SAMPLING AND RECONSTRUCTION OF NON-BANDLIMITED SIGNALS USING SLEPIAN FUNCTIONS

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ABSTRACT

In this paper, we show that the Whittaker-Shannon (WS) sampling theory can be modified for the reconstruction of non-bandlimited signals. According to the uncertainty principle, non-bandlimited signals have finite time support and thus are more common in practical application. Prolate spheroidal wave functions also called Slepian functions have finite time support and maximum energy concentration within a given bandwidth, so instead of infinite length sinc functions, we consider Slepian functions. We show that by projecting non-bandlimited signals onto the space represented by an ortho-normal Slepian basis the minimum sampling rate can be reduced nearly by half, with no aliasing. Moreover, the reconstruction error is much lower than the one obtained by the WS theory. In some cases, depending on the desired reconstruction accuracy, it is possible to lower the rate even further. Simulations show the efficiency of the Slepian functions in the reconstruction of uniformly or non-uniformly sampled bandlimited or non-bandlimited signals.

1. INTRODUCTION

The Whittaker-Shannon (WS) sampling theory is crucial in signal processing and communications. It states that a bandlimited signal can be reconstructed from its samples as an expansion using an ortho-normal sinc basis. In practice, most signals are time-limited which means they cannot be bandlimited due to the uncertainty principle, so it is necessary to apply an antialiasing low-pass filter to satisfy the bandlimited condition.

Over the past 60 years, a great deal of research has been done to improve the WS theory. Some of the extensions of the theory have been about sampling band-pass signals [1], considering non-uniform sampling [2], and joint time-frequency methods for taking into account the non-stationarity of signals [3]. New methods, such as compressive sensing and random filtering exploit the sparseness of the signals to reduce sampling rate [4, 5, 6]. One recent approach applies wavelet structures [7] to develop new sampling and reconstruction techniques. In our contribution, we try to keep the original simplicity of the sampling theory as much as possible while obtaining satisfactory reconstructions for not necessarily bandlimited signals. One simple modification in the WS theory is to interpret the WS sampling (the anti-aliasing pre-filtering and the sampling) as an orthogonal projection operator [8]. This operator provides a minimum error band-limited approximation of a *not necessarily* bandlimited signal onto the space of bandlimited signals. The reconstructed signal, as an expansion of sinc functions, can be made as close as possible to the original signal without being the exact original signal [8]. However, the energy of the sinc function is not concentrated in time, and truncated or windowed sinc functions do not satisfy the partition of unity condition. As a result, the reconstruction error never goes to zero even if the sampling interval tends to zero, indicating that the sinc basis is not the appropriate basis for expansion.

In this paper, we will consider Slepian functions [9, 10] as orthogonal basis for sampling. These functions are time-limited and maximally concentrated within a given bandwidth which make them highly suitable for replacing the sinc basis. We will show that it is possible to reduce the sampling rate and reconstruction error both for bandlimited and non-bandlimited signals. The order of the Slepian expansion can be easily seen in the frequency domain, without considering the time-frequency representation [12]. Once the projection of the input signal onto the Slepian basis is obtained, we consider the cases of uniform and non-uniform sampling and reconstruction. When compared to the WS results, our procedure displays two distinctive advantages. First, it provides a similar structure to the WS theory with reduction in the number of samples needed for reconstruction and smaller reconstruction error with no aliasing. The other advantage is its connection with the compressive sensing and the random filtering, without the requirements of complicated algorithms for the reconstruction.

The rest of the paper is organized as follows: in Section 2, we provide a brief introduction to the Slepian functions, explain the projection method for bandlimited and non-bandlimited signals, show how to reconstruct signals from uniform or non-uniform sampling and provide an error analysis. Simulations and conclusions follow.

