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GENERALIZED CONSISTENT ROBUST CAPON BEAMFORMING FOR ARBITRARILY LARGE ARRAYS

Francisco Rubio and Xavier Mestre

Centre Tecnològic de Telecomunicacions de Catalunya (CTTC)
Parc Mediterrani de la Tecnologia, Av. Canal Olímpic s/n, 08860 Castelldefels (Barcelona, Spain)
URL: http://www.cttc.es/, e-Mail:{francisco.rubio, xavier.mestre}@cttc.es

ABSTRACT

In this paper, the consistency of sample robust Capon beamforming (RCB) solutions that are constructed under signature-mismatch constraints from a set of received array observations is revised and improved. Particular emphasis is given to the class of robust filters heuristically modeling the adverse effects of practical finite sample-size conditions as due to an imperfect knowledge of the effective spatial signature. In contrast, and as in practice, a small sample-size relative to the array dimension is identified in this paper as the actual source of filter estimation errors under unknown second-order statistics. Accordingly, a new alternative random matrix theory based approach to RCB is proposed in this work that explicitly addresses both the signature-mismatch problem and the limitations due to a finite sample-size. An improved performance is demonstrated via numerical simulations in the context of source power estimation in sensor array signal processing.

1. INTRODUCTION

The Capon method has been widely applied in the literature to the problem of identifying the direction of arrival (DoA) and received power of a given number of radiating sources by using an array of passive sensors [1]. In sensor array signal processing, the power and the DoA of the signals impinging on the antenna array are obtained from the evaluation of the estimated angular power spectrum, which is calculated based on the knowledge of the array manifold and the second-order statistics of the observed samples.

Under perfect knowledge of the source spatial signature and assuming a large enough number of snapshots is available so that the true array covariance matrix can be perfectly estimated, the Capon solution is known to offer better resolution than existing data-independent methods. However, these two assumptions are quickly violated in realistic scenarios and, consequently, the filter performance is known to suffer from a severe degradation in practical implementations [2]. Regarding the first type of mismatch, the assumption of perfect knowledge of the source of interest (SOI) steering vector is usually not satisfied due to e.g. inaccuracies in the pointing information, mutual coupling and small calibration errors. In this situations, a considerable risk of SOI power underestimation is faced. Therefore, significant effort has been devoted during the past years to the problem of improving the performance of the conventional Capon solution under an imprecise knowledge of the steering signature vector. In order to specifically cope with the detrimental effects due to this problem, different robust designs particularly involving a diagonal loading factor [3] and essentially based on the mathematical theory of optimization have been recently presented in the literature (see [4]).

As for the second source of error, it is well-known that an insufficient sample-support may cause a considerable mismatch between the true and the sample covariance matrix (SCM). In order to consider the negative consequences of having a limited number of available samples, the main stream of proposed methods heuristically model the small-sample constraint as also due to spatial signature errors. However, the practical implementation of the optimal robust solution relies on the sample estimate of the unknown second-order statistics. The SCM represents a suitable approximation of the actual array covariance matrix under the assumption of a sufficiently large ratio between sample size and dimension. Hence, in practice, the major source of errors in the statistical estimation of filtering solutions can be actually identified with a low sample-size relative to the dimension of the array observation.

In this paper, we follow a similar approach as in [5] to the estimation of the optimal diagonal loading factor (see also [4, Chapter 4]) and propose a new alternative RCB design that explicitly addresses both the signature-mismatch problem and the sample-size limitations. In particular, by resorting to random matrix theory (RMT), we provide a generalization of the conventional the RCB implementation that is consistent under more practical conditions, given by a limited number of samples per observation dimension.

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2. ROBUST CAPON SPATIAL FILTERING

Consider a collection of N multivariate observations obtained by sampling across an antenna array with M sensors. Under the assumption of narrowband signals and linear antenna elements, the array observation can be additively decomposed as

