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MEAN-SQUARE CONSISTENCY AND ASYMPTOTIC NORMALITY OF A HYBRID ESTIMATOR OF THE CYCLIC AUTOCORRELATION FUNCTION OF GENERALIZED ALMOST-CYCLOSTATIONARY PROCESSES

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ABSTRACT

In this paper, the problem of estimating the cyclic autocorrelation function of a continuous-time generalized almost-cyclostationary (GACS) process is addressed. GACS processes in the wide sense have autocorrelation function almost-periodic in time whose generalized Fourier series expansion has both frequencies and coefficients that depend on the lag shifts. Almost-cyclostationary (ACS) processes are obtained as a special case when the frequencies do not depend on the lag shifts. ACS processes filtered by Doppler channels and communications signals with time-varying parameters are further examples. The discrete-time cyclic correlogram of the discrete-time process obtained by uniformly sampling a GACS process is considered as estimator of samples of the continuous-time cyclic autocorrelation function. The asymptotic performance analysis is carried out by resorting to the hybrid cyclic correlogram which is partially continuous-time and partially discrete-time. It is shown that its asymptotic properties are coincident with those of the continuous-time cyclic correlogram. Hence, discrete-time estimation does not give rise to any loose in asymptotic performance with respect to continuous-time estimation.

1. INTRODUCTION

Almost-cyclostationary (ACS) processes are an appropriate model for almost all modulated signals adopted in communications, radar, and telemetry. Almost-cyclostationarity properties have been exploited in signal-selective detection and parameter-estimation algorithms, blind-channel identification and synchronization techniques, and so on [2], [3], [4], [5], [13].

Second-order ACS processes in the wide-sense exhibit the autocorrelation function which is an almost-periodic function of time whose generalized Fourier series expansion has coefficients (the cyclic autocorrelation functions) dependent on the lag parameter and frequencies (the cycle frequencies) not dependent on the lag parameter. In [6], the *generalized almost-cyclostationary (GACS)* processes are introduced. This class of processes extends that of the ACS processes since the autocorrelation function is an almost-periodic function of time whose generalized Fourier series expansion has both coefficients (the generalized cyclic autocorrelation functions) and frequencies (the lag-dependent cycle frequencies) which depend on the lag parameter. Thus, the ACS processes are obtained as a special case of GACS processes when the lag-dependent cycle frequencies are constant with respect to the lag parameter. In [7] and [8], it is shown that GACS processes are generated by the Doppler channel due to relative motion between transmitter and receiver in the case of constant relative radial acceleration and transmitted ACS signal and are an appropriate model to describe chirp signals and several angle-modulated and time-warped communication signals.

The problem of second-order statistical function estimation of GACS processes has recently been addressed in [10], [11].

In [10], under mild assumptions on the regularity of the generalized Fourier series expansions and on the memory of the GACS process, the continuous-time cyclic correlogram is shown to be a mean-square consistent and asymptotically Normal estimator of the cyclic autocorrelation function. These results, generalize to a wider class of processes, namely the class of the GACS processes, well known results for ACS processes [2], [3], [4].

In [9], it is shown that uniformly sampling a continuous-time GACS process gives rise to a discrete-time ACS process. Thus, a discrete-time counterpart of the continuous-time GACS processes does not exist and the GACS or ACS nature of an underlying continuous-time process can only be conjectured starting form the analysis of the discrete-time ACS process. In addition, since GACS processes cannot be strictly band limited [6], [10], unlike the case of ACS or stationary processes, a minimum value of the sampling frequency to completely avoid aliasing in the discrete-time cyclic statistics does not exist. This constitutes a complication for the estimation of the cyclic autocorrelation function of a GACS process starting from the discrete-time process of its samples. Consequently, the discrete-time counterparts of the asymptotic results of [10] are not straightforward.

In [11], the discrete-time cyclic correlogram of the discrete-time process obtained by uniformly sampling a continuous-time GACS process is shown to be a mean-square consistent estimator of samples of the aliased continuous-time cyclic autocorrelation function of the GACS process as the number of data-samples approaches infinity.