2. SLEPIAN BASIS FOR SAMPLING THEORY

According to the WS sampling theory, a *bandlimited* analog signal $x(t)$ with maximum frequency f_{max} (i.e.,$|X(\Omega)| = 0$ for $|\Omega| > 2\pi f_{max}$) can be sampled *uniformly*, $x(t)|_{t=nT_s}$, with sampling rate

$$f_s = \frac{1}{T_s} \geq 2f_{max}$$

and reconstructed from its samples $\{x(nT_s)\}$ by using an *ideal low-pass filter* with the sinc function as its impulse response. Thus a π band-limited signal $x(t)$,sampled uniformly

at a sampling rate of f_s, can be reconstructed exactly as

$$x(t) = \sum_{k=-\infty}^{\infty} x(kT_s)S(t-kT_s) \quad (1)$$

where $\{S(t-kT_s) = \sin(\pi(t-kT_s))/(\pi(t-kT_s))\}$ is the sinc basis used to expand $x(t)$ with coefficients $\{x(kT_s)\}$. The sinc function is well localized in frequency but has poor time decay. In the case of non-bandlimited signals, it is necessary to apply an ideal low-pass filter before sampling to suppress aliasing. A time-limited signal cannot be expressed as in Eq.(1), since a significant part of the energy in the sinc functions would be out of the signal time limits. This is a serious computational handicap for the WS sampling theory since there are no bandlimited functions that are compactly supported [8] as a result of the Paley-Wiener theorem.

The reconstruction error in the WS theory is entirely due to the out of band portion of the signal. Suppose that we choose a certain frequency Ω_M and consider two possible sources for reconstruction error: (i) Ω_M is not the maximum frequency of the analog signal $x(t)$ being sampled, (ii) there is aliasing when the signal is sampled using Ω_M as the wrong maximum frequency. If the Fourier transform of the analog signal, $x(t)$, and the reconstructed signal, $\hat{x}(t)$, are $X(\Omega)$ and $\hat{X}(\Omega)$, respectively, and E_x is the total energy of $x(t)$ then the error for case (i) is

$$\varepsilon_1 = \frac{2}{E_x}\int_{\Omega_M}^{\infty}|X(\Omega)|^2 d\Omega \quad (2)$$

and for case (ii)

$$\varepsilon_2 = \frac{1}{E_x}\int_{-\Omega_M}^{\Omega_M}|\hat{X}(\Omega)-X(\Omega)|^2 d\Omega \quad (3)$$

Then the total normalized reconstruction error would be

$$\begin{aligned}\varepsilon &= \frac{1}{E_x}\int_{-\infty}^{\infty}|x(t)-\hat{x}(t)|^2 dt \\ &= \frac{1}{E_x}\int_{-\infty}^{\infty}|X(\Omega)-\hat{X}(\Omega)|^2 d\Omega \\ &= \varepsilon_1 + \varepsilon_2 \quad (4)\end{aligned}$$

Thus, whenever the signal being sampled is band-limited and $\Omega_M = \Omega_{max}$, then all of the above errors are zero, i.e., the reconstruction is perfect. In the case of non-bandlimited signals, or when the Ω_M does not coincide with the maximum frequency of the signal there will certainly be significant reconstruction errors. A relevant question is then what if $x(t)$ is time-limited and we could project it onto a basis of time-limited functions maximally concentrated within a given bandwidth. Then the error ε_1 and the total normalized error ε would tend to zero which also results in ε_2 being zero or close to zero [12], implying satisfactory reconstruction with no aliasing effects. This corresponds to the case when the Slepian functions are used.

Derived according to the idea of maximum energy concentration, the most important property of Slepian functions is that among all the orthogonal basis defined in a time-limited domain $[-T, T]$, they have the highest energy concentration in a band of frequencies $(-W, W)$. This minimum uncertainty property in the sense of energy concentration is very
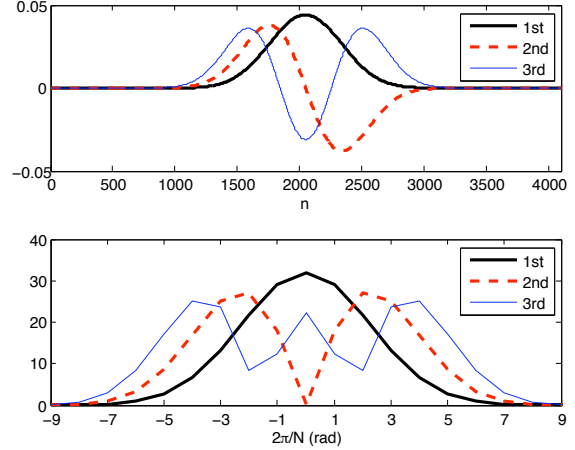

Figure 1: Slepian functions (top) and their spectra (bottom).