$$\mathbf{y}(n) = x(n)\,\mathbf{s} + \mathbf{n}(n), \tag{1}$$

where $x(n) \in \mathbb{C}$ models the signal waveform (or fading channel coefficient) associated with a given signal of interest at the nth discrete-time instant and $\mathbf{s} \in \mathbb{C}^M$ is its spatial signature vector (also steering vector or array transfer vector). A number of K different sources are assumed to impinge on the antenna array from different directions. Accordingly, $\mathbf{n}(n) \in \mathbb{C}^M$ can be modeled as the additive contribution of the interfering sources and background noise, which can be additively decomposed as $\mathbf{n}(n) = \sum_{k=1}^{K-1} x_k(n)\,\mathbf{s}_k + \mathbf{v}(n)$, where, for $k = 1, \ldots, K-1$, $x_k(n) \in \mathbb{C}$ and $\mathbf{s}_k \in \mathbb{C}^M$ are, respectively, the interfering signal processes and associated steering signatures, and $\mathbf{v}(n) \in \mathbb{C}^M$ is the system noise and out-of-system interference. Conventionally, the signals and the noise are assumed to be independent and jointly distributed wide-sense stationary random processes, with SOI power and noise covariance given, respectively, by $\mathbb{E}[x^*(n)\,x(m)] = \sigma_x^2\delta_{m,n}$ and $\mathbb{E}\left[\mathbf{n}(m)\,\mathbf{n}^H(n)\right] = \mathbf{R_n}\delta_{m,n}$.

Consider the problem of estimating the signal waveform of a given source of interest via a linear transformation of the received observations, i.e., $\hat{x}(n) = \mathbf{w}^H\mathbf{y}(n)$. Then, from (1) and $\mathbb{E}\left[|\hat{x}(n)|^2\right] = \mathbf{w}^H\mathbf{R}\mathbf{w}$, we have

$$\sigma^2_{\mathsf{CAPON}} = \min_{\mathbf{w}\in\mathbb{C}^M} \mathbf{w}^H\mathbf{R}\mathbf{w} \quad \text{s.t. } \mathbf{w}^H\mathbf{s} = 1, \tag{2}$$

where $\mathbf{R}$ is the theoretical covariance matrix of the observed samples, i.e.,

$$\mathbf{R} = \sigma_x^2\mathbf{s}\mathbf{s}^H + \mathbf{R_n}. \tag{3}$$

The solution to (2) can be readily seen to deliver a SOI power approximant given by

$$\sigma^2_{\mathsf{CAPON}} = \frac{1}{\mathbf{s}^H\mathbf{R}^{-1}\mathbf{s}}. \tag{4}$$

The Capon SOI power estimate in (4) is well-known to significantly outperform solutions obtained from classical data-independent (phased array) beamforming methods, provided that the actual SOI spatial signature is precisely known. However, in the case that only an inaccurate version of the SOI steering vector is available, as it usually happens in practice, a relatively significant performance degradation is to be expected. In order to alleviate this problem, a number of robust adaptive beamforming techniques has been proposed in the literature that provide a generalization of the original Capon beamformer under an imprecise knowledge of the SOI spatial signature. In particular, different robust solutions have been published that extend the diagonal loading approach by providing an optimum loading level based on pressumed information about the uncertainty of the array steering vector [6, 7, 8, 9, 10, 11]. More specifically, the imperfectly known spatial signature is most often assumed to belong to an uncertainty ellipsoidal set, according to which the corresponding amount of diagonal loading is explicitely calculated.

As in [12], we are interested in the problem of estimating the power of the SOI as in (4) robustly against a mismatch in the spatial signature. Building upon a contrained-covariance-fitting direct derivation of the Capon SOI power estimate provided in [12], a robust Capon estimate is proposed in [6] that uniquely relies on the available imperfect knowledge about the steering vector and the pressumed uncertainty level. In particular, assuming that the steering vector is contained in an ellipsod described by a given positive definite matrix $\mathbf{C} \in \mathbb{C}^{M\times M}$ and centered at a nominal steering vector $\tilde{\mathbf{s}} \in \mathbb{C}^M$, the solution is given by the expression in (4), with the unknown steering vector being replaced by an optimized signature $\mathbf{s}_o \in \mathbb{C}^M$ obtained as

$$\arg\min_{\mathbf{s}\in\mathbb{C}^M} \mathbf{s}^H\mathbf{R}^{-1}\mathbf{s} \quad \text{s.t. } (\mathbf{s}-\tilde{\mathbf{s}})^H\,\mathbf{C}^{-1}(\mathbf{s}-\tilde{\mathbf{s}}) \leq 1. \tag{5}$$

