ADAPTIVE NOTCH SMOOTHING REVISITED

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ABSTRACT

The problem of extraction/elimination of a nonstationary sinusoidal signal from noisy measurements is considered. This problem is usually solved using adaptive notch filtering (ANF) algorithms. It is shown that the accuracy of signal estimates can be significantly increased if the results obtained from ANF are further processed using a cascade of appropriately designed filters. The resulting adaptive notch smoothing (ANS) algorithm can be employed to perform many off-line signal processing tasks, such as elimination of sinusoidal interference from a prerecorded signal. [1]

1. INTRODUCTION

Consider the problem of extraction or elimination of a nonstationary complex sinusoidal signal $s(t)$ (called cisoid) from noisy measurements $y(t)$

$$s(t) = a(t)e^{j\phi(t)}, \quad \phi(t) = \sum_{l=1}^{t} \omega(l)$$
$$y(t) = s(t) + v(t) \ . \tag{1}$$

We will assume that both the complex-valued amplitude $a(t)$ and the real-valued instantaneous frequency $\omega(t) \in (\pi, \pi]$ are slowly-varying quantities and that the measurement noise $\{v(t)\}$ is a zero-mean circular white sequence of random variables with variance σ_v^2.

Since the problem of retrieving/canceling nonstationary cisoids arises in many applications [1], it has attracted a great deal of interest in the signal processing literature over the past 30 years. In a majority of cases this problem is solved using adaptive notch filters (ANFs) – devices that track $a(t)$ and $\omega(t)$ in (1), based on the currently available observation history: $Y_-(t) = \{y(1), \dots, y(t)\}$. ANFs are causal adaptive filters, i.e. they yield estimates $\widehat{a}(t)$ and $\widehat{\omega}(t)$ that are functions of current and past measurements only. While in real-time applications causality is an obvious requirement, in many off-line processing tasks, such as elimination of a sinusoidal interference from a prerecorded signal $\{y(t), t = 1, \dots, N\}$, estimation at the instant t can be based on both past measurements $Y_-(t)$ and on a certain number of "future" data points: $Y_+(t) = \{y(t+1), \dots, y(N)\}$. When appropriately designed, such noncausal estimators, which incorporate smoothing, yield much smaller estimation errors than their causal counterparts.

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For real-valued systems/signals a simple version of an adaptive notch smoother (ANS), obtained by means of compensating estimation delays that arise in the amplitude tracking and frequency tracking loops of ANF algorithms, was proposed in [2] (for complex-valued systems/signals a simplified version of this solution was presented in [3]). The approach pursued in this paper is more sophisticated.
The algorithm that will serve as a basis for our further considerations, further referred to as a pilot ANF, is a normalized version of the ANF algorithm proposed and analyzed in [4]

$$\begin{aligned}
\widehat{f}(t) &= e^{j\widehat{\omega}(t)}\widehat{f}(t-1) \\
\varepsilon(t) &= y(t) - \widehat{a}(t-1)\widehat{f}(t) \\
\widehat{a}(t) &= \widehat{a}(t-1) + \mu \widehat{f}^*(t)\varepsilon(t) \\
\widehat{\omega}(t+1) &= \widehat{\omega}(t) - \gamma \, \mathrm{Im}\left[\frac{\varepsilon^*(t)\widehat{f}(t)}{\widehat{a}^*(t-1)}\right] \\
\widehat{s}(t) &= \widehat{a}(t)\widehat{f}(t)
\end{aligned} \tag{2}$$

where $*$ denotes complex conjugation. Tracking properties of this algorithm are determined by two user-dependent tuning coefficients: the adaptation gain $0 < \mu \ll 1$, which controls the rate of amplitude adaptation, and another adaptation gain $0 < \gamma \ll \mu$, which determines the rate of frequency adaptation.

Based on analysis of tracking properties of the pilot filter (2) we will design a cascade of post-processing filters increasing accuracy of amplitude and frequency estimation. We will show that using such multistage estimation scheme one can significantly improve efficiency of cancellation/retrieval of nonstationary sinusoidal signals buried in noise.

