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COMPRESSIVE SENSING AND RANDOM FILTERING OF EEG SIGNALS USING SLEPIAN BASIS

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ABSTRACT

Electroencephalography (EEG) is a major tool for clinical diagnosis of neurological diseases and brain research. EEGs are often collected over numerous channels and trials, providing large data sets that require efficient collection and accurate compression. Compressive sensing and random filtering —emphasizing signal "sparseness"— enable the reconstruction of signals from a small set of measurements, at the expense of computationally complex reconstruction algorithms. In this paper we show that using Slepian functions, rather than sinc functions, in sampling reduces the minimum Nyquist sampling rate without aliasing. Assuming non-uniform sampling our procedure can be connected with compressive sensing and random filtering. EEG signals are well projected onto a Slepian basis consisting of finite-support functions, with energy optimally concentrated in a band, and related to the sinc function. Our procedure is illustrated using subdural EEG signals, with better performance than that from the conventional compressive sensing and random filtering, without the complex reconstruction of those methods.

1. INTRODUCTION

Electroencephalography (EEG) is a major tool for clinical diagnosis of neurological diseases and brain research [1, 2, 3]. EEG signals are typically collected over many channels in multiple trials, providing large sets of data that require accurate compression not only for storage but for transmission over both wired and wireless channels. For instance, the acquisition of subdural EEG by a small implant inside of the brain requires collection and transmission of data samples in situations far from ideal, and an accurate reconstruction of the data is required outside the human body. Efficient representation of EEG data is also important in brain-computer interfaces (BCI) research which concentrates on developing communication and control technology for those with severe neuromuscular disorders, brainstem stroke, and spinal cord injury.

Conventional data acquisition is based on the Whittaker-Shannon sampling theorem [4], which relies on signals being band-limited and uniformly sampled, and on the classical uncertainty principle. In practice, it is easily encountered that most signals are time-limited rather than band-limited — in fact they are non-stationary, thus requiring continuously-varying sampling rates— and that the sampling is typically done non-uniformly. According to the classical uncertainty principle, for an analog signal $x(t)$ that is essentially zero outside an interval of length Δt, and that has a Fourier transform $X(\Omega)$, also essentially zero outside an interval of length $\Delta\Omega$, the following relation holds

$$\Delta\Omega\Delta t \geq 1,$$

indicating that $x(t)$ and $X(\Omega)$ cannot both be highly concentrated. Donoho [5] recently proposed a more general principle that does not require that the signal and its Fourier transform be concentrated on intervals. According to this new uncertainty principle, it is rather the number of non-zero terms — the signal "sparseness"— in either domain that is important. Thus, if $x(t)$ is practically zero outside a set $\mathscr{T}$, of measure $|\mathscr{T}|$, and $X(\Omega)$ is practically zero outside a set $\mathscr{W}$, of measure $|\mathscr{W}|$, then

$$|\mathscr{T}||\mathscr{W}| \geq 1-\delta,$$

where δ is a small number related to the definition of "practically zero." This new uncertainty principle has significant consequences in the sampling and reconstruction of signals and has led to the new theories of compressive sensing [6, 7] and of random filtering [8].

Compressive sensing enables the reconstruction of sparse or compressible signals from a small set of measurements — much smaller than the required by the Whittaker-Shannon theory. In this paper, we will show that by using as basis the prolate spheroidal wave functions, also called Slepian functions, rather than sinc functions, it is possible to reduce by at least half the minimum sampling rate required for reconstruction by the Whittaker-Shannon theory. Sampling non-uniformly —the sampling times are chosen at random within certain boundaries— provides a random matrix and a random filter, independent of the signal being sampled, for the reconstruction. Despite the theoretical advantages of compressive sensing, it is still not suitable for real-time implementation or application to large data sets, and the reconstruction algorithm is computationally expensive. Although the random filtering approach is less complex it requires the same reconstruction algorithm as compressive sensing.