In [6], the solution of the optimization problem in (5) is found for an uncertainty set $\mathbf{C} = \epsilon\mathbf{I}_M$, yielding as a constraint the sphere $\|\mathbf{s}-\tilde{\mathbf{s}}\| \leq \epsilon$, with $\epsilon \in \mathbb{R}^+$ being a given user parameter. Thus, the optimum robust steering vector is obtained as

$$\mathbf{s}_o = \left(\mathbf{I}_M - (\mathbf{I}_M + \lambda_o\mathbf{R})^{-1}\right)\tilde{\mathbf{s}}, \tag{6}$$

where the parameter λ_o is found as the real positive solution of the following equation in λ, namely,

$$g(\lambda) = \epsilon, \tag{7}$$

where we have defined

$$g(\lambda) = \tilde{\mathbf{s}}^H(\mathbf{I}_M + \lambda\mathbf{R})^{-2}\,\tilde{\mathbf{s}}.$$

Note that $g(\lambda)$ is a monotonically decreasing function of λ for $\lambda > 0$ (see [6] for further details). Hence, using the available erroneous version of the true steering vector in (6), the robust estimate of the SOI power is given by

$$\sigma^2_{\mathsf{CAPON}} = \frac{1}{\mathbf{s}_o^H\mathbf{R}^{-1}\mathbf{s}_o}. \tag{8}$$

In practice, the array observation covariance matrix is not available, and so the implementation of the RCB power estimate must necessarily rely on the SCM, i.e.,

$$\hat{\mathbf{R}} = \frac{1}{N}\sum_{n=1}^{N}\mathbf{y}(n)\,\mathbf{y}^H(n). \tag{9}$$

Since the SCM is a consistent estimator of the theoretical covariance matrix, such an approximation based on directly replacing $\mathbf{R}$ with its sample estimate will yield accurate results provided the number of samples available for computating the SCM is significantly larger than the size of the array. If these conditions hold, large sample-size statistical tools may help in characterizing the actual estimation performance. However, such an assumption hardly matches realistic operation conditions given in a practical setting where both the sample size and dimension are comparable in magnitude.

In order to further improve the power estimation performance under more realisitc finite sample-size conditions, we derive an estimator of the RCB solution in (8) that asymptotically approximates the true power estimate as both the sample and array sizes increase without boud at the same rate. Accordingly, the proposed estimator generalizes conventional implementations by proving to be consistent for an arbitrarily large observation dimension.

3. IMPROVED CONSISTENT RCB ESTIMATION

In this section, we present our generalized consistent estimator of the RCB power estimate in (8). To that effect, we make use of the Stieltjes transform from RMT, which allows us to characterize the asymptotic distribution of the eigenvalues of $\hat{\mathbf{R}}$ in terms of the limiting spectral distribution of $\mathbf{R}$. Our approach is rooted in Girko's general statistical analysis (GSA) of large-dimensional observations [13]. Specifically, we derive an estimator of the optimum parameter λ as well as the class of spectral functions of $\mathbf{R}$ defined by (8) that is consistent in the asymptotic regime described by $M, N \to \infty$, with $M/N \to c < +\infty$. We will refer to this estimators as M,N-consistent as a generalization of traditional N-consistent estimators.

First, observe that from (1) and the statistical assumptions in Section II, we can statistically model the observed samples as $\mathbf{y}(n) = \mathbf{R}^{1/2}\mathbf{u}(n)$, where $\mathbf{u}(n) \in \mathbb{C}^M$, $n = 1, \ldots, N$, is a collection of i.i.d. random vectors, whose entries have zero mean real and imaginary parts with variance $1/2$ and bounded higher moments. Thus, the SCM in (9) will be modeled in the subsequent derivations as

$$\hat{\mathbf{R}} = \frac{1}{N}\mathbf{R}^{1/2}\mathbf{U}\mathbf{U}^H\mathbf{R}^{1/2}, \tag{10}$$

where the matrix $\mathbf{U} \in \mathbb{C}^{M\times N}$ is constructed using as its columns the vectors $\mathbf{u}(n)$, $n = 1, \ldots, N$. In the sequel, given two quantities a, b, $a \asymp b$ will denote the fact that both quantities are asymptotic equivalents, i.e., $|a-b| \overset{a.s.}{\to} 0$, with $a.s.$ denoting almost sure convergence. The following two propositions provide asymptotic equivalents of the optimum parameter defining the robust steering vector in (6) and the SOI power estimate in (8), respectively, as a function of only the SCM.

