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AUTOMATIC CEPSTRUM-BASED SMOOTHING OF THE PERIODOGRAM VIA CROSS-VALIDATION

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ABSTRACT

In this paper we propose a fully automatic method for variance reduction of spectrum estimates. We use the technique of cepstrum thresholding, named SThresh, which is shown to be an effective, yet simple, way of obtaining a smoothed non-parametric spectrum estimate of a stationary signal. We obtain the threshold via a cross-validatory scheme and the results are shown to be in agreement with those obtained when the spectrum is fully known. We name our proposed method CV-SThresh.

1. INTRODUCTION

Let us consider a stationary, discrete-time, real-valued signal $x(t)$, $t = 0,1,2,\ldots$, with covariance sequence $\{r_k\}_{k=-\infty}^{\infty}$ and power spectral density (or spectrum) $\Phi(\omega)$ $(\omega \in [-\pi,\pi])$. The idea in this paper is to estimate the spectrum from a set of observed samples $\{x(t)\}_{t=0}^{N-1}$ of the signal.

An extensively studied and commonly used estimator of $\Phi(\omega)$ is the periodogram given by (see, e.g., [1–3])

$$\hat{\Phi}_P(\omega) = \frac{1}{N}\left|\sum_{t=0}^{N-1} x(t)e^{-i\omega t}\right|^2, \tag{1}$$

where the subscript P denotes the "Periodogram" estimate. $\hat{\Phi}_P(\omega)$ can also be written in terms of the covariance sequence as

$$\hat{\Phi}_P(\omega) = \sum_{k=-(N-1)}^{N-1} \hat{r}_k e^{-i\omega k}, \tag{2}$$

where $\hat{r}_k$ denotes the following estimate of r_k

$$\hat{r}_k = \frac{1}{N}\sum_{t=k}^{N-1} x(t)x(t-k)$$
$$k = 0,...,N-1\ ; \quad \hat{r}_{-k} = \hat{r}_k. \tag{3}$$

Also, let

$$\omega_l = \frac{2\pi}{N}l; \quad l = 0,\ldots,N-1, \tag{4}$$

denote the Fourier grid of the angular frequency axis. As is well known, $\hat{\Phi}_P(\omega_l)$ can be computed efficiently using a fast Fourier transform (FFT) algorithm.

The resulting periodogram estimate is an asymptotically (in N) unbiased but inconsistent estimate of the underlying true spectrum (see, e.g., [1–3]). In particular, the estimate suffers from a high variance which does not converge to zero as N increases, but to $\Phi^2(\omega_l)$. To overcome this problem, many different smoothing techniques have been proposed, such as various windowing methods (in both the time, lag and frequency domain). These techniques suffer from the drawback of having to carefully select the window and the span, for which there are no clear-cut guidelines. Data-dependent choices of the window span are hard to make, due to the complicated statistical properties of the covariance estimates. The same argument also applies to time-domain smoothing. The recently proposed cepstrum based thresholding method [4, 5], called SThresh in [6], uses the very simple statistical properties of the cepstrum estimate [4] [7] to smooth the periodogram in an almost automatic way. We say almost, since the threshold level must be selected manually by a procedure for which there are clear guidelines and for which only minor prior information is needed. In this paper, we will take this cepstrum based thresholding method one step further, by using a cross-validation (CV) scheme to fully automate the smoothing procedure.

The paper is outlined as follows. In the next section we will introduce the cepstrum and the cepstrum based smoothing technique, SThresh. In Section 3 we describe the CV scheme for choosing the optimal threshold and in Section 4 we present some numerical examples to illustrate the benefit of the proposed algorithm.