Different measurements of EEG signals can be projected to a representation with the Slepian basis, consisting of functions of finite support in the time domain and very close to band-limited in the frequency domain. The Slepian functions are optimally obtained and are connected with the sinc functions which are basic in the sampling theory [9, 10].

The rest of the paper is organized as follows. In section 2 we present a brief description of the compressive sensing and random filtering methods. In section 3, we introduce the Slepian functions and their connection with the sinc functions. The orthogonal projection of the signal onto the space spanned by the Slepian basis gives the necessary information to be used in the reconstruction process. Essentially, that at most half of the samples in the WS sampling need to be collected to reconstruct the signal without any aliasing effects. Considering non-uniform sampling, we can obtain the matrix and the random FIR filter to implement compressive sensing and random filtering. In section 4, we illustrate our procedures on EEG signals and compare our results with those from the conventional compressive sensing and random processing. Conclusions follow.

2. COMPRESSIVE SENSING AND RANDOM FILTERING

The recently introduced compressive sensing, which permits jointly sensing and compression, enables the reconstruction of sparse or compressible signals from incomplete observations. Thus a signal having a sparse representation in some basis can be reconstructed from a small set of measurements. Random filtering provides a more computationally tractable implementation in hardware or software, but uses the same reconstruction algorithms.

A real-valued signal $x(n), 0 \leq n \leq N-1$, can be projected onto basis functions $\{\phi_i(n)\}, 0 \leq n \leq N-1$, not necessarily ortho-normal, to obtain a representation

$$\hat{x}(n) = \sum_{i=0}^{K-1} \gamma_i \, \phi_i(n). \tag{1}$$

In the basis functions domain, the signal is called K-sparse if $K < N$ and $\hat{x}(n) = x(n)$. Moreover, a signal is compressible in the basis domain, if the representation in (1) has small coefficients that can be eliminated from the representation —the basic idea of transform coding.

Consider then a transformation of $\hat{x}(n)$ into a measurement signal $y(n)$ by means of a fixed $K \times N$ matrix $\mathbf{\Phi}$, independent of $\hat{x}(n)$, so that the measurements $y(n)$ are K different randomly weighted linear combinations of $\hat{x}(n)$. The measurement equation is given in matrix form as

$$\mathbf{y} = \mathbf{\Phi}\hat{\mathbf{x}} = \mathbf{\Phi}\mathbf{\Psi}\gamma = \mathbf{\Theta}\gamma, \tag{2}$$

where $\mathbf{y}$ and $\hat{\mathbf{x}}$ are vectors containing the samples of $y(n)$ and $\hat{x}(n)$. Under these conditions the reconstruction of the signal $\hat{x}(n)$ can be obtained as the solution of an optimization problem that considers the K measurements, the random matrix $\mathbf{\Phi}$ and the basis functions represented by a matrix $\mathbf{\Psi}$.

If γ is obtained from the measurements, then reconstruction of the original signal is possible. This has been approached in two different ways:

- Minimum square norm, ℓ_2: the solution is of the form

$$\hat{\gamma} = \mathbf{\Theta}^{\dagger}\mathbf{y},$$

  where $\mathbf{\Theta}^{\dagger}$ is the pseudo-inverse of $\mathbf{\Theta}$.

- Minimum absolutely summable norm, ℓ_1: the solution here is obtained by convex optimization methods [7].

The random filtering approach makes the $\mathbf{\Phi}$ matrix a banded Toeplitz matrix, where each of its rows is a shifted copy of the row above it, or equivalently the convolution with an FIR filter with random impulse response. The reconstruction still uses the ℓ_1 methods. The results are very similar for these two procedures, although the random filtering method is easier to implement. We will see in the following section that using the Slepian basis allow us to obtain the $\mathbf{\Phi}$ matrix and the random filter for the compressive sensing and the random filtering, and the reconstruction is done with less computational expense.

3. SLEPIAN SAMPLING OF EEG SIGNALS

Different types of signals are obtained when recording EEGs and it is hard to imagine that these signals are sparse in any particular domain. There have been some attempts to characterize EEG signals using non-orthogonal Gabor basis [3]. This basis is non orthogonal, with compact supports in both time and frequency. The Slepian sequences constitute a basis for signals with finite energy that are time-limited and their energy is optimally concentrated in a given bandwidth.