2. FREQUENCY SMOOTHING

To analyze frequency tracking properties of the pilot algorithm (2), we will assume that the signal amplitude is unknown but constant $a(t) = a, \forall t$, and that the instantaneous frequency $\omega(t)$ changes according to the random-walk model

$$\omega(t) = \omega(t-1) + w(t) \tag{3}$$

where $\{w(t)\}$, independent of $\{v(t)\}$, is a zero-mean white sequence of real-valued random variables with variance σ_w^2. Such model of frequency variation is often used in tracking studies as it leads to analytical results. It will allow us to reveal important features of the frequency tracking loop.

Denote by $\Delta\widehat{\omega}(t)=\widehat{\omega}(t)-\omega(t)$ the frequency estimation error. Using the approximating linear filter technique, one can show that [4]

$$\Delta\widehat{\omega}(t)=\frac{F(q^{-1})-1}{1-q^{-1}}w(t)+(1-q^{-1})F(q^{-1})z(t) \qquad (4)$$

where $z(t)=\text{Im}[f^*(t)v(t)/a]$, q^{-1} denotes the backward shift operator, $F(q^{-1})=\gamma q^{-1}/[1-(\lambda+\delta)q^{-1}+\lambda q^{-2}]$ and $\delta=1-\gamma$, $\lambda=1-\mu$. Note that, similarly as $\{v(t)\}$, $\{z(t)\}$ is a zero-mean circular white sequence with variance $\sigma_z^2=\sigma_v^2/(2|a|^2)$.

Due to mutual orthogonality of $\{w(t)\}$ and $\{z(t)\}$, the mean-squared signal estimation error can be expressed in the form

$$\text{E}\{|\Delta\widehat{\omega}(t)|^2\}\cong\frac{1}{2\pi}\int_{-\pi}^{\pi}h\left[F(e^{-j\xi})\right]d\xi \qquad (5)$$

where ξ denotes standard Fourier-domain normalized angular frequency variable,

$$h[F]=\sigma_w^2(F-1)(F^*-1)\Delta\Delta^*+\sigma_z^2\frac{HH^*}{\Delta\Delta^*}$$

and $\Delta(q^{-1})=1/(1-q^{-1})$. Using residue calculus one obtains

$$\text{E}\{|\Delta\widehat{\omega}(t)|^2\}\cong\left[\frac{\mu}{2\gamma}+\frac{1}{2\mu}\right]\sigma_w^2+\frac{\gamma^2}{2\mu}\sigma_z^2\,. \qquad (6)$$

The best tracking results can be obtained for

$$\mu_o^\omega\cong\sqrt[4]{4\kappa_\omega}\,,\quad \gamma_o^\omega\cong\sqrt{\kappa_\omega}$$

where $\kappa_\omega=\sigma_w^2/\sigma_z^2=2|a|^2\sigma_w^2/\sigma_v^2$.

To obtain a smoothed estimate of $\omega(t)$, further denoted by $\widetilde{\omega}(t)$, we will pass the estimates $\widehat{\omega}(t)$ through a noncausal filter $G(q^{-1})=\ldots+g_{-1}q^{-1}+g_0+g_1q^1+\ldots$

$$\widetilde{\omega}(t)=G(q^{-1})\widehat{\omega}(t)\,. \qquad (7)$$

The filter $G(q^{-1})$ will be designed so as to minimize the mean-squared frequency estimation error $\text{E}\{[\Delta\widetilde{\omega}(t)]^2\}$ where $\Delta\widetilde{\omega}(t)=\widetilde{\omega}(t)-\omega(t)$. Combining (4) with (7) one arrives at

$$\Delta\widetilde{\omega}(t)=\frac{X(q^{-1})-1}{1-q^{-1}}w(t)+(1-q^{-1})X(q^{-1})z(t) \qquad (8)$$

$$\text{E}\{[\Delta\widetilde{\omega}(t)]^2\}\cong\frac{1}{2\pi}\int_{-\pi}^{\pi}h\left[X(e^{-j\xi})\right]d\xi \qquad (9)$$

where

$$X(q^{-1})=F(q^{-1})G(q^{-1})\,. \qquad (10)$$

Minimization of (9) is pretty straightforward – the problem can be solved by minimizing $h[X(e^{-j\xi})]$ for every value of $\xi\in[-\pi,\pi]$. Requiring that $\partial h/\partial X^*|_{X=X_s}=0$, where $X_s(q^{-1})$ is the optimal transfer function, and using standard rules of symbolic differentiation with respect to complex variables [6], known as Wirtinger calculus, one obtains

$$X_s(q^{-1})=\frac{\kappa_\omega}{\kappa_\omega+(1-q^{-1})^2(1-q)^2} \qquad (11)$$