2. SMOOTHED SPECTRAL ESTIMATION VIA CEPSTRUM THRESHOLDING – STHRESH

Given a signal $x(t)$, the cepstral coefficients are defined as

$$c_k = \frac{1}{N}\sum_{l=0}^{N-1} \ln[\Phi(\omega_l)]e^{i\omega_l k}$$
$$k = 0,\ldots,N-1, \tag{5}$$

where it is assumed that $\Phi(\omega_l) > 0, \forall l$. The cepstral coefficients have several interesting features, one of which is a

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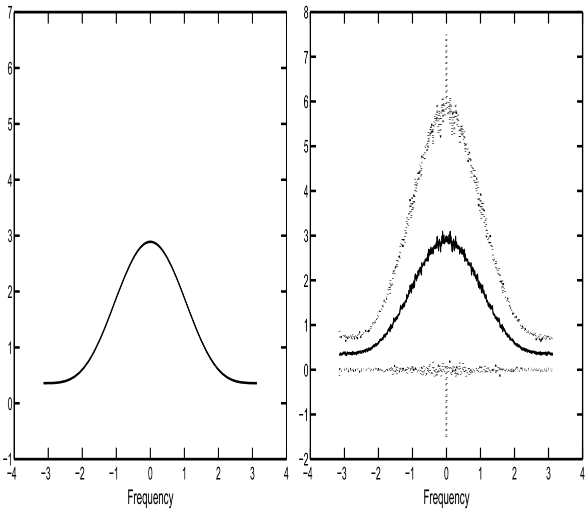
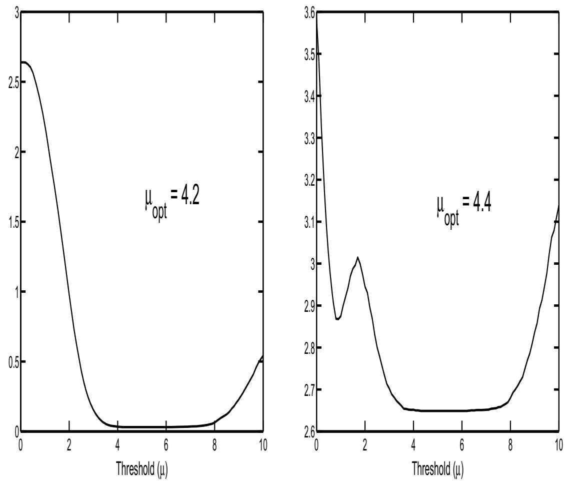


(a) (b)

Figure 1: (a) True spectrum, $\Phi(\omega)$, and (b) the periodogram estimate, mean$[\hat{\Phi}_P(\omega)] \pm$ st.dev.$[\hat{\Phi}_P(\omega)]$, versus frequency, for a broadband MA signal. $N = 512$.



(a) IMSE (b) CVMSE

Figure 2: (a) IMSE($\hat{\Phi}_{cep}(\omega)$) and (b) the cross-validatory estimate of IMSE, CVMSE($\hat{\Phi}_{cep}(\omega)$), versus μ, for the simulated broadband MA signal. $N = 512$.

mirror symmetry:

$$c_{N-k} = c_k \qquad k = 0, 1, ..., \frac{N}{2}, \tag{6}$$

which means that only half of the sequence, $c_0, ..., c_{(N/2)}$, is distinct. The other half is obtained from $c_1, ..., c_{(N/2)-1}$ via (6).

Using the periodogram estimate in (1), a common estimate of the cepstral coefficients is obtained by replacing $\Phi(\omega)$ in (5) with $\hat{\Phi}_P(\omega)$, which gives [4] [7]

$$\hat{c}_k = \frac{1}{N} \sum_{l=0}^{N-1} \ln[\hat{\Phi}_P(\omega_l)] e^{i\omega_l k} + \gamma \delta_{k,0} \qquad k = 0, ..., M, \tag{7}$$

where

$$\delta_{k,0} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{else,} \end{cases} \tag{8}$$

$M = \frac{N}{2}$ and $\gamma = 0.577216\ldots$ (the Euler's constant).