3.1 Slepian Functions

The Slepian functions $\{\varphi_m(t),\ 0 \leq m \leq M-1\}$, also known as Prolate Spheroidal Wave (PSW) functions, form an orthogonal and complete set for finite energy signals in $L^2(-\tau,\tau)$, for finite or infinite τ, and are optimally concentrated in a frequency band $(-W,W)$ related to τ and M [9]. These functions are connected with the sinc functions as eigenfunctions of the integral operator

$$\begin{aligned}\varphi_n(t) &= \frac{1}{\lambda_n}\int_{-\tau}^{\tau} \varphi_n(x)S(t-x)dx \\ &= \int_{-\infty}^{\infty} \varphi_n(x)S(t-x)dx,\end{aligned}$$

where $S(t)$ =sinc(t), and the eigenvalues $\{\lambda_n\}$, determining the energy of the Slepian functions, are ordered as

$$1 > \lambda_0 > \cdots > \lambda_{M-1} > 0.$$

Consider a signal $x(t)$, and assume that it is sampled according to the Whittaker-Shannon theory. If the signal is band-limited or it is processed by an antialiasing filter to make it band-limited, then the signal can be reconstructed by an interpolation using sinc functions. Assuming that the antialiasing filter is an ideal low-pass filter of normalized cutoff frequency $\Omega_c = \pi$, then

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(kT_s)S(t-kTs)$$

is a projection of $x(t)$ onto a basis of sinc functions $\{S(t-k)\}$. Although the basis is orthogonal, the energy of the sincs is not concentrated in time. Using the orthogonality of the Slepian functions in an infinite domain, it can be shown that the sinc functions are given by [10]

$$S(t-kT_s) = \sum_{n=0}^{\infty} \varphi_n(kT_s)\varphi_n(t)$$

Thus a signal $x(t)$ which is π band-limited with a *concentration interval* $(0,\tau)$, $\tau = (N_n-1)T_s$, where T_s is the Nyquist sampling period and N_n is the corresponding minimum number of samples for perfect reconstruction, can be represented as

$$\begin{aligned} x(t) &= \sum_{k=0}^{N_n-1} x(kT_s)\left[\sum_{m=0}^{\infty} \varphi_m(kT_s)\varphi_m(t)\right] \\ &\approx \sum_{m=0}^{M-1} \underbrace{\left[\sum_{k=0}^{N_n-1} \varphi_m(kT_s)x(kT_s)\right]}_{\gamma_m} \varphi_m(t) \end{aligned}$$

where the value of M is chosen so that energy of the signal

$$\begin{aligned} \int_0^{\tau} |x(t)|^2 dt &= \sum_{m=0}^{\infty} \lambda_m |\gamma_m|^2 \\ &\approx \sum_{n=0}^{M-1} \lambda_m |\gamma_m|^2 \end{aligned}$$

where we use the eigenvalues corresponding to the Slepian functions to determine the energy concentration of the signal. The above equation gives us the projection of the samples of $x(t)$ in terms of the Slepian functions, indeed setting $t = nT_s$ in the above equation we have the projection

$$x(nT_s) = \sum_{m=0}^{M-1} \gamma_m \varphi_m(nT_s) \qquad 0 \le n \le N_n - 1 \tag{3}$$

and making the maximum frequency Ω_{max} of $x(t)$ coincide with that of $\varphi_{M-1}(t)$ we have

$$\Omega_{max} T_s = \pi = \frac{2\pi}{N_n}(M-1) \tag{4}$$

or $M = 0.5N_n + 1$. Although the number of samples N_n, required by the Whittaker-Shannon theory for perfect reconstruction, is reduced in half when using Slepian sequences, it is usually necessary to increase M by a few additional samples to get a better reconstruction as then the corresponding eigenvalues become almost zero (See Fig. 1). Also, depending on the signal, it is possible to ignore some of the coefficients in the projection because they are very small, reducing further the number of samples. The computation of M this way does not require the time-frequency approach suggested in [11].