where $\kappa_\omega=\sigma_w^2/\sigma_z^2$. It can be shown that

$$X_s(q^{-1})=F_s(q^{-1})F_s(q) \qquad (12)$$

where $F_s(q^{-1})=F(q^{-1}|\mu=\mu_s^\omega,\gamma=\gamma_s^\omega)$ and μ_s^ω, γ_s^ω denote the optimal values of μ, γ that can be obtained by solving

$$\frac{(\gamma_s^\omega)^2}{1-\mu_s^\omega}=\kappa_\omega\,,\quad \frac{(\mu_s^\omega)^2}{2-\mu_s^\omega}=\gamma_s^\omega\,.$$

When $\kappa_\omega\ll 1$ (the slow variation condition) it holds that $1-\mu_s^\omega\cong 1$ and $2-\mu_s^\omega\cong 2$ which results in

$$\gamma_s^\omega\cong\sqrt{\kappa_\omega}=\gamma_o^\omega\,,\quad \mu_s^\omega\cong\sqrt{2\gamma_s^\omega}=\sqrt[4]{4\kappa_\omega}=\mu_o^\omega$$

and $X_s(q^{-1})\cong F_o(q^{-1})F_o(q)$, where $F_o(q^{-1})=F(q^{-1}|\mu=\mu_o^\omega,\gamma=\gamma_o^\omega)$. Hence, when the tracking algorithm (2) is optimally tuned, i.e. $F(q^{-1})=F_o(q^{-1})$, the transfer function of the optimal smoothing filter is given by $G(q^{-1})\cong X_s(q^{-1})/F_o(q^{-1})\cong F_o(q)$. Moreover, one can show that, under Gaussian assumptions imposed on $\{v(t)\}$ and $\{w(t)\}$, the mean-squared estimation error takes for $\widetilde{\omega}(t)=F_o(q)\widehat{\omega}(t)$ its minimum value known as Bayesian (or posterior) Cramér-Rao bound [8], [5].

The results presented above suggest the following form of frequency smoothing

$$\widetilde{\omega}(t)=F(q)\widehat{\omega}(t)\,. \qquad (13)$$

Since the filter $F(q)$ is anticausal, the smoothed estimate $\widetilde{\omega}(t)$ can be obtained by means of backward-time filtering of the estimates yielded by the pilot algorithm:

$$\begin{aligned}\widetilde{\omega}(t)&=(\lambda+\delta)\widetilde{\omega}(t+1)-\lambda\widetilde{\omega}(t+2)\\&\quad+\gamma\widehat{\omega}(t+1),\quad t=N-2,\ldots,1\end{aligned} \qquad (14)$$

with initial conditions set to: $\widetilde{\omega}(N)=\widehat{\omega}(N)$, $\widetilde{\omega}(N-1)=\widehat{\omega}(N-1)$.

Since the smoothing formula (13) was derived under idealized assumptions some robustness analysis is needed to confirm its usefulness under more realistic conditions, e.g. for frequency changes that are not governed by the random-walk model and/or in the presence of amplitude variations.

First, it should be noticed that the relationship (4), which is the cornerstone of the smoothing procedure, remains valid even if the sequence of one-step frequency changes $\omega(t)-\omega(t-1)=w(t)$ is not a white noise process, i.e. it holds for *arbitrary* slow frequency variations.

Second, and equally importantly, careful examination of the derivation presented in [4] shows that the relationship (4) remains approximately valid even if the signal amplitude $a(t)$

is not constant, but slowly varies with time – the only thing that should be changed in this more general case is the definition of $z(t)$: $z(t) = \mathrm{Im}[f^*(t)v(t)/a(t)]$. This observation is consistent with the known fact that the results of frequency estimation of narrow-band signals are usually pretty insensitive to the results of their amplitude estimation (but not *vice versa*!) [7].

For zero-mean measurement noise it holds that $\mathrm{E}[z(t)] = 0$, hence the relationship (4) entails

$$\mathrm{E}[\widehat{\omega}(t)|\omega(s), s \le t] \cong F(q^{-1})\omega(t) \; .$$

Since $F(q^{-1})$ is a lowpass filter with unity static gain $F(1) = 1$, when the instantaneous frequency varies slowly with time, the mean path of frequency estimates is roughly the time-delayed version of the true traqjectory

$$\mathrm{E}[\widehat{\omega}(t)|\omega(s), s \le t] \cong \omega(t - t_\omega)$$

where $t_\omega = \mathrm{int}[\mu/\gamma]$ denotes nominal (low-frequency) delay introduced by the filter $F(q^{-1})$ [3]. The lag error results in estimation bias which, especially for small adaptation gains, may severely degrade cancellation/extraction efficiency of the pilot algorithm.