It can be shown (see, e.g., [4]) that in large samples, the estimated cepstral coeffcients $\{\hat{c}_k\}_{k=0}^M$ are independent normally distributed random variables:

$$\hat{c}_k \sim \mathcal{N}(c_k, s_k^2) \tag{9}$$

with

$$s_k^2 = \begin{cases} \frac{\pi^2}{3N} & \text{if } k = 0, M \\ \frac{\pi^2}{6N} & \text{if } k = 1, .., M-1. \end{cases} \tag{10}$$

With the above equations in mind, the idea behind cepstrum thresholding is straightforward. Let $\tilde{c}_k$ be a new estimate of c_k and note that $\tilde{c}_k = 0$ has a mean squared error (MSE) equal to c_k^2. This estimate is preferred to $\hat{c}_k$ as long as $c_k^2 \le s_k^2$. Now let

$$S = \{k \in [0, M] \,|\, c_k^2 \le s_k^2\} \tag{11}$$

and let $\tilde{S}$ be an estimate of the set S. Thresholding $\{\hat{c}_k\}_{k \in \tilde{S}}$ gives the following new estimates of c_k:

$$\tilde{c}_k = \begin{cases} 0 & \text{if } k \in \tilde{S} \\ \hat{c}_k & \text{else} \end{cases} \qquad k = 0, ...M. \tag{12}$$

A good estimate of S is given by (see [5] for details):

$$\tilde{S} = \{k \in [0, M] \,|\, |\hat{c}_k| \le \mu s_k\} \tag{13}$$

where the parameter μ controls the risk of concluding that $|c_k|$ is "significant" when this is not true, the so called "false alarm probability". The following values of μ are recommended in [4,6] for $N \in (128, 2048)$:

$$\mu = \mu_0 + \frac{N - 128}{1920}, \tag{14}$$

where

$$\mu_0 = \begin{cases} 4 & \text{for a broadband signal with} \\ & \text{small dynamic range} \\ 3 & \text{for a broadband signal with} \\ & \text{large dynamic range} \\ 2 & \text{for a narrowband signal with} \\ & \text{very large dynamic range.} \end{cases} \tag{15}$$

This means that μ will belong to the interval $(\mu_0, \mu_0 + 1)$. For other intervals of the sample length, N, similar rules can be given.

The smoothed spectral estimate corresponding to $\{\tilde{c}_k\}$ is given by:

$$\tilde{\Phi}_{cep}(\omega_l) = \exp\left[\sum_{k=0}^{N-1} \tilde{c}_k e^{-i\omega_l k}\right] \qquad l = 0, ..., N-1, \tag{16}$$

where the subscript cep signifies its cepstrum dependence.

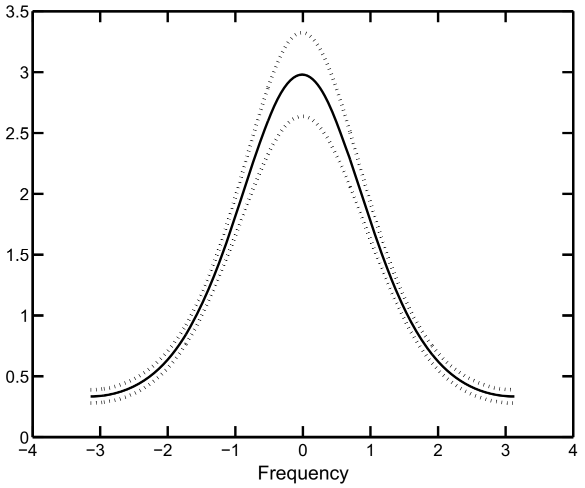


Figure 3: Smoothed spectrum, mean$[\hat{\Phi}_{cep}(\omega)] \pm$ st.dev.$[\hat{\Phi}_{cep}(\omega)]$, for the simulated broadband MA signal. $N = 512$, $\mu_{opt} = 4.4$.