In this paper, the asymptotic statistical analysis of the discrete-time cyclic correlogram of the ACS process obtained by uniformly sampling a continuous-time GACS process is carried out when the number of data samples approaches infinity (to get consistency) and the sampling period approaches zero (to counteract aliasing). It is pointed out that the discrete-time cyclic correlogram has the drawback that, when the sampling period approaches zero, the unnormalized lag parameter approaches zero and the unnormalized cycle frequency approaches infinity. A general procedure to carry out the asymptotic analysis as the number of data-samples approaches infinity and the sampling period approaches zero is proposed in the paper by resorting to the hybrid cyclic correlogram. It is called "hybrid" since some parameters are continuous-time and others are discrete-time. In the paper, it is shown that the hybrid cyclic correlogram is a mean-square consistent and asymptotically Normal estimator of the cyclic autocorrelation function when the number of samples approaches infinity and the sampling period approaches zero in such a way that the overall data-record length approaches infinity. Thus, it is shown that the mean-square error between the discrete-time cyclic correlogram and samples of the continuous-time cyclic autocorrelation can be made arbitrarily small provided that the number of data samples is sufficiently large and the sampling period is sufficiently small. Moreover, it is shown that the asymptotic bias, covariance and distribution of the hybrid cyclic correlogram are coincident with those of the continuous-time cyclic correlogram. Finally, it is pointed that the proposed asymptotic analysis can be applied also to the case of non strictly band limited ACS and stationary processes.

2. GENERALIZED ALMOST-CYCLOSTATIONARY PROCESSES

Let superscript $(*)$ denote an optional complex conjugation. A finite-power complex-valued continuous-time stochastic process $x(t)$, $t \in \mathbb{R}$, is said to be *second-order GACS in the wide sense* [10] if for both conjugation configurations

$$\mathcal{R}_{xx^{(*)}}(t,\tau) \triangleq \mathrm{E}\left\{x(t+\tau)\,x^{(*)}(t)\right\} \tag{1}$$

is an almost-periodic function of time. If $(*)$ is present, $\mathcal{R}_{xx^*}(t,\tau)$ is the *autocorrelation function*, if $(*)$ is absent, $\mathcal{R}_{xx}(t,\tau)$ is the *conjugate autocorrelation function*. Both second-order moments are necessary for a complete second-order characterization of complex processes [14]. From the almost periodicity in t it follows that, for each fixed τ, $\mathcal{R}_{xx^{(*)}}(t,\tau)$ is the limit of an uniformly convergent sequence of trigonometric polynomials in t which can be written in the two following equivalent forms [6], [7]:

$$\begin{aligned}\mathcal{R}_{xx^{(*)}}(t,\tau) &= \sum_{\alpha\in A_\tau} R_{xx^{(*)}}(\alpha,\tau)\,e^{j2\pi\alpha t} && \text{(2a)}\\ &= \sum_{n\in\mathbb{I}} R^{(n)}_{xx^{(*)}}(\tau)\,e^{j2\pi\alpha_n(\tau)t} && \text{(2b)}\end{aligned}$$

where coefficients and frequencies in both Fourier series depend on the possible complex conjugation. In (2a), the real numbers α and the complex-valued functions $R_{xx^{(*)}}(\alpha,\tau)$, referred to as *(conjugate) cycle frequencies* and *(conjugate) cyclic autocorrelation functions*, are the frequencies and coefficients, respectively, of the generalized Fourier series expansion of $\mathcal{R}_{xx^{(*)}}(t,\tau)$ that is,

$$R_{xx^{(*)}}(\alpha,\tau) \triangleq \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\mathcal{R}_{xx^{(*)}}(t,\tau)\,e^{-j2\pi\alpha t}\,\mathrm{d}t\,. \tag{3}$$