The situation is different when smoothing is applied. Since the nominal delay of the filter $F(q^{-1})F(q)$ is equal to zero, one arrives at the following approximate relationship

$$\mathrm{E}[\widetilde{\omega}(t)|\omega(s), s \le t] \cong F(q^{-1})F(q)\omega(t) \cong \omega(t)$$

which shows that smoothing, in the proposed form, reduces estimation bias *irrespective* of the shape of the estimated frequency trajectory. Additionally, by the very nature of smoothing, the variance component of the mean-squared estimation error is also reduced.

3. AMPLITUDE SMOOTHING

The smoothed frequency estimates $\widetilde{\omega}(t)$ can be used to obtain more accurate estimates of signal amplitudes. A simple way of doing this, suggested in [3], is run – in addition to (2) – the following frequency-guided ANF filter

$$\begin{aligned} \widetilde{f}(t) &= e^{j\widetilde{\omega}(t)}\widetilde{f}(t-1) \\ \bar{\varepsilon}(t) &= y(t) - \bar{a}(t-1)\widetilde{f}(t) \\ \bar{a}(t) &= \bar{a}(t-1) + \mu \widetilde{f}^*(t)\bar{\varepsilon}(t) \\ \bar{s}(t) &= \bar{a}(t)\widetilde{f}(t) \; . \end{aligned} \tag{15}$$

We will analyze amplitude tracking properties of this algorithm under the assumption that the time-varying instantaneous frequencies are known exactly, i.e. $\widetilde{\omega}(t) = \omega(t), \forall t$. Even though obviously violated for the pilot algorithm (2), this assumption is approximately fulfilled (for small values of μ and for sufficiently small frequency variations) by the frequency-guided filter (15). Additionally, we will assume that signal amplitude $a(t)$ evolves according to the random-walk model

$$a(t) = a(t-1) + n(t) \tag{16}$$

where $\{n(t)\}$, independent of $\{v(t)\}$, is a zero-mean circular white sequence of complex-valued random variables with variance σ_n^2.

After setting $\widetilde{\omega}(t) \equiv \omega(t)$ in (15) one arrives at

$$\begin{aligned} f(t) &= e^{j\omega(t)}f(t-1) \\ \bar{\varepsilon}(t) &= y(t) - \bar{a}(t-1)f(t) \\ \bar{a}(t) &= \bar{a}(t-1) + \mu f^*(t)\bar{\varepsilon}(t) \\ \bar{s}(t) &= \bar{a}(t)f(t) \; . \end{aligned} \tag{17}$$

Denote by $\Delta\bar{a}(t) = \bar{a}(t) - a(t)$ and $\Delta\bar{s}(t) = \bar{s}(t) - a(t)$ the amplitude tracking and signal tracking errors, respectively. Observe that $|\Delta\bar{s}(t)| = |\Delta\bar{a}(t)|$. Combining (1) with (16) and (17), one obtains

$$\Delta\bar{a}(t) = \frac{H(q^{-1}) - 1}{1 - q^{-1}} n(t) + H(q^{-1})v(t) \tag{18}$$

where $H(q^{-1}) = \mu/1 - \lambda q^{-1}$.

Due to mutual orthogonality of $\{n(t)\}$ and $\{v(t)\}$, the mean-squared signal estimation error can be expressed in the form

$$\mathrm{E}\{|\Delta\bar{s}(t)|^2\} = \mathrm{E}\{|\Delta\bar{a}(t)|^2\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g\left[H(e^{-j\xi})\right] d\xi \tag{19}$$

where

$$g[H] = \sigma_n^2 (H-1)(H^*-1)\Delta\Delta^* + \sigma_v^2 HH^* \; . \tag{20}$$

Using residue calculus one obtains

$$\mathrm{E}\{|\Delta\bar{s}(t)|^2\} = \frac{\sigma_n^2}{2\mu} + \frac{\mu\sigma_v^2}{2} \tag{21}$$

Denote by μ_o^a the value of μ that minimizes (21). Simple calculations yield $\mu_o^a = \sigma_n/\sigma_v$.

