COMPUTATIONALLY EFFICIENT ONLINE PHASE-BASED FREQUENCY ESTIMATION OF A SINGLE TONE

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ABSTRACT

Computationally efficient estimation of the frequency of a single complex sinusoid in white Gaussian noise is an important problem in a wide variety of applications. A variety of low-cost batch-processing methods, the so-called phase-based frequency estimators, have been proposed during the recent decades. Although quite efficient in offline data processing, these algorithms are ill-suited for online estimation, as they require the reprocessing of the entire data with each new measurement. In this work, we develop a time-recursive phase-based estimator that is statistically efficient and computationally well-suited for online processing of data.

1. INTRODUCTION

Mathematically, the problem of single frequency estimation can be formulated as follows; given a data sequence

$$y(t) = \alpha e^{i(\omega t+\theta)} + v(t), \quad t = 0, \ldots, N-1 \quad (1)$$

where $\alpha e^{i(\omega t+\theta)}$ represents a complex sinusoid, with α and $\theta \in [0,\ 2\pi)$ denoting an unknown deterministic (and real-valued) amplitude and an unknown initial phase, respectively, corrupted by a circularly symmetric zero-mean white Gaussian noise, $v(t)$, with variance σ_v^2. Then, the objective is to find an estimate of the frequency, ω, that is computationally cheap and statistically efficient. This simple problem of rapidly estimating a single (dominant) frequency from noisy data appears in a wide variety of application areas including array signal processing, spectral estimation, digital communications and biomedicine. Consequently, one finds a rich collection of single tone frequency estimators in the recent literature (see, e.g., [1–17]). Among these, the statistically efficient methods such as the approximate maximum likelihood (AML) approach in [2] and the iterative linear prediction (ILP) approach in [13], are (relatively) computationally expensive, requiring at least $\mathcal{O}(N \log_2 N)$ operations.

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To reduce the computational complexity, several phase-based methods requiring only $\mathcal{O}(N)$ operations have been developed. Among these, the so-called hybrid method [15] is of particular interest as it has been shown to have a performance close to AML, but with a computational complexity of only $\mathcal{O}(N)$.

All the methods mentioned above, operate on block data, i.e., the total number of samples, N, is fixed. However, in many real-time applications, N may grow over time with the online arrival of new data. In such cases, it is desirable to have a recursive approach for low-cost updating of the frequency estimates. For instance, if the data size increases from N to $N+1$, the hybrid method would require $\mathcal{O}(N+1)$ operations to recompute the updated frequency estimate obtained from the N initial samples. In this work, we develop a recursive version of the hybrid algorithm that allows for computationally cheap updates of the frequency estimates as N grows, while maintaining the hybrid estimator's AML-like performance. The performance of the proposed recursive method is demonstrated with the help of numerical examples.

A word on notation: $(\cdot)^T$ and $(\cdot)^*$ are used to represent the transpose and the complex conjugate, respectively. Vectors are denoted with bold letters, $\mathbf{y}$, while scalars are in light-face, y.

2. THE HYBRID PHASE-BASED FREQUENCY ESTIMATOR

In this section, we present a brief review of the hybrid phase-based frequency estimator proposed in [15]. This will be helpful in deriving the recursive formulation in Section 3. We note, however, that for the sake of brevity only the main steps of the hybrid estimator are presented here. Interested readers are referred to [15] for more details. The hybrid method first forms an initial coarse estimate of the unknown frequency, ω, using the uniformly weighted linear predictor method [1, 4]

$$\hat{\omega}_c = \arg\left[y_c\right], \quad (2)$$

where

$$y_c = \frac{1}{N-1}\sum_{t=0}^{N-2} y^*(t)y(t+1). \qquad (3)$$

This estimate can be shown to be unbiased, with a variance [4, 7]

$$\mathrm{var}(\hat{\omega}_c) = \frac{1}{(N-1)^2\,\mathrm{SNR}}, \qquad (4)$$

where SNR is the signal-to-noise ratio, defined as $\mathrm{SNR} = \alpha^2/\sigma_v^2$. We note that for high SNR scenarios, one may use the multiple correlations approach of [18] to obtain a slightly improved coarse estimate. The coarse estimate, $\hat{\omega}_c$, is used to form a downshifted signal, $y_d(t)$, to remove the frequency dependency of the SNR threshold [14]