Furthermore, in (2a) and (2b),

$$\begin{aligned}A_\tau &\triangleq \{\alpha\in\mathbb{R} : R_{xx^{(*)}}(\alpha,\tau)\neq 0\} && \text{(4a)}\\ &= \bigcup_{n\in\mathbb{I}}\{\alpha\in\mathbb{R} : \alpha=\alpha_n(\tau)\} && \text{(4b)}\end{aligned}$$

is a countable set, $\mathbb{I}$ is also countable, the real-valued functions $\alpha_n(\tau)$ are referred to as *(conjugate) lag-dependent cycle frequencies* and the complex-valued functions $R^{(n)}_{xx^{(*)}}(\tau)$, referred to as *(conjugate) generalized cyclic autocorrelation functions*, are defined as

$$R^{(n)}_{xx^{(*)}}(\tau) \triangleq R_{xx^{(*)}}(\alpha,\tau)\big|_{\alpha=\alpha_n(\tau)} \tag{5}$$

for all values of τ such that two different lag-dependent cycle frequencies do not intersect [7], [10]. It can be shown that [6], [7]

$$R_{xx^{(*)}}(\alpha,\tau) = \sum_{n\in\mathbb{I}} R^{(n)}_{xx^{(*)}}(\tau)\,\delta_{\alpha-\alpha_n(\tau)} \tag{6}$$

where δ_γ denotes Kronecker delta, that is, $\delta_\gamma = 1$ for $\gamma = 0$ and $\delta_\gamma = 0$ for $\gamma \neq 0$. That is, the lag-dependent cycle-frequency curves $\alpha = \alpha_n(\tau)$, $n \in \mathbb{I}$, describe the support of the (conjugate) cyclic autocorrelation function $R_{xx^{(*)}}(\alpha,\tau)$.

The second-order wide-sense ACS processes are obtained as a special case of GACS processes when the lag-dependent cycle frequencies are constant with respect to τ and, hence, are coincident with the cycle frequencies [6]. In such a case,

$$\mathcal{R}_{xx^{(*)}}(t,\tau) = \sum_{n\in\mathbb{I}} R^{\alpha_n}_{xx^{(*)}}(\tau)\,e^{j2\pi\alpha_n t}\,. \tag{7}$$

Moreover, $R_{xx^{(*)}}(\alpha,\tau) = R^{\alpha_n}_{xx^{(*)}}(\tau)$ for $\alpha = \alpha_n \in A$, and $R_{xx^{(*)}}(\alpha,\tau) = 0$ otherwise, with $A = \{\alpha_n\}_{n\in\mathbb{I}}$ countable.

The GACS model turns out to be appropriate in mobile communications systems when the channel cannot be modeled as almost-periodically time-variant [7], [8]. For example, the output complex envelope $y(t)$ of the Doppler channel existing between a stationary transmitter and a moving receiver with constant relative radial acceleration is GACS when the input complex envelope $x(t)$ is ACS. In such a case, the transmitted signal experiences a quadratically time-variant delay. Under the "narrow-band" approximation [15], the time-varying component of the delay in the complex envelope $x(\cdot)$ can be neglected obtaining the chirp-modulated signal

$$y(t) = a\,x(t-d_0)\,e^{j2\pi\nu t}\,e^{j\pi\zeta t^2} \tag{8}$$

where a is the complex gain, d_0 the constant delay, ν the frequency shift, and ζ the chirp rate. If $x(t)$ is ACS (with autocorrelation function (7)), then $y(t)$ is GACS [10] with lag-dependent cycle frequencies $\alpha_n + \zeta\tau$ and generalized cyclic autocorrelation functions

$$R^{(n)}_{yy^*}(\tau) = |a|^2\,R^{\alpha_n}_{xx^*}(\tau)\,e^{j2\pi\nu\tau}\,e^{j\pi\zeta\tau^2}\,e^{-j2\pi\alpha_n d_0}\,. \tag{9}$$

Further examples of GACS processes are angle modulated signals and communications signals with slowly time-varying parameters such as baud rate and carrier frequency [6].