Consider the smoothed estimate of $a(t)$

$$\widetilde{a}(t) = E(q^{-1})\bar{a}(t) \; . \tag{22}$$

where $E(q^{-1})$ is a transfer function of a linear noncausal filter. The filter $E(q^{-1})$ will be designed so as to minimize the mean-squared estimation error $\mathrm{E}\{|\Delta\widetilde{s}(t)|^2\} = \mathrm{E}\{|\Delta\widetilde{a}(t)|^2\}$ where $\Delta\widetilde{s}(t) = \widetilde{s}(t) - s(t)$, $\widetilde{s}(t) = \widetilde{a}(t)f(t)$ and $\Delta\widetilde{a}(t) = \widetilde{a}(t) - a(t)$. Combining (18) with (22), one obtains

$$\Delta\widetilde{a}(t) = \frac{Y(q^{-1}) - 1}{1 - q^{-1}} n(t) + Y(q^{-1})v(t) \tag{23}$$

$$\mathrm{E}\{|\Delta\widetilde{s}(t)|^2\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g\left[Y(e^{-j\xi})\right] d\xi \tag{24}$$

where

$$Y(q^{-1}) = H(q^{-1})E(q^{-1}) \; . \tag{25}$$

Minimization of (24) can be carried in an analogous way as minimization of (9). Requiring that $\partial g/\partial Y^*|_{Y=Y_s} = 0$, where $Y_s(q^{-1})$ is the optimal transfer function, one obtains

$$Y_s(q^{-1}) = \frac{\kappa_a}{\kappa_a + (1-q^{-1})(1-q)} \tag{26}$$

where $\kappa_a = \sigma_n^2/\sigma_v^2$. It can be checked that

$$Y_s(q^{-1}) = H_s(q^{-1})H_s(q) \tag{27}$$

where $H_s(q^{-1}) = H(q^{-1}|\mu = \mu_s^a)$ and μ_s^a – the optimal value of μ – can be obtained by solving the following equation

$$\frac{(\mu_s^a)^2}{1-\mu_s^a} = \kappa_a \, .$$

When $\kappa_a \ll 1$ (the slow variation condition) it holds that $\mu_s^a \ll 1$ and $\mu_s^a \cong \sqrt{\kappa_a} = \mu_o^a$ leading to $Y_s(q^{-1}) = H_o(q^{-1})H_o(q)$, where $H_o(q^{-1}) = H(q^{-1}|\mu = \mu_o^a)$. Therefore, according to (23), when the tracking algorithm (17) is optimally tuned, i.e. $H(q^{-1}) = H_o(q^{-1})$, transfer function of the optimal smoothing filter (21) is given by $E(q^{-1}) = H_o(q)$. Similarly as in the case of frequency tracking one can show that, under Gaussian assumptions imposed on $\{v(t)\}$ and $\{n(t)\}$, the mean-squared estimation error attains for $\widetilde{a}(t) = H_o(q)\bar{a}(t)$ the Bayesian Cramér-Rao bound.

The analytical results presented above suggest the following form of amplitude smoothing

$$\widetilde{a}(t) = H(q)\bar{a}(t) \, . \tag{28}$$

Since the filter $H(q)$, similarly as $F(q)$, is anticausal, the smoothed estimate $\widetilde{a}(t)$ can be obtained by means of backward-time filtering of the estimates yielded by the frequency-guided algorithm

$$\widetilde{a}(t) = \lambda\widetilde{a}(t+1) + \mu\bar{a}(t), \quad t = N-1,\ldots,1 \tag{29}$$

with initial condition set to: $\widetilde{a}(N) = \bar{a}(N)$.

The results of robustness analysis of the amplitude smoothing procedure resemble those obtained for frequency smoothing. Since the relationship (18) remains valid for arbitrary amplitude changes (the sequence of one-step amplitude changes $a(t) - a(t-1) = n(t)$ must not be white for (18) to hold), for zero-mean measurement noise and slow amplitude variations one arrives at

$$\mathrm{E}[\bar{a}(t)|a(s), s \le t] = H(q^{-1})a(t) \cong a(t - t_a)$$

where $t_a = \mathrm{int}[\lambda/\mu]$ is the nominal delay introduced by the lowpass filter $H(q^{-1})$.