useful for efficient signal representation, modeling, filter design [11], theory of laser resonators, etc. Figure 1 shows the first three Slepian function and their spectra for arbitrarily chosen length $N = 4096$ and a bandwidth of $32\pi/N$. The application of Slepian functions to sampling was initially considered in [10]. In this paper, we extend the application to the non-uniform sampling of non-bandlimited signals and show that Slepian bases allow us to reduce the minimum Nyquist rate by at least half while having smaller reconstruction errors than the ones obtained using the conventional sinc functions. The Slepian functions $\{\varphi_n(t)\}$ form an orthogonal and complete set for functions in $L^2(-T, T)$, for finite as well as infinite T, that are close to bandlimited in $(-W, W)$. They are connected with the sinc function $S(t)$ as the eigenfunctions of the integral operator

$$\begin{aligned}\varphi_n(t) &= \frac{1}{\lambda_n}\int_{-T}^{T}\varphi_n(x)S(t-x)dx \\ &= \int_{-\infty}^{\infty}\varphi_n(x)S(t-x)dx\end{aligned}$$

showing double orthogonality in finite and infinite intervals. The concentration of energy in $(-W, W)$ of the eigenfunctions $\{\varphi_n(x)\}$ is given by the eigenvalues $\{\lambda_n\}$, which are ordered as $1 > \lambda_0 > \cdots > \lambda_{N-1} > 0$. For λ_0 the energy concentration is maximum.

2.1 Orthogonal Projection

In the WS theory, sampling a non-bandlimited signal $x(t)$ requires pre-filtering by a low-pass filter of impulse response $h(t)$. The combination of the pre-filtering and the sampling operations can be expressed as an inner product [8]:

$$\begin{aligned}(h*x)(t)|_{t=kT_s} &= \int x(\tau)h(kT_s-\tau)d\tau \\ &= <x, \tilde{\phi}_k >\end{aligned}$$

so that the reconstructed signal is

$$\hat{x}(t)\,|_{t=kT_s} = \sum_k <x, \tilde{\phi}_k > \tilde{\phi}_k(kT_s) \quad (5)$$

The WS sampling system, including and anti-aliasing pre-filter, can thus be interpreted as an orthogonal projection [8].

This operator computes the bandlimited approximation of a *not necessarily* band-limited signal $x(t)$. Using this approach, it will be satisfactory to obtain a reconstructed signal which is as close as possible to the original signal instead of being the exact original signal.

If the anti-aliasing filter is an ideal low-pass filter, then $h(t) = S(t)$ and Eq. (5) is an orthogonal expansion using the time-shifted sinc functions. Instead of using the sinc functions, we obtain the projection in terms of the Slepian basis, due to the reasons mentioned above. Indeed, there is an expression for the sinc function in terms of the Slepian functions [10]:

$$S(t) = \sum_{m=0}^{\infty} a_m \varphi_m(t) \tag{6}$$

The orthogonality of the Slepian functions gives the following representation for the shifted sinc

$$S(t-k) = \sum_{m=0}^{\infty} \varphi_m(k) \varphi_m(t) \tag{7}$$

where the expansion coefficients are $a_m = \varphi_n(0)$. The reconstruction formula in Eq. (1) takes the following form

$$x(t) = \sum_{k=-\infty}^{\infty} x(kT_s) \left[\sum_{m=0}^{\infty} \varphi_m(kT_s) \varphi_m(t) \right] \tag{8}$$

Assume the sampled signal is $x(t)\,|_{t=nT_s}$, where T_s is the Nyquist sampling period. If $x(t)$ is time-limited with a support $(0,T)$, then $T = (N_n - 1)T_s$, where N_n is the minimum number of samples needed for reconstruction in the WS theory. Then

$$
\begin{aligned}
x(nT_s) &= \sum_{m=0}^{\infty} \sum_{k=-\infty}^{\infty} x(kT_s) \varphi_m(kT_s) \varphi_m(nT_s) \\
&\approx \sum_{m=0}^{M-1} \underbrace{\left[\sum_{k=0}^{N_n-1} \varphi_m(kT_s) x(kT_s) \right]}_{\gamma_m} \varphi_m(nT_s)
\end{aligned}
$$

where truncation of the sum with respect to k is due to the finite length of $x(nT_s)$, and the truncation to M for the other sum is by letting the maximum discrete frequency (assuming f_{max} is the maximum frequency of $x(t)$)

$$2\pi f_{max} T_s = \pi \le \frac{2\pi}{N_n}(M-1) \tag{9}$$

or $M \ge N_n/2 + 1$.