The final scaled spectrum estimate $\hat{\Phi}_{cep}(\omega_l)$ is then given by

$$\hat{\Phi}_{cep}(\omega_l) = \hat{\alpha}\tilde{\Phi}_{cep}(\omega_l) \quad l = 0,\dots,N-1, \tag{17}$$

where

$$\hat{\alpha} = \frac{\sum_{l=0}^{N-1} \hat{\Phi}_P(\omega_l)\tilde{\Phi}_{cep}(\omega_l)}{\sum_{l=0}^{N-1} \tilde{\Phi}^2_{cep}(\omega_l)}. \tag{18}$$

The above outlined smoothing scheme was called Simple Thresholding, or SThresh, in [6]. We now proceed to automate the selection of μ, using a cross-validation scheme.

3. CROSS-VALIDATION BASED THRESHOLD SELECTION – CV-STHRESH

The idea of using cross-validation for selection of smoothing parameters is quite appealing. For example, in [8, 9], the selection of the bandwidth of estimates that are based on a discrete periodogram average, by means of cross-validation, is discussed. Here we apply a similar idea to find an optimum of the threshold μ in (13). This optimal threshold will then be used to smooth the spectrum with the method outlined in Section 2.

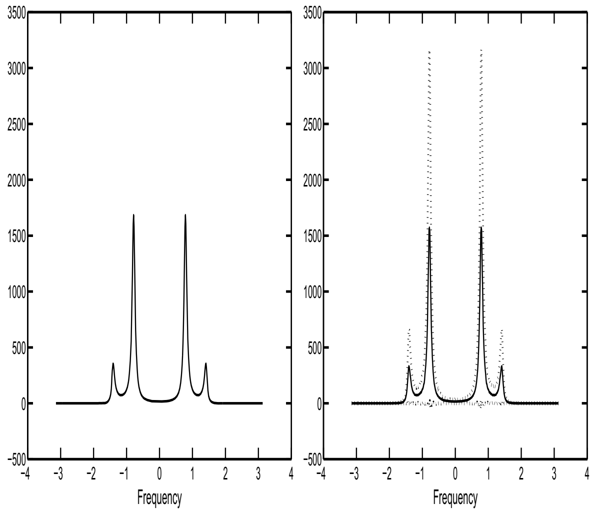
If we knew the true underlying spectrum $\Phi(\omega)$, we would find μ by minimizing the integrated mean squared error (IMSE) of $\hat{\Phi}_{cep}$ with respect to μ, where

$$IMSE(\hat{\Phi}_{cep}) = \frac{1}{N}\sum_{j=0}^{N-1} E\left[\hat{\Phi}_{cep}(\omega_j) - \Phi(\omega_j)\right]^2 \tag{19}$$

and $E[\cdot]$ denotes the expectation.

Unfortunately, in most practical situations, $\Phi(\omega)$ is unknown and computing the IMSE is therefore not possible. However, cross-validation schemes can be utilized to find an estimate of IMSE, for instance, the following estimate given by (see [8, 9])

$$CVMSE(\hat{\Phi}_{cep}) = \frac{1}{N}\sum_{j=0}^{N-1} \left[\hat{\Phi}^{-j}_{cep}(\omega_j) - \hat{\Phi}_P(\omega_j)\right]^2. \tag{20}$$



(a) (b)

Figure 4: (a) True spectrum, $\Phi(\omega)$, and (b) the periodogram estimate, mean$[\hat{\Phi}_p(\omega)] \pm$ st.dev.$[\hat{\Phi}_p(\omega)]$, versus frequency, for a narrowband ARMA signal.$N = 512$.