$$y_d(t) = y(t)e^{-i\hat{\omega}_c t}. \qquad (5)$$

Next, the downshifted signal is passed through a K-tap averaging filter to further reduce the SNR threshold [9]

$$y_f(t) = \frac{1}{K}\sum_{k=0}^{K-1} y_d(t+k). \qquad (6)$$

The adjacent phase difference of the filtered signal, $y_f(t)$, is then formed as

$$\Delta\phi_f(t) = \arg\left[y_f^*(t)y_f(t+1)\right] = \omega_f + u_c(t), \qquad (7)$$

where $u_c(t)$ is a noise process which will be coloured due to the average filtering [12]. The frequency correction term, ω_f, is then estimated from (7), using the filter bank approach of [10, 12] that takes into account the coloration of $u_c(t)$, giving

$$\hat{\omega}_f = \sum_{t=1}^{\lfloor (N-K)/K \rfloor} q(t,N)\sum_{m=1}^{K-1}\Delta\phi_f(tK-m), \qquad (8)$$

where $\lfloor\cdot\rfloor$ represents the floor operation, and

$$q(t,N) = \frac{6tK(N-tK)}{N^3 - NK^2}. \qquad (9)$$

Finally, the correction term, (8), is added to the coarse estimate, (2), to form the hybrid frequency estimate

$$\hat{\omega}_h = \hat{\omega}_c + \hat{\omega}_f. \qquad (10)$$

Since this approach uses the complete data block for the frequency estimation, we refer to it here as the *block-hybrid* estimator. It is worth stressing that the block-hybrid estimator has been shown to closely follow the Cramér-Rao lower bound (CRLB) at a low SNR threshold and to be essentially independent of the true frequency, ω. These are some features that we would like to retain in an online-hybrid estimator.

3. ONLINE HYBRID PHASE-BASED FREQUENCY ESTIMATOR

To motivate the development of an online phase-based estimator, we consider a real-time measurement scenario where a new data point is available after each sampling interval. At an arbitrary time, say $t = N-1$, the block-hybrid algorithm provides a frequency estimate, $\hat{\omega}_h^{(N)}$, based on the data vector

$$\mathbf{y}^{(N)} \triangleq \left[y(0), y(1), \ldots, y(N-1)\right]^T, \qquad (11)$$

where the superscript indicates the length of the data vector used for the estimation[1]. Assuming that a new data point, $y(N)$, is made available at the next instant, $t = N$, similar to other existing efficient algorithms, the block-hybrid algorithm would require the reprocessing of the whole data to obtain the improved frequency estimate, $\hat{\omega}_h^{(N+1)}$, making such an update computationally expensive in an online applications. To alleviate this problem, we develop a recursive hybrid phase-based frequency estimator that incorporates new data into the frequency estimation in a computationally efficient manner. To begin the recursive formulation, we note that the extended data vector containing the sample $y(N)$ may be written in terms of (11) as

$$\mathbf{y}^{(N+1)} = \begin{bmatrix} \mathbf{y}^{(N)} \\ y(N) \end{bmatrix}. \qquad (12)$$

Using (2), the extended correlator may be obtained as

$$\begin{aligned} y_c^{(N+1)} &= \frac{1}{N}\sum_{t=0}^{N-1} y^*(t)y(t+1) \\ &= \frac{N-1}{N}\left(\frac{1}{N-1}\sum_{t=0}^{N-2} y^*(t)y(t+1)\right) \\ &\quad + \frac{1}{N}y^*(N-1)y(N). \end{aligned} \qquad (13)$$

Noting that the term in the large parenthesis above is the correlation at $t = N-1$, i.e.,

$$y_c^{(N)} = \frac{1}{N-1}\sum_{t=0}^{N-2} y^*(t)y(t+1), \qquad (14)$$

we obtain the recursive formulation

$$y_c^{(N+1)} = \frac{N-1}{N}y_c^{(N)} + \frac{1}{N}y^*(N-1)y(N), \qquad (15)$$

[1] Similar notation is used for other estimates in the rest of this paper.