Proposition 1 *Let ϵ, $\tilde{\mathbf{s}}$, $\mathbf{R}$, λ_o and $\hat{\mathbf{R}}$ be defined as above. Under the previous statistical assumptions, as $M, N \to +\infty$, with $M/N \to c < +\infty$, we have that $\lambda_o \asymp \breve{\lambda}$, where*

$$\breve{\lambda} = \frac{\breve{\eta}}{g_2(\breve{\eta})}, \tag{11}$$

and $\breve{\eta}$ is the unique solution to the following equation in η, namely, $\breve{g}(\eta) = \epsilon$, where $\breve{g}(\eta)$ is equal to

$$\frac{\eta^2\left[g_2(\eta)\,\tilde{\mathbf{s}}^H\left(\hat{\mathbf{R}} - \eta\mathbf{I}_M\right)^{-2}\tilde{\mathbf{s}} - g_1(\eta)\,\tilde{\mathbf{s}}^H\left(\hat{\mathbf{R}} - \eta\mathbf{I}_M\right)^{-1}\tilde{\mathbf{s}}\right]}{g_2(\eta) + \eta g_1(\eta)}, \tag{12}$$

where

$$\begin{aligned} g_1(\eta) &= \frac{1}{N}\operatorname{Tr}\left[\hat{\mathbf{R}}\left(\hat{\mathbf{R}} - \eta\mathbf{I}_M\right)^{-2}\right], \\ g_2(\eta) &= 1 - \frac{c}{M}\operatorname{Tr}\left[\hat{\mathbf{R}}\left(\hat{\mathbf{R}} - \eta\mathbf{I}_M\right)^{-1}\right], \end{aligned}$$

and such that

$$1 \le \eta\frac{g_1(\eta)}{g_2(\eta)}.$$

Proof. The result follows from the asymptotic characterization of the eigenvalues and eigenvectors of SCM-type matrices via the Stieltjes transform. In particular, under the assumptions of the proposition, as $M, N \to \infty$, with $M/N \to c < +\infty$, for all $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im}\{z\} > 0\}$ [14]

$$\frac{1}{M}\operatorname{Tr}\left[\left(\hat{\mathbf{R}} - z\mathbf{I}_M\right)^{-1}\right] \asymp \frac{1}{M}\operatorname{Tr}\left[\left(w(z)\,\mathbf{R} - z\mathbf{I}_M\right)^{-1}\right],$$

where $w(z) = 1 - c - czm(z)$ and $m = m(z)$ is the unique solution in the set $\{m \in \mathbb{C} : -(1-c)/z + cm \in \mathbb{C}^+\}$ to the following equation in m, namely,

$$m = \frac{1}{M}\sum_{m=1}^{M}\frac{\lambda_m(\mathbf{R})}{\lambda_m(\mathbf{R})(1 - c - czm) - z}.$$

Furthermore, for two M dimensional deterministic vectors $\mathbf{a}_1, \mathbf{a}_2$ with uniformly bounded Euclidean norm for all M [15, Theorem 1.5]

$$\mathbf{a}_1^H\left(\hat{\mathbf{R}} - z\mathbf{I}_M\right)^{-1}\mathbf{a}_2 \asymp \mathbf{a}_1^H\left(w(z)\,\mathbf{R} - z\mathbf{I}_M\right)^{-1}\mathbf{a}_2.$$

Equivalently, if we consider $f(z) = z/w(z)$, we have $w(z) \asymp \hat{w}(z)$, and $f(z) \asymp \hat{f}(z)$, where

$$\hat{w}(z) = 1 - \frac{c}{M}\sum_{m=1}^{M}\frac{\lambda_m\left(\hat{\mathbf{R}}\right)}{\lambda_m\left(\hat{\mathbf{R}}\right) - z},$$

and

$$\hat{f}(z) = \frac{z}{\hat{w}(z)}, \tag{13}$$

respectively. On the other hand, let $f(z) = -\frac{1}{\lambda}$ and write (7) as

$$f(z)^2 \, \tilde{\mathbf{s}}^H \left(\mathbf{R} - f(z)\,\mathbf{I}_M\right)^{-2} \tilde{\mathbf{s}} = \epsilon. \tag{14}$$

Furthermore, note that the identity in (14) can be written as

$$\frac{f(z)^2}{f'(z)} \frac{\partial}{\partial z} \left\{ \tilde{\mathbf{s}}^H \left(\mathbf{R} - f(z)\,\mathbf{I}_M\right)^{-1} \tilde{\mathbf{s}} \right\} = \epsilon. \tag{15}$$