Considering $\hat{x}(t)$, the projected signal, the signal we wish to recover (as the antialiasing filtering does not permit us to recover the original signal), we can write it in the matrix form

$$\hat{\mathbf{x}} = \mathbf{\Phi}\gamma \tag{5}$$

where $\hat{\mathbf{x}}$ is a $N_n \times 1$ vector containing samples $\{x(nT_s)\}$, γ is the vector formed by the coefficients resulting from the projection with respect to the Slepian sequences. The $N_n \times M$ matrix is given by

$$\mathbf{\Phi} = \begin{bmatrix} \varphi_0(t_0) & \varphi_1(t_0) & \cdots & \varphi_{M-1}(t_0) \\ \varphi_0(t_1) & \varphi_1(t_1) & \cdots & \varphi_{M-1}(t_1) \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_0(t_{N_n-2}) & \varphi_1(t_{N_n-2}) & \cdots & \varphi_{M-1}(t_{N_n-2}) \\ \varphi_0(t_{N_n-1}) & \varphi_1(t_{N_n-1}) & \cdots & \varphi_{M-1}(t_{N_n-1}) \end{bmatrix}$$

where $t_n = nT_s$, $0 \le n \le N_n - 1$.

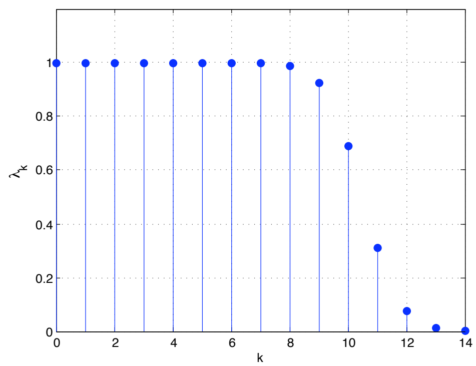

Figure 1: Eigenvalues $\{\lambda_k\}$, for the Slepian functions $\varphi_k(t)$, corresponding to the energy concentrated in their bandwidth $2\pi(M-1)/N_n$.

3.2 Non-uniform Slepian Sampling

Suppose the samples are not taken uniformly at $\{nT_s\}$, but at random times around these values, i.e., at $\hat{t}_0 = 0$ and $\hat{t}_n = cnT_s + \Delta$, $0 < n \le M-1$, with $c = N_n/M$ and Δ is a random variable uniformly distributed in $[-0.5cT_s, 0.5cT_s]$. Using the M-orthogonal expansion, the non-uniform samples can be written as

$$\begin{bmatrix} \hat{x}(\hat{t}_0) \\ \hat{x}(\hat{t}_1) \\ \vdots \\ \hat{x}(\hat{t}_{M-2}) \\ \hat{x}(\hat{t}_{M-1}) \end{bmatrix} = \begin{bmatrix} \varphi_0(\hat{t}_0) & \cdots & \varphi_{M-1}(\hat{t}_0) \\ \varphi_0(\hat{t}_1) & \cdots & \varphi_{M-1}(\hat{t}_1) \\ \vdots & \cdots & \vdots \\ \varphi_0(\hat{t}_{M-2}) & \cdots & \varphi_{M-1}(\hat{t}_{M-2}) \\ \varphi_0(\hat{t}_{M-1}) & \cdots & \varphi_{M-1}(\hat{t}_{M-1}) \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{M-2} \\ \gamma_{M-1} \end{bmatrix}$$

Writing the above equations in a matrix form as the measurement equations

$$\hat{\mathbf{x}}(\hat{\mathbf{t}}_{\mathbf{i}}) = \varphi(\hat{\mathbf{t}}_{\mathbf{i}})\gamma$$

the matrix $\varphi(\hat{\mathbf{t}}_{\mathbf{i}})$ of dimension $M \times M$ is random, given the random nature of the sampling, and independent of the signal.