Thus, we need only half of the samples required by the WS sampling theory to reconstruct the signal. If we let M be a value slightly larger than $N_n/2 + 1$ the reconstruction error is greatly reduced, this is due to the behavior of the eigenvalues that sharply change to zero as we increase the value of M (See Fig. 2). Likewise, there will be cases when it is possible to reduce the value of M given the sparsity of the signal [5] with respect to the Slepian basis.

2.2 Reconstruction from Uniform/Nonuniform Sampling

Although the sampling in the WS theory is uniform, i.e., the samples are taken at time nT_s, $0 \le n \le N_n - 1$, in practice the implementation of uniform sampling is not realistic.
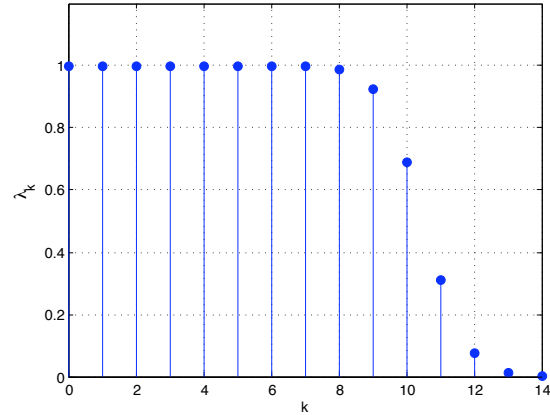

Figure 2: Eigenvalues λ_k for an $M = N_n/2 + 1 = 10$ projection.

Thus, we consider that the samples are taken at random times around the nT_s values without changing the number of total samples N_n. The samples are taken at time $t_i = icT_s + \Delta$ where $c = N_n/M$ and Δ is a random variable uniformly distributed in $[-0.5cT_s, 0.5cT_s]$, except at the two extremes of the time support, i.e., $t_0 = 0$ and $t_{N-1} = (N_n - 1)T_s$. Thus the uniform sampling is a special case of the non-uniform sampling having $\Delta = 0$ with probability of *one*.

The projection of $x(t)$ onto the space spanned by $\{\varphi_m(t)\}$ for $m=0,...,M-1$

$$x_M(t) = \sum_{m=0}^{M-1} \gamma_m \varphi_m(t)$$

makes $x_M(t)$ optimally concentrated on $[-\Omega_{max}, \Omega_{max}]$ at level M. The M sampling points of

$$\mathbf{x_M} = [x_M(t_0), x_M(t_1), ..., x_M(t_{M-1})]^T$$

can be chosen such that the locations of t_m are uniform or arbitrary. Letting the coefficient vector be

$$\gamma_{\mathbf{M}} = [\gamma_0, \gamma_1, ..., \gamma_{M-1}]^T$$

we can write the projection computed at times

$$\mathbf{t_i} = [t_0, t_1, \cdots, t_{M-1}]$$

as

$$\mathbf{x_M} = \Psi(\mathbf{t_i}) \gamma_{\mathbf{M}}$$

where $\Psi(t_i)$ is an $M \times M$ matrix

$$\Psi(\mathbf{t_i}) = \begin{bmatrix} \varphi_0(t_0) & \varphi_1(t_0) & \cdots & \varphi_{M-1}(t_0) \\ \varphi_0(t_1) & \varphi_1(t_1) & \cdots & \varphi_{M-1}(t_1) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_0(t_{M-1}) & \varphi_1(t_{M-1}) & \cdots & \varphi_{M-1}(t_{M-1}) \end{bmatrix}$$

Since there is no way to guarantee that this matrix is invertible, especially in the non-uniform sampling, the coefficients $\gamma_{\mathbf{M}}$ are obtained by using the pseudo-inverse matrix $\Theta = \Psi(\Psi\Psi^T)^{-1}$, such that

$$\gamma_{\mathbf{M}} = \Theta \mathbf{x_M} \tag{10}$$

The signal is then recovered as

$$x(t) \approx x_M(t) = \sum_{m=0}^{M-1} \gamma_m \varphi_m(t) \qquad (11)$$

Clearly, in the non-uniform sampling the matrix $\boldsymbol{\Psi}(\mathbf{t_i})$ is random, and its pseudo-inverse is also random having many zero elements. If we multiply it by a matrix $\boldsymbol{\Phi}$ formed by the Slepian basis, we will get a solution similar to the one obtained by the compressive sensing method. Likewise, it is possible to obtain a random filter from that matrix product to get similar results to the ones we may obtain by the random filtering method. We will explore these connections in another paper.