Moreover, the LHS of (15) can be (asymptotically) equivalently expressed as

$$\frac{f(z)^2}{f'(z)} \frac{\partial}{\partial z} \left\{ \tilde{\mathbf{s}}^H \left(\mathbf{R} - f(z)\,\mathbf{I}_M\right)^{-1} \tilde{\mathbf{s}} \right\} \asymp \frac{\hat{f}(z)^2}{\hat{f}'(z)} \frac{\partial}{\partial z} \left\{ \hat{w}(z)\, \tilde{\mathbf{s}}^H \left(\hat{\mathbf{R}} - z\mathbf{I}_M\right)^{-1} \tilde{\mathbf{s}} \right\}.$$

The RHS of the expression above depends only on the SCM and defines an asymptotically equivalent equation (7) that can be expressed, after some analysis and defining $z = \eta$, as $\breve{g}(\eta) = \epsilon$, with $\breve{g}(\eta)$ being given by (12). Thus, the M, N-consistent estimator of the optimum parameter can be obtained by first solving for the value of η satisfying $\breve{g}(\eta) = \epsilon$, denoted by $\breve{\eta}$, and then finding an asymptotic equivalent of $-\frac{1}{f(\breve{\eta})}$ (see [15] for further details). ■

Note that the function $\breve{g}(\eta)$ is also monotonically decreasing for η smaller than the minimum eigenvalue of $\hat{\mathbf{R}}$. Moreover, regarding the robust SOI power estimate, we have

Proposition 2 *Under the assumptions and definitions above, as $M, N \to +\infty$, with $M/N \to c < +\infty$, we have that $\sigma^2_{\mathsf{CAPON}} \asymp \breve{\sigma}^2_{\mathsf{CAPON}}$, where*

$$\breve{\sigma}^2_{\mathsf{CAPON}} = \frac{\dfrac{1}{\breve{\mu}\breve{\lambda}} \left(\dfrac{g_1(\breve{\mu})}{\lambda_o} - 1 \right)}{\tilde{\mathbf{s}}^H \left(\hat{\mathbf{R}} - \breve{\mu}\mathbf{I}_M\right)^{-1} \hat{\mathbf{R}} \left(\hat{\mathbf{R}} - \breve{\mu}\mathbf{I}_M\right)^{-1} \tilde{\mathbf{s}}}, \tag{16}$$

where $\breve{\lambda}$ is given as in Proposition 1 and $\breve{\mu}$ is the solution to the following equation in μ, namely,

$$\mu = -\frac{1}{\breve{\lambda}} g_2(\mu).$$

Proof. Observe that we can write $\mathbf{s}_o^H \mathbf{R}^{-1} \mathbf{s}_o$ as

$$\tilde{\mathbf{s}}^H \left(\mathbf{R} - f(z)\,\mathbf{I}_M\right)^{-1} \mathbf{R} \left(\mathbf{R} - f(z)\,\mathbf{I}_M\right)^{-1} \tilde{\mathbf{s}}, \tag{17}$$

where we have fixed $f(z) = -\frac{1}{\lambda_o}$. Now, an asymptotic equivalent of the RHS of (17) can be found as

$$\tilde{\mathbf{s}}^H \left(\mathbf{R} - f(z)\,\mathbf{I}_M\right)^{-1} \mathbf{R} \left(\mathbf{R} - f(z)\,\mathbf{I}_M\right)^{-1} \tilde{\mathbf{s}} \asymp \frac{1}{\hat{f}'(z)} \tilde{\mathbf{s}}^H \left(\hat{\mathbf{R}} - z\mathbf{I}_M\right)^{-1} \hat{\mathbf{R}} \left(\hat{\mathbf{R}} - z\mathbf{I}_M\right)^{-1} \tilde{\mathbf{s}}.$$

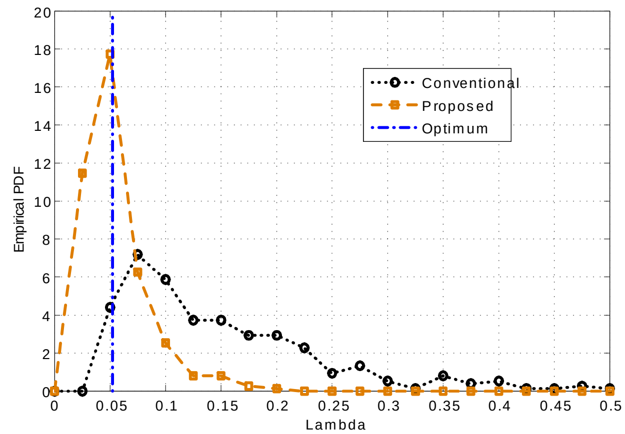

Fig. 1. Distribution of estimators of λ_o.