Here, $\hat{\Phi}^{-j}_{cep}(\omega_j)$ is the leave-one-out (crossvalidated) version of $\hat{\Phi}_{cep}(\omega_j)$, constructed such that $\hat{\Phi}^{-j}_{cep}(\omega_j)$ is independent of $\hat{\Phi}_P(\omega_j)$ for all j. This is achieved by first calculating

$$\hat{c}^{-j}_k = \frac{1}{N}\sum_{\substack{l=0 \\ l\neq j}}^{N-1} \ln[\hat{\Phi}_P(\omega_l)]e^{i\omega_l k}. \tag{21}$$

This leave-one-out estimate can be efficiently obtained from $\hat{c}_k$ in (7) as follows

$$\hat{c}^{-j}_k = \hat{c}_k - \ln[\hat{\Phi}_P(\omega_j)]e^{i\omega_j k}. \tag{22}$$

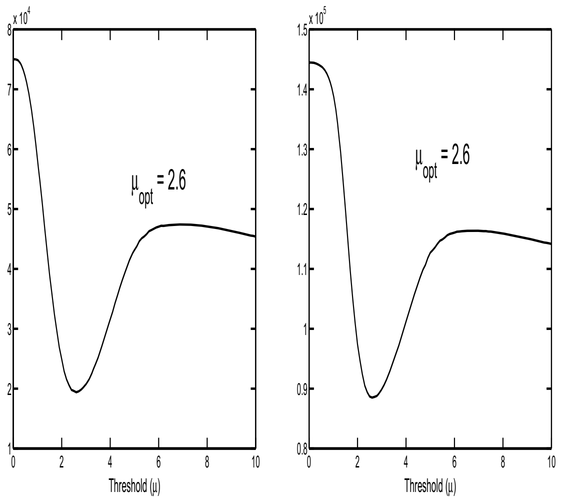
The cepstral coefficients are then thresholded according to (12) for a particular choice of μ. Finally, the leave-one-out estimate in (20) is given by

$$\hat{\Phi}^{-j}_{cep}(\omega_j) = \hat{\alpha}\exp\left[\sum_{k=0}^{N-1} \tilde{c}^{-j}_k e^{-i\omega_j k}\right] \quad j = 0,\dots,N-1, \tag{23}$$

where $\tilde{c}^{-j}_k$ denotes the thresholded version of $\hat{c}^{-j}_k$ and $\hat{\alpha}$ is obtained by replacing $\tilde{\Phi}_{cep}(\omega)$ with its leave-one-out estimate $\tilde{\Phi}^{-j}_{cep}(\omega)$ in (18). The optimal thresholding parameter μ is then the one that minimizes the criterion (20).

In summary, the proposed smoothing scheme, named CV-SThresh makes use of the following steps to find the optimal threshold parameter, say μ_{opt}:

1. From $x(t)$, $t = 0,1,\dots,N-1$, compute $\hat{\Phi}_P(\omega)$ and $\hat{c}_k$.
2. Choose a $\mu \in [0, \mu_{max}]$. Empirical studies have shown that taking $\mu_{max} = 10$ is a good general choice.
3. Find an estimate of $IMSE(\hat{\Phi}_{cep}(\omega))$ by efficiently evaluating $CVMSE(\hat{\Phi}_{cep}(\omega))$ in (20). The efficient way of obtaining $\hat{c}^{-j}_k$ in (22) reduces computation by a factor of $\log N$.



(a) IMSE (b) CVMSE.

Figure 5: (a) IMSE($\hat{\Phi}_{cep}(\omega)$), and (b) the cross-validatory estimte of IMSE, CVMSE($\hat{\Phi}_{cep}(\omega)$), versus μ, for the simulated narrowband ARMA signal. $N = 512$.

4. Repeat the above step over the range $[0, \mu_{max}]$.
5. Find the value of μ that minimizes $CVMSE(\hat{\Phi}_{cep}(\omega))$, i.e., $\mu_{opt} = \arg\min_{\mu \in [0,\mu_{max}]} CVMSE(\hat{\Phi}_{cep}(\omega))$.
6. Finally compute the smoothed spectrum as done in Section 2, using μ_{opt} as a threshold.

4. NUMERICAL EXAMPLES

We will illustrate the performance of the proposed CV-SThresh method by using the broadband and narrowband examples in [4] and the ocean wave data example in [6].

