in (7), it can be shown that with additional constraints $\mathbf{T}_\beta \mathbf{1}_\beta = \mathbf{1}_\beta$ and $\mathbf{T}_\beta^{-T} \mathbf{1}_\beta = \mathbf{1}_\beta$, the regularity conditions simplify to $\mathbf{E}_0 \mathbf{1}_M = c\mathbf{e}_0$ and $\mathbf{E}_0^{-T} \mathbf{1}_M = d\mathbf{e}_0$, which has been studied in [21, 11, 13]. Such $\mathbf{E}_0$ can be characterized by using the LDU matrix factorization [21] or Householder transforms [11]. The additional constraints on $\mathbf{T}_\beta$ can be enforced structurally if we constrain $\mathbf{T}_\beta$ to be an orthogonal or symmetric matrix with sum of each row to be unity, or more generally a generalized doubly stochastic matrix [22].

6. DESIGN EXAMPLE

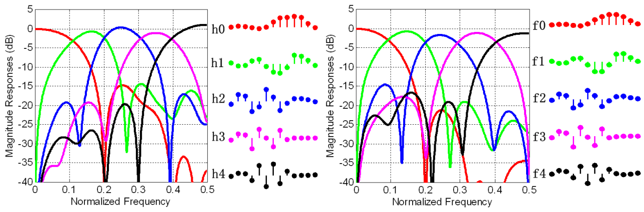
In this section, we show a design example for PRFB with equal length $L = KM + \beta$ designed by using the proposed lattice structures. A good performance Fb can be obtained through unconstrained nonlinear optimization due to the lattice factorization (7) where the lattice parameters are the free parameters for optimization. In different applications, various objective functions could be employed for optimization of the filter coefficients. One objective function is minimization of the stopband attenuation and/or passband ripple for ideal filter shape, which is a classical one in FB design and optimization. Another optimization criterion which measures the energy compaction capability of FBs and is related to efficient signal compression, is the generalized coding gain (CG), which for a signal is taken as

$$C_{\mathrm{CG}} = 10\log_{10}\left[\sigma_x^2 / \left(\prod_{i=0}^{M-1} \sigma_i^2 \|f_i\|^2\right)^{1/M}\right] \qquad (9)$$

where σ_x^2 is the input signal variance, σ_i^2 is the variance of the ith subband signal and $\|f_i\|^2$ is the norm of the ith synthesis filter impulse response $f_i[n]$. We consider the popular AR(1) process with correlation coefficient $\rho = 0.95$ as our source model.

For our design example, the free invertible matrix of the lattice structure such as $\mathbf{E}_0$ is decomposed by SVD [22] through orthogonal and diagonal matrices, where the orthogonal matrix is further decomposed by Givens rotation. Thus, the lattice parameters for an $M \times M$ invertible matrix are the rotation angles and singular values. The invertibility of square matrices is ensured as long as the singular values in the diagonal matrix are nonzero. In addition, the degree parameter r_i for the building block $\mathbf{G}_i(z)$ is chosen to be $r_i = 2$ in our design example. Besides the PR property, we further impose one-degree of regularity, i.e., no DC leakage, for our designed FB.

We show a design example for arbitrary equal-length odd channel PRFB with $M = 5$ and $L = 13$ ($\beta = 3$), optimized for both ideal filter shape and coding gain in Fig. 2. The unconstrained optimization procedure used for this design example is the simplex search function *fminunc* available in MATLAB Optimization Toolbox since the lattice factorization (7) can structurally enforce the desired PR, regularity property and filter length L. The magnitude responses of the analysis



(a) analysis FB (b) synthesis FB

Figure 2: Design example of a five-channel PRFB with length $L = 13$

bank and the corresponding synthesis bank are shown in Fig. 2(a) and Fig. 2(b), respectively. At the right of both magnitude responses, we show the optimized filter coefficients in the wavy lines with filled circles for the corresponding analysis and synthesis filters, respectively. Note that the very strong attenuation at the zero frequency for all the bandpass and highpass filters in the magnitude responses of both analysis and synthesis banks, due to the structural enforcement of one-degree of regularity.

7. CONCLUSION

We have presented the existence conditions, lattice factorizations and designs for a class of M-channel FIR FBs with arbitrary equal lengths. The desired properties like PR or PU can be structurally enforced into the proposed lattice structures. The arbitrary filter length L can be easily achieved by appropriately choosing parameters β and K. In contrast to very limited possible designs by most traditional methods, our design approaches are more general and can considerably expand the designer's choices. In addition, it can be shown that the proposed lattice factorization can completely span the whole space of M-channel FIR PUFBs, and PRFBs with lengths $L \leq 2M$. To our knowledge, these are the most general lattice factorizations for multichannel FIR PU and PR filter banks in the literature.

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