Then, using this asymptotic equivalence, along with $\lambda_o \asymp \breve{\lambda}$ and the derivative of (13), namely,

$$\hat{f}'(z) = \frac{1 - \hat{f}(z)\,\hat{w}'(z)}{\hat{w}(z)},$$

where

$$\hat{w}'(z) = \frac{1}{N} \operatorname{Tr}\left[\hat{\mathbf{R}} \left(\hat{\mathbf{R}} - z\mathbf{I}_M\right)^{-2} \right],$$

we obtain the result in the proposition (see [15] for further details). ■

From Proposition 2, note that the random quantity $\breve{\sigma}^2_{\mathsf{CAPON}}$ in (16) is a strongly consistent estimator of $\hat{\sigma}^2_{\mathsf{CAPON}}$.

4. NUMERICAL EVALUATIONS

In the following, we consider a typical array processing application concerning the estimation of the SOI power. Instead of relying on the availability of an accurately known SOI spatial signature, we assume that an erroneous measurement or estimate of the steering vector is available and allow for a certain degree of uncertainty level in its knowledge. In particular, we numerically compare the performance of both the *conventional* (based on the direct substitution of $\mathbf{R}$ with $\hat{\mathbf{R}}$) and our *proposed* implementations of the robust Capon power estimate in (8). Specifically, we consider a scenario consisting of $K = 5$ sources impinging on an array with $M = 30$ sensor elements from angles (degrees) $\{0, 20, 30, 50, 60\}$ and powers (dB) $\{10, 5, 30, 10, 25\}$ over the noise level $\sigma_n^2 = 1$. Moreover, a number of observed samples equal to $N = 20$ is assumed to be available for SCM computation. Note that, although $N < M$, interestingly enough, the power estimate (involving the inverse of the covariance matrix) can be realized

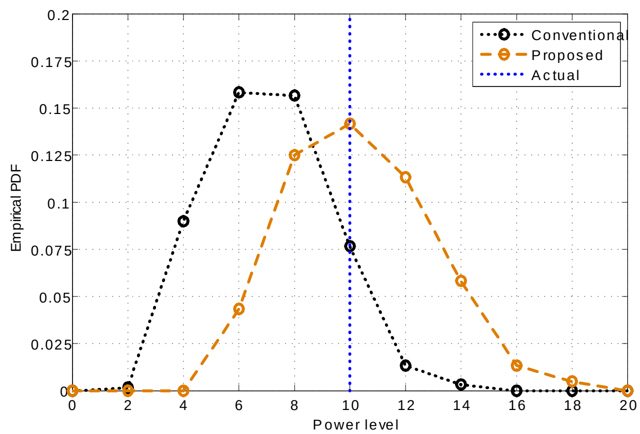

Fig. 2. Distribution of estimators of (8).

by using the matrix inversion lemma as in (17). Finally, the constant scalar defining the uncertainty level is fixed to $\epsilon = 1$.

Figure 1 shows the simulations results for the estimation of the optimum parameter λ_o. In particular, the empirical probability density function (PDF) of both conventional and proposed estimators is depicted versus the theoretical value of λ obtained by solving equation (7) using the true covariance matrix. Observe that, even in the considered adverse (undersampled) estimation conditions, an apparent nearly unbiased behavior and a lower variance can be empirically appreciated for the proposed estimator against a highly biased and variant performance of the conventional implementation.

On the other hand, in Figure 2 the normalized histograms of the robust SOI power estimate obtained via the conventional as well as the proposed methods are depicted. The actual SOI power value is also shown. As before, the conventional estimator is characterized by an empirically observed unbiased behavior against the conventional implementation.

5. CONCLUSIONS

In this paper, an alternative construction of the robust Capon power estimate has been proposed that provides an improved estimation performance under comparatively large sample size and dimension. Our approach explicitly addresses both the signature-mismatch problem and the limitations due to a finite sample-size. For our purposes, we resorted to RMT in order to derive a generalized consistent RCB estimator for arbitrarily high dimensional observations. Finally, regarding the algorithmic complexity, the leading computational constraint representing the bulk of the computation of the proposed generalized M,N-consistent estimator essentially involves implementing the EVD of the SCM. Thus, the number of required arithmetic operations is of the same order of magnitude as that corresponding to the original N-consistent RCB solution.

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