3.3 Mean Square Solution

Although the matrix $\varphi(\hat{\mathbf{t}}_{\mathbf{i}})$ is a square matrix, it is not always possible to expect it to be invertible. Thus we find the coefficients of the projection using its pseudo-inverse,

$$\gamma = \left[\varphi(\hat{\mathbf{t}}_{\mathbf{i}})\right]^{\dagger} \hat{\mathbf{x}}(\hat{\mathbf{t}}_{\mathbf{i}}).$$

Then, the reconstructed signal will be given by

$$\begin{aligned} \mathbf{x_r} &= \mathbf{\Phi}\left[\varphi(\hat{\mathbf{t}}_{\mathbf{i}})\right]^{\dagger} \hat{\mathbf{x}}(\hat{\mathbf{t}}_{\mathbf{i}}) && (6) \\ &= \mathbf{\Theta}\, \hat{\mathbf{x}}(\hat{\mathbf{t}}_{\mathbf{i}}) && (7) \end{aligned}$$

where $\mathbf{\Phi}$ is the $N_n \times M$ matrix shown above. The random matrix

$$\mathbf{\Theta}^{\mathbf{T}} = [\mathbf{\Theta_1} \ \ \mathbf{\Theta_2}]$$

is a sparse matrix, with significant values around the diagonal of the $M \times M$ submatrix $\mathbf{\Theta_1}$. Such a diagonal, which is a random sequence, is used as the impulse response of the random filter. Up-sampling the measurement signal and using this random filter, the result is similar to the one obtained with a random matrix $\mathbf{\Theta}$.

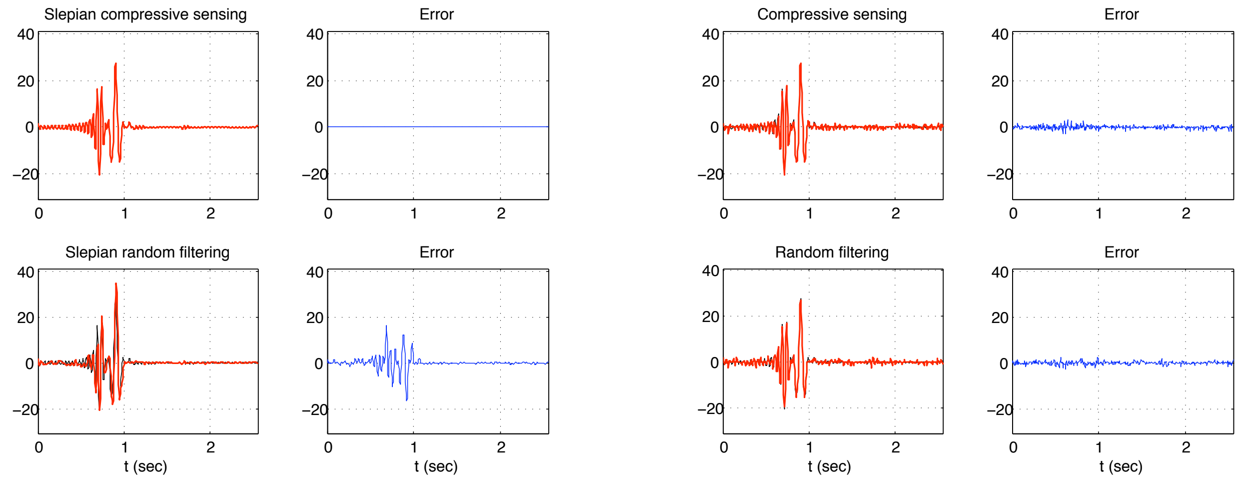

Figure 2: Subdural EEG signal (fast): Slepian compressive sensing and random filtering (left); conventional compressive sensing and random filtering (right).