When smoothing is applied one obtains

$$\mathrm{E}[\widetilde{a}(t)|a(s), s \le t] = H(q^{-1})H(q)a(t) \cong a(t)$$

which stems from the fact that the nominal delay of the filter $H(q^{-1})H(q)$ is zero and $H(1) = 1$. Therefore, whether optimal or not, for small adaptation gains the proposed smoothing procedure can significantly improve accuracy of amplitude estimation.

4. ADAPTIVE NOTCH SMOOTHER

After combining the results of frequency smoothing and amplitude smoothing, the smoothed estimate of the signal $s(t)$ can be obtained in the form

$$\begin{aligned} \widetilde{f}(t) &= e^{j\widetilde{\omega}(t)}\widetilde{f}(t-1) \\ \widetilde{s}(t) &= \widetilde{a}(t)\widetilde{f}(t) \, . \end{aligned} \tag{30}$$

The proposed adaptive notch smoothing algorithm is a five-step procedure, combining results yielded by the: pilot ANF algorithm (2), frequency smoother (14), frequency-guided ANF algorithm (15), amplitude smoother (29) and output filter (30). One can easily check that all algorithms mentioned above can be rewritten in an equivalent form that alleviates the need to compute the quantities $\widehat{f}(t)$, $\widehat{a}(t)$, $\bar{f}(t)$, $\bar{a}(t)$ and $\widetilde{a}(t)$. This reduces the computational burden by approximately 40% to 28 real multiplications and 1 real division per time update (1 complex multiplication is counted as an equivalent of 4 real ones). The resulting cost-optimized ANS algorithm is listed below:

pilot filter:

$$\begin{aligned} \varepsilon(t) &= y(t) - e^{j\widehat{\omega}(t)}\widehat{s}(t-1) \\ \widehat{s}(t) &= e^{j\widehat{\omega}(t)}\widehat{s}(t-1) + \mu\varepsilon(t) \\ \widehat{\omega}(t+1) &= \widehat{\omega}(t) - \gamma\,\mathrm{Im}\left[\frac{\varepsilon^*(t)e^{j\widehat{\omega}(t)}}{\widehat{s}^*(t-1)}\right] \\ t &= 1,\ldots,N \end{aligned}$$

frequency smoother:

$$\begin{aligned} \widetilde{\omega}(N) &= \widehat{\omega}(N) \\ \widetilde{\omega}(N-1) &= \widehat{\omega}(N-1) \\ \widetilde{\omega}(t) &= (\lambda+\delta)\widetilde{\omega}(t+1) - \lambda\widetilde{\omega}(t+2) \\ &\quad + \gamma\widehat{\omega}(t+1) \\ t &= N-2,\ldots,1 \end{aligned}$$

frequency-guided filter:

$$\begin{aligned} \bar{\varepsilon}(t) &= y(t) - e^{j\widetilde{\omega}(t)}\bar{s}(t-1) \\ \bar{s}(t) &= e^{j\widetilde{\omega}(t)}\bar{s}(t-1) + \mu\bar{\varepsilon}(t) \\ t &= 1,\ldots,N \end{aligned}$$

output filter:

$$\begin{aligned} \widetilde{s}(N) &= \bar{s}(N) \\ \widetilde{s}(t) &= \lambda e^{-j\widetilde{\omega}(t+1)}\widetilde{s}(t+1) + \mu\bar{s}(t) \\ t &= N-1,\ldots,1 \end{aligned}$$

Using the framework described in [3], the proposed ANS algorithm can be easily extended to the multiple frequencies case.

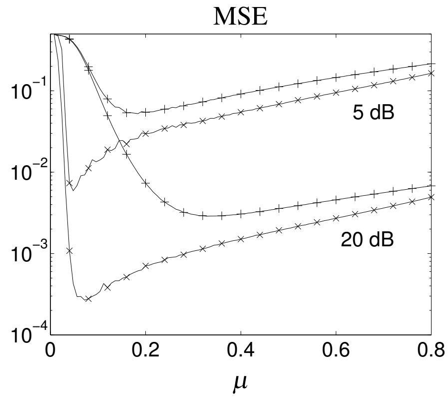

Fig. 1. Dependence of the mean-squared cancellation error on adaptation gain μ ($\gamma = \mu^2/2$) for the pilot estimate $\widehat{s}(t)$ (+) and smoothed estimate $\widetilde{s}(t)$ ($\times$), for two signal-to-noise ratios: 5 dB (two upper plots) and 20 dB (two lower plots). All plots (solid lines) were evaluated on a grid of 100 equidistant values of μ.