3. CONTINUOUS- AND DISCRETE-TIME ESTIMATORS OF THE CYCLIC AUTOCORRELATION FUNCTION

In [10], the (conjugate) continuous-time cyclic correlogram

$$R_{xx^{(*)}}(\alpha,\tau;t_0,T) \triangleq \int_{\mathbb{R}} w_T(t-t_0)\,x(t+\tau)\,x^{(*)}(t)\,e^{-j2\pi\alpha t}\,\mathrm{d}t \tag{10}$$

where $w_T(t)$ is a unit-area data-tapering window, is shown to be a mean-square consistent and asymptotically Normal estimator of the (conjugate) cyclic autocorrelation function.

Let

$$x_{\mathrm{d}}(n) \triangleq x(t)|_{t=nT_s} \tag{11}$$

be the discrete-time sequences obtained by uniformly sampling with period $T_s = 1/f_s$ the continuous-time zero-mean GACS processes $x(t)$. In [11], it is shown that the *(conjugate) discrete-time cyclic correlogram* at cycle frequency $\widetilde{\alpha}$

$$\widetilde{R}_{x_{\mathrm{d}}x_{\mathrm{d}}^{(*)}}(\widetilde{\alpha},m;n_0,N) \triangleq \sum_{n=-N}^{N} v_N(n-n_0)\, x_{\mathrm{d}}(n+m)\, x_{\mathrm{d}}^{(*)}(n)\, e^{-j2\pi\widetilde{\alpha}n} \tag{12}$$

where $v_N(n)$ is a data-tapering window, is a mean-square consistent estimator of the discrete-time (conjugate) cyclic autocorrelation function

$$\begin{aligned} &\widetilde{R}_{x_{\mathrm{d}}x_{\mathrm{d}}^{(*)}}(\widetilde{\alpha},m) \\ &\triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \mathrm{E}\left\{x_{\mathrm{d}}(n+m)\, x_{\mathrm{d}}^{(*)}(n)\right\} e^{-j2\pi\widetilde{\alpha}n} && \text{(13a)} \\ &= \sum_{p=-\infty}^{+\infty} R_{xx^{(*)}}\left((\widetilde{\alpha}+p)f_s, mT_s\right) && \text{(13b)} \end{aligned}$$

which is an aliased version of samples of the continuous-time (conjugate) cyclic autocorrelation function.

The mean-square consistency results in [10] and [11] are based on assumptions on the almost-periodic structure of the second- and fourth-order cumulants of the continuous-time process $x(t)$, on the regularity of the generalized Fourier series expansions of such almost-periodic functions, and on the finite or practically finite memory of the process expressed in terms of summability of its second- and fourth-order cumulants (Assumption 1). In addition, regularity properties of the data-tapering window need to be assumed (Assumption 2). Finally, asymptotic Normality is proved for the continuous-time estimator in [10] under further regularity assumptions on higher-order statistics of $x(t)$ (Assumption 3).

Assumption 1 For any choice of z_1 and z_2 in $\{x, x^*\}$, it results ***a)***

$$\mathrm{E}\{z_1(t+\tau)\, z_2(t)\} = \sum_n R^{(n)}_{z_1z_2}(\tau)\, e^{j2\pi\alpha^{(n)}_{z_1z_2}(\tau)t} \tag{14}$$

$$\text{with} \quad \sum_n \|R^{(n)}_{z_1z_2}\|_\infty < \infty \quad \text{and} \quad \sum_n \int_{\mathbb{R}} \left|R^{(n)}_{z_1z_2}(s)\right| \mathrm{d}s < \infty$$

and ***b)***

$$\begin{aligned} &\mathrm{cum}\{x(t+\tau_1), x^*(t+\tau_2), z_1(t+\tau_3), z_2(t)\} \\ &\quad = \sum_n C^{(n)}_{xx^*z_1z_2}(\tau_1,\tau_2,\tau_3)\, e^{j2\pi\beta_n(\tau_1,\tau_2,\tau_3)t} \qquad (15) \end{aligned}$$

$$\text{with} \quad \sum_n \|C^{(n)}_{xx^*z_1z_2}\|_\infty < \infty$$

$$\text{and} \quad \sum_n \int_{\mathbb{R}} \left|C^{(n)}_{xx^*z_1z_2}(s+\tau_1, s, \tau_2)\right| \mathrm{d}s < \infty \quad \forall \tau_1, \tau_2 \in \mathbb{R}\,.$$

In addition, the process $x(t)$ has uniformly bounded fourth-order absolute moment.