2.3 Error Analysis

As indicated before, the reconstruction error ε in the WS procedure depends on whether or not the signal is band-limited and on the aliasing caused by possible non-bandlimited conditions. In the Slepian basis approach, the value of M is chosen such that the corresponding eigenvalue is zero or close to zero, and the reconstruction error is zero or close to zero. Using the following upper bound for the reconstruction error with the Slepian basis [12],

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \leq \frac{\varepsilon_1}{1 - \lambda_{M-1}} \qquad (12)$$

where λ_{M-1} is the eigenvalue corresponding to the Slepian function $\varphi_{M-1}(t)$. If $\varepsilon = 0$ then by adjusting the value of M such that $\lambda_{M-1} = 0$ will imply $\varepsilon_2 = 0$. So if we have a time-limited signal and choose the value of M so that the condition in Eq.(12) is satisfied and the corresponding $\lambda_{M-1} = 0$, the signal will be exactly reconstructed and there will be no error due to aliasing. Furthermore, if we choose M so that $\lambda_{M-1} = 0$, then independent of whether the signal being sampled is band-limited or not, the process will not have any aliasing errors, this is because if $\lambda_{M-1} = 0$, then

$$\varepsilon_1 + \varepsilon_2 \leq \varepsilon_1$$

which can only be satisfied when $\varepsilon_2 = 0$, since these are quadratic errors.

3. SIMULATIONS

To illustrate the Slepian reconstruction we explore both uniform and non-uniform sampling of band-limited and non-bandlimited signals. We will compare our results with the ones obtained using the sinc interpolation for the WS method using the Nyquist rate. Three test signals are used: (i) sum of three sinusoids of frequencies with different amplitudes and low frequencies (ii) the same sum of sinusoids embedded in Gaussian noise, (iii) a linear frequency-modulated chirp with a known maximum frequency. The first two of these signals are band-limited with known maximum frequency, and the third test signal is approximately band-limited (See Fig. 7). From the maximum frequency we calculate the value of M as indicated above and find the corresponding Slepian projection. The reconstruction error is computed for both our method and WS.

Figure 3 shows the result of using non-uniform Slepian sampling using half the Nyquist rate, and the corresponding WS reconstruction using the Nyquist rate with the corresponding reconstruction errors. When we let $M = N_n/3$ the
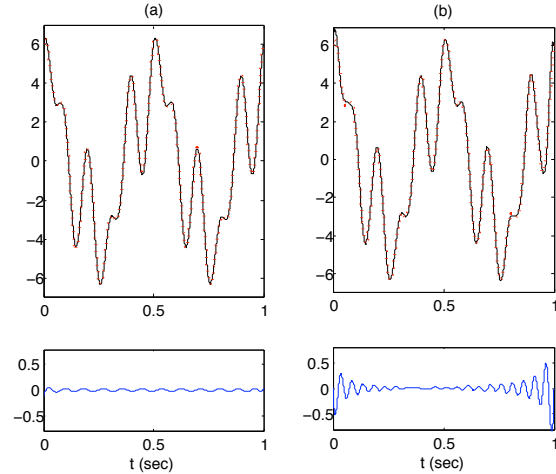

Figure 3: (a) Top: original and reconstructed signals using half Nyquist rate. Bottom: reconstruction error. (b) Top: original and WS reconstructed signals with Nyquist rate. Bottom: reconstruction error.

spectrum of the M-Slepian does not cover that of the original (See Fig. 4) and the reconstruction error becomes a version of the third sinusoid. Adding noise to the signal as well as reducing M to about $N_n/3$ increases the reconstruction error in our method compared to the WS but the reconstructed signal follows the original. Finally, applying our method to the FM signal, after choosing the value of M from the spectrum of the original and the projected signals (See Fig. 7) gives similar results for the uniform or non-uniform cases, but not for the WS as shown in Fig. 6. The effect on the reconstruction error by varying the value of M is shown in Fig. 8 for the sinusoidal and the chirp test signals.
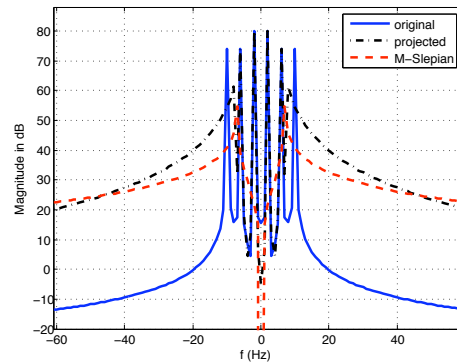

Figure 4: Spectra of the original signal and its projection, and of the M-Slepian function with $M = N_n/3$.