5. COMPUTER SIMULATIONS

To check performance of the proposed ANS algorithm a noisy quasiperiodically varying signal (1) was generated with sinusoidal amplitude and frequency changes

$$a(t) = \cos(2\pi t/2000) \, , \quad \omega(t) = \sin(2\pi t/2000) \, .$$

Fig. 1 shows comparison of the steady-state mean-squared signal estimation errors, yielded by the pilot ANF algorithm and by the proposed ANS algorithm, for different values of the adaptation gain μ and for two noise intensities: $\sigma_v = 0.56$ (SNR=5 dB) and $\sigma_v = 0.01$ (SNR=20 dB). To reduce the number of design degrees of freedom the adaptation gain γ was set to $\mu^2/2$ – see [4] for further explanations. All MSE values were obtained by means of joint time averaging (the evaluation interval [2001,4000] was placed inside a wider analysis interval [1,6000]) and ensemble averaging (100 realizations of measurement noise were used). As expected, the ANS algorithm yielded uniformly better results than the ANF algorithm. The peak-to-peak (or, more adequately, bottom-to-bottom) variance reduction is approximately equal to 10 dB.

Fig. 2 shows true signal and its estimates obtained for a typical realization of measurement noise (SNR=10 dB) in the case where $\mu = 0.08$. Close-up views of these plots are shown in Fig. 3.

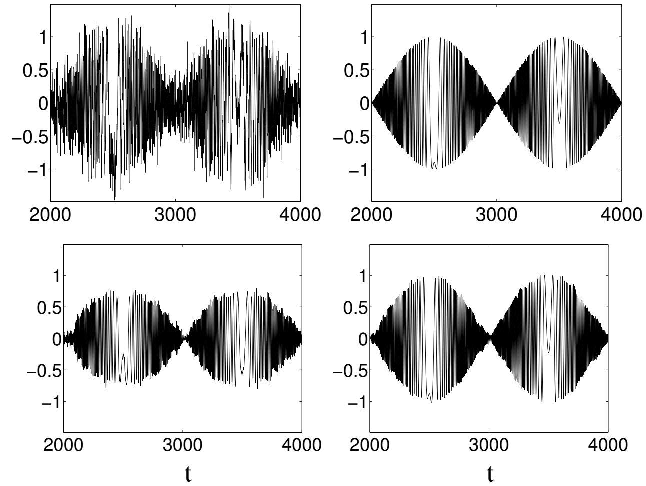

Fig. 2. Real parts of the noisy signal $y(t)$ (top left figure), noiseless signal $s(t)$ (top right figure), pilot estimate $\widehat{s}(t)$ (bottom left figure) and smoothed estimate $\widetilde{s}(t)$ (bottom right figure).

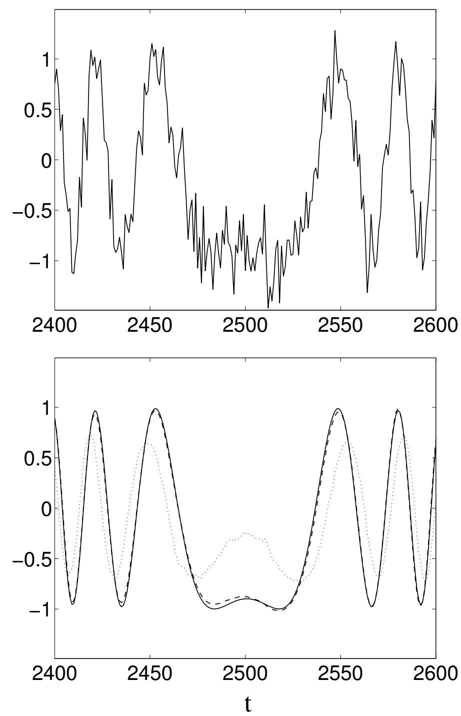

Fig. 3. Real parts of a selected fragment of the noisy signal $y(t)$ (top figure), noiseless signal $s(t)$ (bottom figure - solid line), pilot estimate $\widehat{s}(t)$ (bottom figure - dotted line) and smoothed estimate $\widetilde{s}(t)$ (bottom figure - broken line).

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