Assumption 2 The continuous-time data-tapering window $w_T(t)$ is nonzero in $[-T/2, T2]$ and can be expressed as $w_T(t) = a(t/T)/T$, with $a(t) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, continuous a.e., and with unit area. The discrete-time data-tapering window $v_N(n)$ is nonzero in $\{-N,\ldots,N\}$ and can be expressed as $v_N(n) = a(n/(2N+1))/(2N+1)$.

Assumption 3 Let $z_i(t) \triangleq [x(t+\tau_i)\, x^{(*)}(t)]^{[*]_i}, i = 1,\ldots,k$, with $[*]_i$ optional complex conjugation. For every integer k, the magnitude of the kth-order cumulant function $\mathrm{cum}\{z_k(t), z_i(t+s_i),\ i = 1,\ldots,k-1\}$ is uniformly bounded by a positive Riemann-integrable function of $s_1,\ldots,s_{k-1}$. In addition, for every integer k, the processes $z_i(t)$ have bounded absolute kth-order cross-moments.

The discrete-time (conjugate) cyclic correlogram provides a reliable estimator of samples of the continuous-time (conjugate) cyclic autocorrelation function, provided that the aliasing effect in (13b) due to replicas with $p \neq 0$ is negligible. This is assured by the following assumption.

Assumption 4 For every $\widetilde{\alpha}$ and m there exists a sequence $\{M_p\}_{p\in\mathbb{Z}}$ of positive numbers such that $\left|R_{xx^{(*)}}\left((\widetilde{\alpha}+p)f_s, mT_s\right)\right| \leqslant M_p$ and $\sum_{p=-\infty}^{+\infty} M_p < \infty$.

Assumption 4 is verified, for example, by GACS process with a finite number of lag-dependent cycle frequencies (see, e.g., the chirp-modulated PAM signal with Nyquist pulse [10]). In such a case we have that the aliasing terms ($p \neq 0$) in (13b) can be made arbitrarily small by taking the sampling period T_s sufficiently small. In addition, in [11], the following result is proved.

Theorem 1 Under Assumptions 1, 2, and 4 it results that

$$\lim_{T_s\to 0} \lim_{N\to\infty} \mathrm{E}\left\{\left|\widetilde{R}_{x_{\mathrm{d}}x_{\mathrm{d}}^{(*)}}(\widetilde{\alpha},m;n_0,N) - R_{xx^{(*)}}(\alpha,\tau)\Big|_{\substack{\alpha=\widetilde{\alpha}f_s \\ \tau=mT_s}}\right|^2\right\} = 0 \tag{16}$$

where the order of the two limits cannot be interchanged. □

4. ASYMPTOTIC ANALYSIS BY THE HYBRID CYCLIC CORRELOGRAM

From Theorem 1 it follows that the mean-square error between the discrete-time cyclic correlogram and samples of the (continuous-time) cyclic autocorrelation function can be made arbitrarily small, provided that the number of data samples is sufficiently large and the sampling period is sufficiently small. However, such asymptotic result has the drawback that, for fixed $\widetilde{\alpha}$ and m, when $T_s \to 0$ it follows $\alpha = \widetilde{\alpha} f_s \to \infty$ and $\tau = mT_s \to 0$. Thus, this analysis turns out to be unuseful if the asymptotic (as $N \to \infty$ and $T_s \to 0$) bias and covariance are needed and asymptotic Normality needs to be proved. Note that such a drawback is also present if the discrete-time cyclic correlogram is adopted to estimate samples of the continuous-time cyclic autocorrelation function of a non band-limited ACS process. Furthermore, the same problem is encountered with the discrete-time correlogram estimate ($\widetilde{\alpha} = 0$ in (12)) of the autocorrelation function of a non band-limited wide-sense stationary process. This problem does not arise if continuous- and discrete-time estimation are treated separately, and the aliasing problem arising from sampling is not addressed [1].