4. CONCLUSIONS

In this paper we showed the performance of the Slepian functions in the sampling and reconstruction of reconstruction of uniformly or non-uniformly sampled bandlimited and non-bandlimited signals. Not only Slepian basis modification of the WS system provides reduction in the number of samples with smaller reconstruction errors but also it keeps the sim-
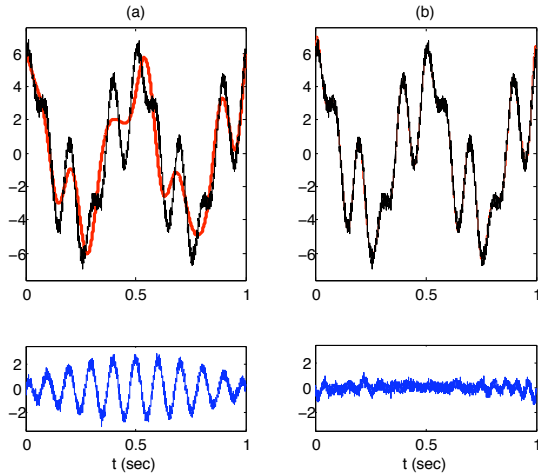
Figure 5: (a) Top: noisy and reconstructed signals using $1/3$ Nyquist rate. Bottom: reconstruction error. (b) Top: noisy and WS reconstructed signals using Nyquist rate. Bottom: reconstruction error.
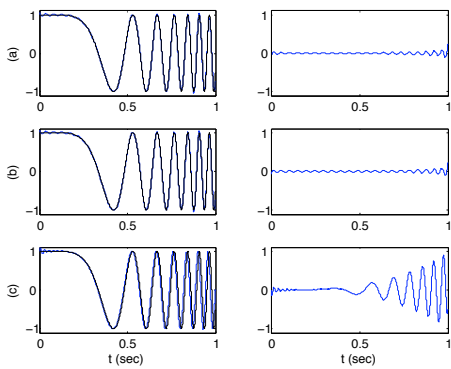

Figure 6: Linear FM chirp: Slepian reconstruction from (a) uniform sampling, (b) non-uniform sampling with corresponding errors. (c) WS reconstruction with corresponding reconstruction error.

plicity of original WS theorem. This modification is open for further development to obtain similar results to compressive sampling. Slepian functions can also be exploited for developing random filters which seem very promising due to computational matters.

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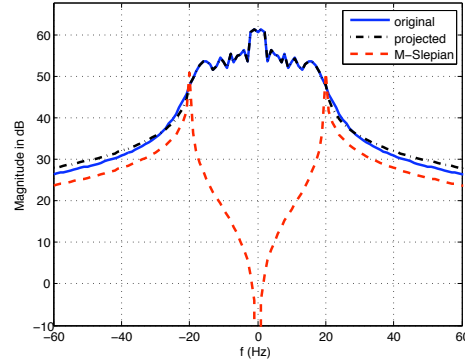[4] R. G. Baraniuk, "Compressive sensing," in *IEEE Signal Processing Magazine*, July 2007, pp. 118–121.

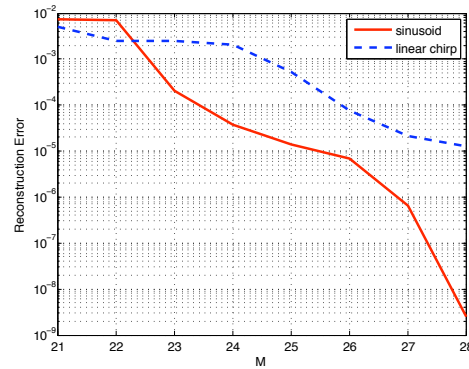Figure 7: Spectra of the linear FM chirp signal, its projection and of the M-Slepian function.

Figure 8: Reconstruction error vs. M as variable.

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