suggesting the updated frequency estimate

$$\hat{\omega}_c^{(N+1)} = \arg\left[y_c^{(N+1)}\right]. \quad (16)$$

The updated downshifted signal is then obtained as

$$\mathbf{y}_d^{(N+1)} = \mathbf{y}^{(N+1)} e^{-i\hat{\omega}_c^{(N+1)} t}. \quad (17)$$

Since (17) would require the updating of all the previous samples, we propose to replace $\hat{\omega}_c^{(N+1)}$ by a stable estimate $\bar{\omega}_c$. We remark that such an estimate would either be available from previously processed data, or be evaluated from (16) for a relatively small N. Therefore, we assume that a stable coarse estimate is available at $N = \bar{N}$. Using $\bar{\omega}_c$, an update equation for the downshifted signal is formed as

$$\mathbf{y}_d^{(N+1)} \approx \begin{bmatrix} \mathbf{y}^{(N)} \\ y(N) \end{bmatrix} e^{-i\bar{\omega}_c t}, \quad (18)$$

which, using $\mathbf{y}_d^{(N)} = \mathbf{y}^{(N)} e^{-i\bar{\omega}_c t}$, may be written as

$$\mathbf{y}_d^{(N+1)} \approx \begin{bmatrix} \mathbf{y}_d^{(N)} \\ y(N) e^{-i\bar{\omega}_c t} \end{bmatrix}. \quad (19)$$

The updated downshifted signal is then filtered using a K-lag averaging filter. This leads to the updated filtered data vector

$$\mathbf{y}_f^{(N+1)} = \begin{bmatrix} \mathbf{y}_f^{(N)} \\ y_f^{(N+1)}(N-K+1) \end{bmatrix}, \quad (20)$$

where $\mathbf{y}_f^{(N)}$ is the filtered signal of length $N-K+1$, obtained previously from (6) as

$$\mathbf{y}_f^{(N)} = \begin{bmatrix} \frac{1}{K}\sum_{k=0}^{K-1} y_d^{(N)}(k) \\ \vdots \\ \frac{1}{K}\sum_{k=0}^{K-1} y_d^{(N)}(N-K+k) \end{bmatrix}, \quad (21)$$

and

$$y_f^{(N+1)}(N-K+1) = \frac{1}{K}\sum_{k=0}^{K-1} y_d^{(N+1)}(N-K+1+k). \quad (22)$$

The update of the phase difference equation may now be formed as

$$\boldsymbol{\Delta\phi_f}^{(N+1)} = \begin{bmatrix} \boldsymbol{\Delta\phi_f}^{(N)} \\ \Delta\phi_f^{(N+1)}(N-K) \end{bmatrix}, \quad (23)$$

where $\boldsymbol{\Delta\phi_f}^{(N)}$ is the vector of the adjacent phase differences available from previous iteration, i.e.,

$$\boldsymbol{\Delta\phi_f}^{(N)} = \begin{bmatrix} \arg\left[y_f^{*(N)}(0) y_f^{(N)}(1)\right] \\ \vdots \\ \arg\left[y_f^{*(N)}(N-K-1) y_f^{(N)}(N-K)\right] \end{bmatrix},$$

and

$$\Delta\phi_f^{(N+1)}(N-K) = \arg\left[y_f^{*(N)}(N-K) y_f^{(N+1)}(N-K+1)\right]. \quad (24)$$

From (9)

$$q(t, N+1) = \frac{6tK(N+1-tK)}{(N+1)^3 - (N+1)K^2}, \quad (25)$$

and by defining

$$\Phi^{(N+1)}(t) = \sum_{m=1}^{K-1} \Delta\phi_f^{(N+1)}(tK-m), \quad (26)$$

we form the updated frequency correction term

$$\hat{\omega}_f^{(N+1)} = \sum_{t=1}^{\lfloor (N+1-K)/K \rfloor} q(t, N+1)\Phi^{(N+1)}(t). \quad (27)$$

Finally, we form the updated hybrid estimate by adding the correction term, (27), to $\bar{\omega}_c$

$$\hat{\omega}_h^{(N+1)} = \bar{\omega}_c + \hat{\omega}_f^{(N+1)}. \quad (28)$$

We term this approach of frequency estimation as the *online-hybrid* estimator.