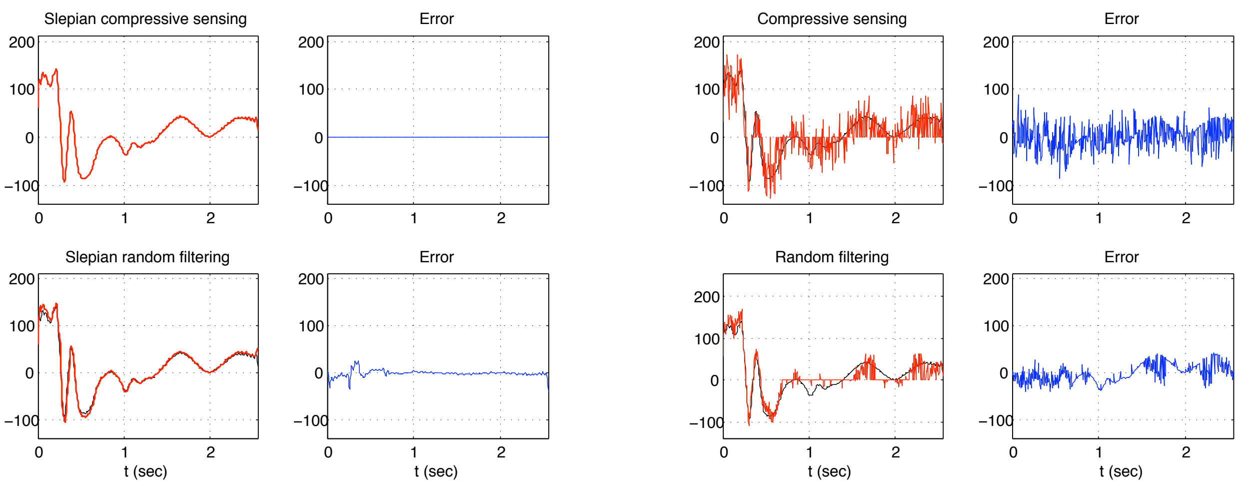

Figure 3: Subdural EEG signal (slow): Slepian compressive sensing and random filtering (left); conventional compressive sensing and random filtering (right).

4. SIMULATIONS WITH EEG SIGNALS

To illustrate the Slepian reconstruction we consider three subdural EEG signals recorded from a patient with epilepsy, and shown in Fig. 5. These signals have significant differences in smoothness, scarcity and amplitude. Windowing these signals into smaller segments, each becomes time-limited and as such the WS sampling would not apply. The orthogonal projection of each of these signals onto the Slepian basis is equivalent to filtering and sampling. The maximum frequency for the EEG is taken as $f_{max} = f_s/4 = 50$ Hz, from which we compute the minimum number of samples N_n required by the WS sampling for reconstruction. We then find the projection using $M = N_n/2+1$ Slepian functions. These values are used in our procedures as well as those using compressive sensing and random filtering.

Given the optimal energy concentration of the Slepian functions in frequency, the projected signal is also very concentrated within the bandwidth of the Slepian functions. For $M = 0.5N_n + 1$ the bandwidth of the Slepian basis equals that of the sampled signal. Due to the orthogonality of the basis the random matrix $\mathbf{\Theta}$ in Eq. (7) has a large number of zeros, i.e., highly sparse. See Fig. (6). The reconstructed signal is found directly from $\mathbf{\Theta}$, a Slepian compressive sensing, and by up-sampling and filtering with an FIR random filter obtained from the diagonal of the random matrix $\mathbf{\Theta_1}$. The results for 3 segments are shown in Figs. 2 to 4.

To compare our results with those obtained from conventional compressive sensing and random filtering for the Slepian basis, we generated an $M \times M$ random matrix $\hat{\varphi}$ and use the matrix $\mathbf{\Phi}$ from the Slepian basis. Likewise, to simulate the random filtering we generated a random FIR filter from a Gaussian random sequence. The reconstruction follows the methods suggested in [6, 7] and [8]. The results for the same signals considered above, are given in Figs. 2 to 4.