In this section, an asymptotic analysis not suffering of such drawback is carried out by the *hybrid cyclic correlogram* at cycle frequency $\alpha \in \mathbb{R}$ and lag $\tau \in \mathbb{R}$ defined as

$$\rho_{x_{\rm d} x_{\rm d}^{(*)}}(\alpha, \tau; n_0, N, T_s) \triangleq \sum_{n=-N}^{N} v_N(n-n_0)\, x_{{\rm d}\tau}(n)\, x_{\rm d}^{(*)}(n)\, e^{-j2\pi\alpha n T_s} \quad (17)$$

where

$$x_{{\rm d}\tau}(n) \triangleq x(t+\tau)|_{t=nT_s} \qquad \tau \in \mathbb{R}\,. \quad (18)$$

It is referred to as "hybrid" since data samples are discrete-time, but the lag parameter and the cycle frequency are not normalized to the sampling period T_s and the sampling frequency f_s, respectively, and are the same as those of the continuous-time cyclic correlogram. In (17), the delay of the continuous-time process does not need to be an integer multiple of the sampling period T_s. Consequently, it can be retained constant when $T_s \to 0$ avoiding the drawback $\tau \to 0$ as $T_s \to 0$ as in Theorem 1. Analogous considerations hold for the cycle frequency.

Note that the hybrid cyclic correlogram turns out to be useful just to analytically carry out the asymptotic analysis, whereas, in practice, the discrete-time cyclic correlogram is implemented.

For finite N and T_s, the expected value and the covariance of the hybrid cyclic correlogram can be obtained by those of the discrete-time cyclic correlogram reported in [11, Theorem 1] and [11, Theorem 2], respectively, with the replacements $\tau = mT_s$ and $\alpha = \widetilde{\alpha} f_s$ in (21) of [11]. Furthermore, the asymptotic results as $N \to \infty$ for the hybrid cyclic correlogram are obtained, with minor changes, as those for the discrete-time cyclic correlogram. In particular, the asymptotic expected value of the hybrid cyclic correlogram is given in [11, Theorem 3] with the replacements $\tau = mT_s$ and $\alpha = \widetilde{\alpha} f_s$ in (23) of [11] and the asymptotic covariance is given in [11, Theorem 4] with the replacements $\tau_1 = m_1 T_s$, $\alpha_1 = \widetilde{\alpha}_1 f_s$, $\tau_2 = m_2 T_s$, and $\alpha_2 = \widetilde{\alpha}_2 f_s$.

In the following, asymptotic results as $N \to \infty$ and $T_s \to 0$ are provided. Condition $N \to \infty$ needs to get consistency for the discrete-time estimator, whereas condition $T_s \to 0$ assures lack of aliasing. Note that, in order to have asymptotically an infinitely long data-record length, condition $T = (2N+1)T_s \to \infty$ needs to be verified. Consequently, in the following asymptotic results, we have that first $N \to \infty$ and then $T_s \to 0$, that is, the order of the two limits as $N \to \infty$ and $T_s \to 0$ cannot be interchanged.

In order to establish the rate of convergence to zero of the bias of the hybrid cyclic correlogram and its asymptotic Normality, as $N \to \infty$ and $T_s \to 0$, a further assumption on the lack of cluster of cycle frequencies is needed. For this purpose, let us define the set

$$\bar{A}_\tau \triangleq \left\{\beta \in \mathbb{R} : \sum_{p=-\infty}^{+\infty} R_{xx^{(*)}}(\beta + p f_s, \tau) \neq 0\right\} \quad (19a)$$

$$= \left\{\beta \in \mathbb{R} : \beta = \alpha \bmod f_s, \quad \alpha \in A_\tau\right\} \quad (19b)$$

$$= A_\tau \bmod f_s \quad (19c)$$

Assumption 5 For every τ, the cycle-frequency set $\bar{A}_\tau$ does not contain any cluster of cycle frequencies. That is, let