4. NUMERICAL EXAMPLES

We now demonstrate the effectiveness of the proposed recursive approach using simulated data. We compare the mean square error (MSE) of the frequency estimates obtained using the online-hybrid and block-hybrid methods against the CRLB, at different SNR levels and for different selections of the actual frequency, ω. Starting with $N = 1$, an online data collection scenario is simulated, where a new sample arrives at every iteration. At every iteration, the block-hybrid estimator process the complete updated data vector. On the other hand, the online-hybrid estimator uses the low-cost update relations developed in Section 3. Figures 1, 2, and 3 show the MSE of the frequency estimates obtained using the online-hybrid and the block-hybrid estimators at SNR = 2dB, 4dB, and 6dB, respectively, using 500 Monte-Carlo simulations. The

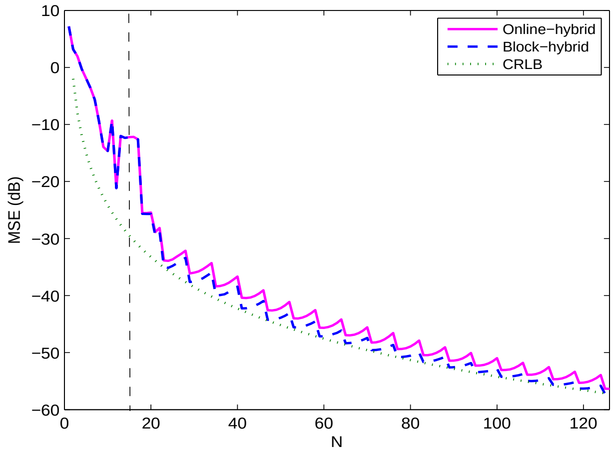

Figure 1: MSE of the frequency estimates at SNR = 2dB and $\omega = 0.75\pi$, against number of steps, N.

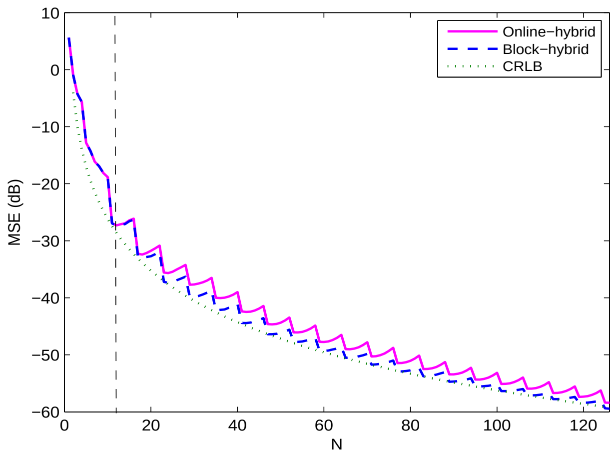

Figure 2: MSE of the frequency estimates at SNR = 4dB and $\omega = 0.75\pi$, against number of steps, N.

CRLB is also plotted for reference. The dashed vertical line shows the point $N = \bar{N}$ at which the variance of the coarse frequency estimate, $\hat{\omega}_c$, drops below a predetermined threshold, leading to a stable estimate, $\bar{\omega}_c$. Initially, for $N < \bar{N}$, the online-hybrid method works in the computationally expensive block mode, switching to the more efficient recursive estimation at $N = \bar{N}$. As is clear from the figures, the proposed recursive approach provides performance very close to the block-hybrid estimator, both following the CRLB closely. We remark that the 'steps' appearing in the figures are due to the floor operation in (8) and (27), due to which only samples up to a multiple of K are processed. Figure 4 shows the MSE of the frequency estimates at SNR = 2dB against the true frequency. As is clear from the figure, much like the