5. CONCLUSIONS

In this paper we have shown that the compressive and random filtering sampling procedures can be efficiently implemented using the Slepian basis. This is particularly true for EEG signals which can be accurately represented as a projection of

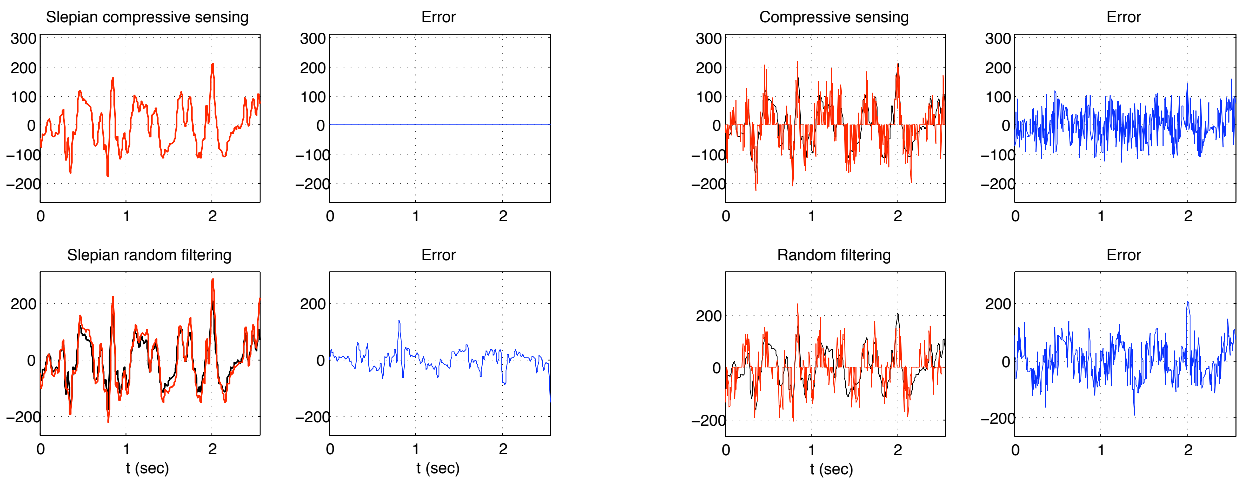

Figure 4: Subdural EEG signal (primary): Slepian compressive sensing and random filtering (left); conventional compressive sensing and random filtering (right).

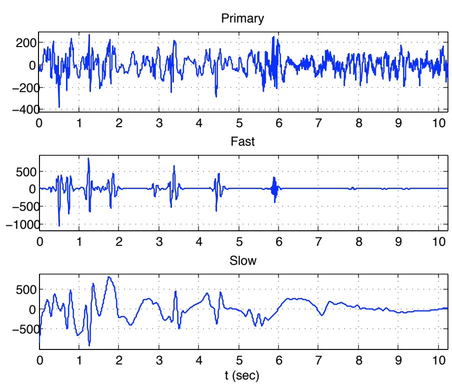

Figure 5: Three subdural EEG signals: primary (top), fast and slow.

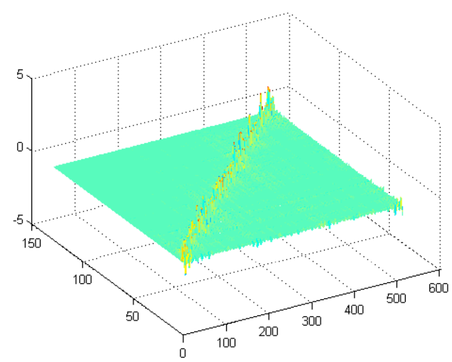

Figure 6: Slepian compressive sensing matrix Θ.

these functions, given their finite time support and optimal concentration within the bandwidth of the EEG signals. Different from the compressive sensing and the ramdom filtering, the reconstruction is efficiently done in the mean-square sense rather than using the complex optimization in the absolutely summable sense. Our approach provides better results independent of whether the signal is sparse or not, given the additional information in the measuring matrix. The compressive sensing and the random filtering methods are less noisy when reconstructing sparse signals than smooth signals. Our procedure has the advantages that it could be realized in real time to process time-limited signals —given the characteristics of the Slepian functions.

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