$$\mathbb{J}_{\alpha,\tau,T_s} \triangleq \left\{k \in \mathbb{I} : \alpha_k(\tau) = \alpha \bmod f_s,\ R^{(k)}_{xx^{(*)}}(\tau) \neq 0\right\} \quad (20)$$

then, for every $k \notin \mathbb{J}_{\alpha,\tau,T_s}$, no curve $\alpha_k(\tau)$ is such that the value $\alpha_k(\tau) \bmod f_s$ can be arbitrarily close to the cycle frequency α. Thus, for every α and τ it results

$$h_{\alpha,\tau,T_s} \triangleq \inf_{k \notin \mathbb{J}_{\alpha,\tau,T_s}} |[\alpha - \alpha_k(\tau)]\, T_s \bmod 1| > 0\,. \quad (21)$$

This assumption means that there is no cluster of lag-dependent cycle-frequency curves, where cycle frequencies are considered modulo f_s. Thus, it is stronger than [10, Assumption 4.4] made to state the rate of convergence of the bias of the *continuous-time* cyclic correlogram. A sufficient condition assuring that Assumption 5 is satisfied is that the set $\mathbb{I}$ is finite.

Let $A(f)$ be the Fourier transform of $a(t)$. It can be shown that, under Assumption 2, there exists $\gamma > 0$ such that $A(f) = \mathcal{O}(|f|^{-\gamma})$ as $|f| \to \infty$. Starting from the expression of the asymptotic expected value as $N \to \infty$, the following result can be proved [12], where the made assumptions allow to interchange the order of limit, sum, and expectation operations.

Theorem 2 Under Assumptions 1a and 2, assuming that the number of lag-dependent cycle frequencies is finite (so that also Assumptions 4 and 5 are verified), and provided that $\gamma > 1$ (or $\gamma = 1$ in the special case of $a(t) = \mathrm{rect}(t)$), it results

$$\lim_{T_s \to 0} \lim_{N \to \infty} \left[(2N+1)T_s\right]^\gamma \left| \mathrm{E}\left\{\rho_{x_{\rm d} x_{\rm d}^{(*)}}(\alpha,\tau;n_0,N,T_s)\right\} - R_{xx^{(*)}}(\alpha,\tau)\right| = \mathcal{O}(1) \quad (22)$$

where the order of the two limits cannot be interchanged. □

By expressing the covariance of the product $x_{{\rm d}\tau}(n)\, x_{\rm d}^{(*)}(n)$ in terms of second-order moments and a fourth-order cumulant, the following result can be proved [12], where the made assumptions allow to interchange the order of limit, sum, and expectation operations and the order of multiple-index sum operations.

Theorem 3 Under Assumptions 1 and 2, the asymptotic ($N \to \infty$ and $T_s \to 0$ with $NT_s \to \infty$) covariance of the hybrid cyclic correlogram is coincident with the asymptotic ($T \to \infty$) covariance of the continuous-time cyclic correlogram given in [10, eqs. (70)-(73)]. □

As an easy corollary of Theorem 2 we have that the hybrid cyclic correlogram is an asymptotically ($N \to \infty$ and $T_s \to 0$ with $NT_s \to \infty$) unbiased estimator of the continuous-time cyclic autocorrelation function. Moreover, from Theorem 3 it follows that the hybrid cyclic correlogram has asymptotically vanishing variance. Therefore, the hybrid cyclic correlogram is a *mean-square consistent* estimator of the continuous-time cyclic autocorrelation function.