4.1 Broadband example

The first example is a broadband, second-order MA process. The signal $x(t)$ was generated using the moving average equation

$$x(t) = e(t) + 0.55e(t-1) + 0.15e(t-2) \quad t = 0,\ldots,N-1, \tag{24}$$

where $e(t)$ is a zero mean, unit variance normal white noise. We generated 1000 realizations, each of length $N = 512$ of the process. In Fig. 1 we show the true spectrum and the mean and standard deviation of the periodogram. These figures should be compared with the spectrum smoothened via CV-SThresh, shown in Fig. 3. We clearly see that the variance of the smoothed spectrum is significantly smaller than that of the peridogram. The optimal μ can be obtained from Fig. 2(a) by finding the μ corresponding to the minimum value of IMSE. It is clearly seen from Fig. 2 that the true and the CV estimate of IMSE have a minimum at almost the same point. Cross-validation therefore gives a nearly optimal value of μ for thresholding.

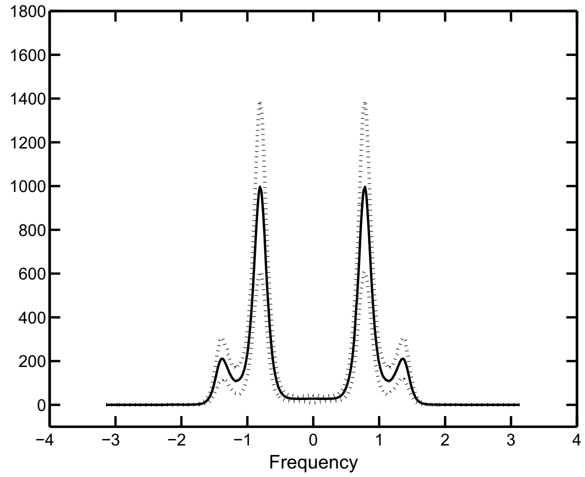


Figure 6: Smoothed spectrum, mean[$\hat{\Phi}_{cep}(\omega)$] $\pm$ st.dev.[$\hat{\Phi}_{cep}(\omega)$], for the simulated narrowband ARMA signal. $N = 512$, $\mu_{opt} = 2.6$.

4.2 Narrowband example

The above simulations are repeated for a narrowband ARMA process represented by

$$\begin{aligned} x(t) - 1.6408x(t-1) + 2.2044x(t-2) - 1.4808x(t-3) \\ +0.8145x(t-4) = e(t) + 1.5857e(t-1) \\ +0.9604e(t-2) \quad t = 0,\ldots,N-1 \end{aligned} \tag{25}$$

where $e(t)$ is again a zero mean, unit variance normal white noise. We generated 1000 Monte-Carlo simulations, each of length $N = 512$. As described in the broadband example, the optimal threshold parameter, μ_{opt}, has been obtained as the threshold that yields the minimum cross-validatory estimate of IMSE. For comparison, we have plotted the true spectrum together with the mean and standard deviation of the periodogram in Fig. 4. Fig. 5 shows the IMSE and CVMSE curves versus μ, used to find μ_{opt}. Again, the IMSE and its estimate CVMSE give similar results. Fig. 6 shows the mean of the CV-SThresh smoothed spectrum together with its standard deviation, obtained using μ_{opt}. Comparing Fig. 6 with Fig. 4, we see that the variance has been reduced but a bias has been introduced. When smoothing the spectrum via cepstrum thresholding, some of the energy is lost due to the truncation of $\hat{c}_k$ to zero in (12). For broadband signals, very few of the cepstral estimates are truncated so only a small bias is introduced, whereas for narrowband signals, many more coefficients are set to zero, see [4], thus causing the bias seen in Fig. 6.