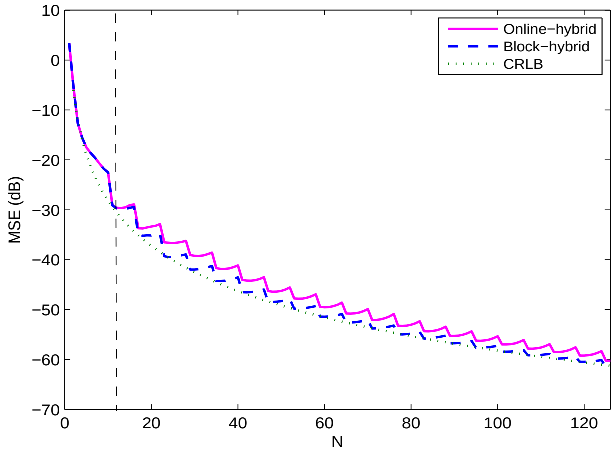

Figure 3: MSE of the frequency estimates at SNR = 6dB and $\omega = 0.75\pi$, against number of steps, N.

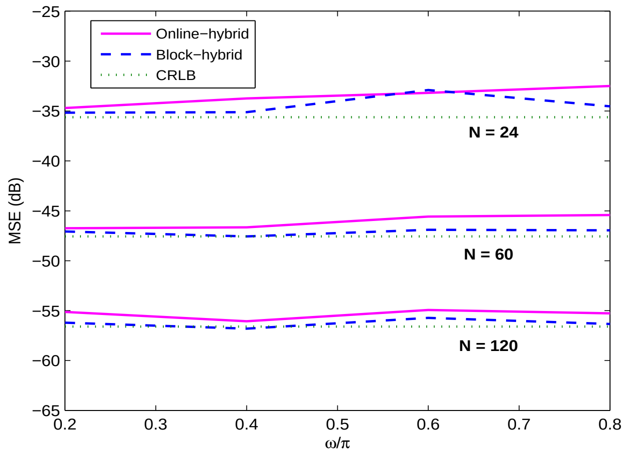

Figure 4: MSE of the frequency estimates at SNR = 2dB, and $N = 24, 60, 120$, against true sinusoidal frequency.

block-hybrid estimator, the online-hybrid estimator is essentially independent of the true sinusoidal frequency. Finally, Figure 5 compares the time taken by the block-hybrid and online-hybrid estimators to estimate the frequency as the data size increases from 1 to N. The CPU times, measured using MATLAB, for $N = 100$ to 1000 (in 10 steps) are shown. Since the recursive approach uses fewer update operations at each step, it outperforms the block-hybrid approach.

5. CONCLUSION

In this work, a time-recursive phase-based estimator that is statistically efficient and computationally well-suited for real-time processing of data was

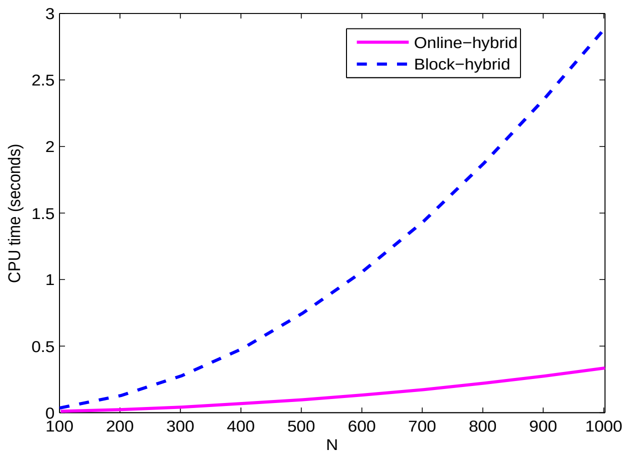

Figure 5: Processing times for the compared estimators as data increases from 1 to N.

developed. It was shown with the help of numerical examples that the proposed online-hybrid estimator provides frequency estimates quite similar to the block-hybrid algorithm, while significantly reducing the computational cost of time-updating the estimate

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