The proof of the zero-mean joint complex asymptotic Normality of the random variables

$$V_i^{(N,T_s)} \triangleq \sqrt{(2N+1)T_s}\left[\rho_{y_{\rm d} x_{\rm d}^{(*)}}(\alpha_i,\tau_i;n_0,N,T_s) - R_{yx^{(*)}}(\alpha_i,\tau_i)\right] \quad (23)$$

is made in Theorem 4 showing that asymptotically ($N \to \infty$ and $T_s \to 0$ with $NT_s \to \infty$)

1) $\mathrm{cum}\{V_i^{(N,T_s)}\} \equiv \mathrm{E}\{V_i^{(N,T_s)}\} = 0$

2a) the covariance $\mathrm{cov}\{V_i^{(N,T_s)}, V_j^{(N,T_s)}\}$ is finite;

2b) the conjugate covariance $\mathrm{cov}\{V_i^{(N,T_s)}, V_j^{(N,T_s)*}\}$ is finite;

3) $\mathrm{cum}\{V_{i_1}^{(N,T_s)[*]_1}, \dots, V_{i_k}^{(N,T_s)[*]_k}\} = 0$ for $k \geqslant 3$.

with superscript $[*]_h$ denoting optional complex conjugation. Condition 1) follows from Theorem 2 on the rate of decay to zero of the bias of the hybrid cyclic correlogram. Condition 2a) is a consequence of Theorem 3 on the asymptotic covariance of the hybrid cyclic correlogram. Condition 2b) follows from an analogous result that can be proved for the asymptotic conjugate covariance. Finally, Condition 3) follows from Lemma 1 on the rate of decay to zero of the joint cumulant of the hybrid cyclic correlograms.

Lemma 1 Under Assumptions 2 and 3, for every $k \geqslant 2$ and $\varepsilon > 0$ it results that

$$\lim_{T_s \to 0} \lim_{N \to \infty} [(2N+1)T_s]^{k-1-\varepsilon} \mathrm{cum}\left\{\rho^{[*]_i}_{y_\mathrm{d} x_\mathrm{d}^{(*)}}(\alpha_i, \tau_i; n_{0i}, N, T_s),\ i = 1, \dots, k\right\} = 0 \quad (24)$$

where the order of the two limits cannot be interchanged. □

Theorem 4 Under Assumptions 1, 2, and 3 on the continuous-time process and under Assumptions 4 and 5, and provided that $\gamma > 1$ (or $\gamma = 1$ in the special case of $a(t) = \mathrm{rect}(t)$), the random variables $V_i^{(N,T_s)}$ defined in (23) are asymptotically (as $N \to \infty$ and $T_s \to 0$ with $NT_s \to \infty$) zero-mean jointly complex Normal with asymptotic covariance matrix with entries

$$\Sigma_{ij} = \lim_{T_s \to 0} \lim_{N \to \infty} \mathrm{cov}\left\{V_i^{(N,T_s)}, V_j^{(N,T_s)}\right\} \quad (25)$$

which can be shown to be coincident with those given in [10, eqs. (70)-(73)] and asymptotic conjugate covariance matrix with entries

$$\Sigma_{ij}^{(c)} = \lim_{T_s \to 0} \lim_{N \to \infty} \mathrm{cov}\left\{V_i^{(N,T_s)}, V_j^{(N,T_s)*}\right\} \quad (26)$$

which can be shown to be coincident with those given in [10, eqs. (D6)-(D9)]. □

Theorems 2, 3, and 4 show that the hybrid cyclic correlogram has the same asymptotic bias, covariance, and distribution of the continuous-time cyclic correlogram. Thus, there is non lose in asymptotic performance by carrying out discrete-time estimation instead of continuous-time estimation of the cyclic autocorrelation function.

5. CONCLUSIONS

Continuous-time GACS processes are an appropriate model to describe the output signal of some Doppler channels excited by ACS signals. The discrete-time cyclic correlogram of the ACS sequence obtained by uniformly sampling a continuous-time GACS process is proposed as an estimator of the continuous-time cyclic autocorrelation function and the hybrid cyclic correlogram is adopted to analyze its asymptotic performance as the number of data samples approaches infinity and the sampling period approaches zero, provided that the overall data-record length approaches infinity. The hybrid cyclic correlogram has been shown to be a mean-square consistent and asymptotically Normal estimator of the continuous-time cyclic autocorrelation function. Moreover, it has been shown that its asymptotic performance is the same as that of the continuous-time cyclic correlogram.

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