4.3 Ocean wave data example

A real-life data set has been used as a third example. The data is a time series recorded in the Pacific Ocean by a wave-follower. Every $1/4$ second the sea level is measured as the wave-follower moves up and down following the water surface. The data consists of $N = 1024$ data points that were low-pass filtered using an antialiasing filter with a cutoff frequency of approximately 1 Hz. The data was originally collected to investigate whether the rate at which the spectrum

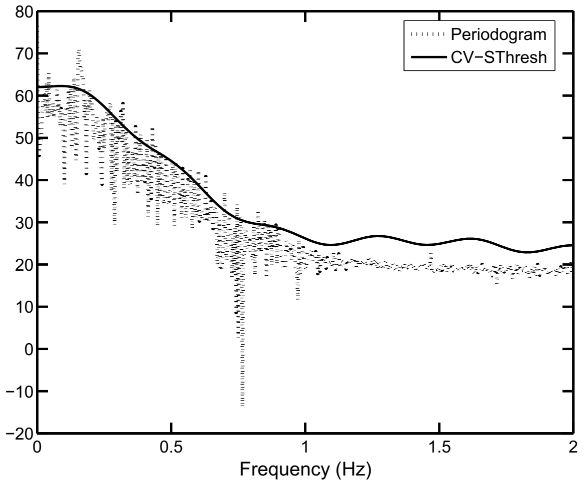


Figure 7: The log-periodogram of the ocean wave data ($N = 1024$) together with the smoothed spectrum obtained via CV-SThresh.

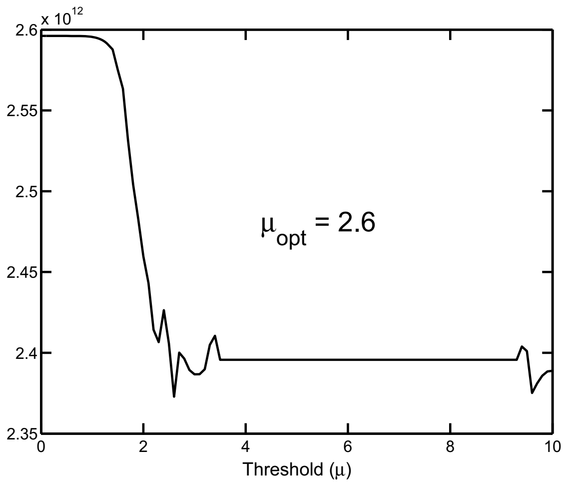


Figure 8: The CVMSE for the ocean-wave data.

decreases in the interval 0.2 to 1.0 Hz was consistent with a physical model. The behaviour for frequencies above 1 Hz was of little interest since it was mainly determined by instrumentation and preprocessing. For a detailed explanation on this ocean wave data see [2]. Applying CV-SThresh to the ocean wave data, we obtain the smoothed spectrum that can be seen in Fig. 7 together with the periodogram estimate of the spectrum. We conclude that using the smoothed spectrum makes the computation of the slope in the range of 0.2 to 1.0 Hz very easy compared to using the highly erratic periodogram estimate. In Fig. 8 we plot the curve of CVMSE versus μ, used to find μ_{opt}.

5. CONCLUDING REMARKS

In this paper we have proposed CV-SThresh, a new data-driven method for threshold selection in smoothed non-parametric spectral estimation. The proposed cross-validation based method has been applied to a broadband, a narrowband and a real-life broadband example. The results obtained conform with the results in [4, 6], derived using some apriori knowledge about the true spectrum. While the method proposed here is complete, there are a few steps of it that can be explored, namely the criterion used in the threshold selection and the cross-validation scheme. The criterion used in this paper is a cross-validatory estimate of the minimum mean square error. There is no particular argument regarding the choice of this criterion, so the possibility for using other criteria can be explored. Regarding the cross-validation method employed, given N periodogram ordinates ($\hat{\Phi}_P(\omega_k)$) there are many cross validation schemes available: leave-one-out, leave-two-out, etc. In this paper the focus was on the leave-one-out scheme, mainly because it gives good results with few computations.

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