ESTIMATION QUALITY OF A WEIGHTED LEAST-SQUARE PARAMETER ESTIMATION METHOD BASED ON BINARY OBSERVATIONS

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ABSTRACT

In this paper, we investigate the quality of a weighted least-square (WLS) parameter estimation method based on binary observations when only a finite number of samples are available. An upper bound of the number of samples that are necessary for identifying system with a given accuracy is theoretically derived. The accuracy is defined in the sense of correlation coefficient between the system parameters and our estimated system parameters. Furthermore, we compare theoretical results with simulations in order to study the validity of the results practically.

1. INTRODUCTION

Today, system identification based on binary-valued output observations play an important role in many applications in different domains such as switching sensors and industry sensors in automotive applications, chemical process sensors for vacuum, pressure, and power levels; traffic condition indicators in the asynchronous transmission mode (ATM) networks; gas content sensors in gas and oil industry. In medical applications, estimation and prediction of causal effects with dichotomous outcomes are closely related to binary-valued output systems [1]. In the context of micro devices, it can be also used to estimate the parameters of MEMS and NEMS [2, 3]. Other applications can be found in [1].

In 1998, Wigren has developed a least-mean-squares (LMS) approach to the problem of online parameter estimation from quantized observations [4]. This method is based on an approximation of the quantizer, which makes it possible to define an approximate gradient of the least-squares criterion [4, 5]. In [1], another method for parameter estimation from binary (or quantized) data was introduced. The unknown system is excited by a periodic signal and, as in [6, 7]; the threshold of the quantizer is randomly specified by a partially known dithering signal. This approach is generalized in [8], where it is shown that the cumulative distribution function (cdf) of the threshold does not have to be known a priori: it can be estimated along with the parameters of the system. This work has also been extended from finite impulse response (FIR) systems to infinite impulse response (IIR) systems and to nonlinear Wiener systems [9].

In [2], we recently presented an alternative approach to estimate the parameters of a finite impulse response system using binary observations. This method relies on the minimization of a weighted least-squares (WLS) criterion where the parameter-dependent weights are chosen in order to smooth out the discontinuities of the unweighted criterion (classical criterion [10, 11]). The consistency of this approach can be guaranteed, even in the presence of measurement noise, provided the signal at the quantizer's input is Gaussian and centred. This method is also adapted to the test of microelectronic devices such as MEMS and NEMS [2, 3]. Therefore, in this paper we study the quality of this WLS approach when only a finite number of samples are available as it hasn't been already investigated.

It should be mentioned that the Cramer-Rao bound (CRB) is usually used to determine the efficiency and quality of estimation methods in previous papers [1, 8, 9]. However, the CRB is difficult to establish in our proposed approach, because there exists no analytical expression of the optimal parameters in this technique [2]. Therefore, we define another criterion to analyze the quality and performance of this method based on estimation accuracy and the number of necessary samples to identify the system in the noise-free case.

This paper is organized as follows. Section 2 introduces the framework and our WLS method to estimate the system parameters based on binary data, while Section 3 presents the theoretical results to investigate the quality of our estimation method. Section 4 resumes some simulations to study the validity of the results which are established in Section 3. Finally, conclusions and perspectives are drawn in Section 5.

2. PRELIMINARIES

2.1 Framework and notations

Let us consider a discrete-time invariant linear system H. We assume H has a finite impulse response of length L, i.e. the impulse response can be represented by a column vector $\theta = (\theta_l)_{l=1}^{L}$. Let u_l be the known scalar value of the system input at time l. We also define y_l as the (scalar) value of the system output, so that:

$$y_l = \phi_l^T \theta,$$

where $\phi_l = (u_k)_{k=l-L+1}^{l}$ is the (column) vector of observations at time l.

Let d_l be a known additive dithering signal at the quantizer's input. The system output is measured via a 1-bit ADC so that only the sign $s_l = S(z_l)$ of the system output is known,

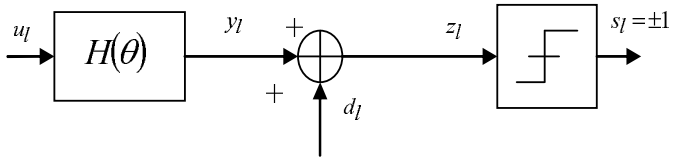

Figure 1: Block diagram of the system.

where

$$\begin{cases} S(x) = 1 & \text{, if } x \geq 0 \\ S(x) = -1 & \text{, otherwise,} \end{cases}$$

and $z_l = y_l + d_l$.

We are interested in finding an estimate $\hat{\theta}$ of θ, based on N observations of u_l, s_l and, if need be, d_l. Let us consider that u_l and d_l are stationary, independent, white Gaussian and centred. It is also assumed that $\hat{\theta}$ has length L and $\sigma_u = 1$. The estimated quantities are denoted by a hat (e.g. $\hat{s}_l = S(\hat{z}_l)$). These notations are summed up in Fig. 1.

2.2 WLS approach for parameter estimation based on binary observation

In [2], we proved that the problem of parameter estimation from binary measurement can be treated by minimizing WLS criteria of the form:

$$J_p^N(\hat{\theta}) = \frac{1}{4} \frac{\sum_{l=1}^{N} \hat{z}_l^{2p} (s_l - \hat{s}_l)^2}{\sum_{l=1}^{N} \hat{z}_l^{2p}}, p \geq 1 \tag{1}$$

in which the term $\hat{z}_l^{2p}$ acts as a (positive) weight to the binary-valued error $(s_l - \hat{s}_l)^2$ and smoothes out the discontinuities of the unweighted criterion (classical criterion).

We have already established the properties of J_0^N and J_1^N such as convexity and estimator consistency under a probabilistic framework, when N goes to infinity in [2]. Some analytical expression of J_0^∞ and J_1^∞ were also derived, which are given here:

$$J_0^\infty(\hat{\theta}) = \frac{1}{\pi} \arccos(r),$$

$$J_1^\infty(\hat{\theta}) = \frac{1}{\pi} \left(\arccos(r) - r\sqrt{1 - r^2} \right),$$

where r is the correlation coefficient of z and $\hat{z}$. In the noise-free case, this boils down to:

$$r(\theta, \hat{\theta}) = \frac{\sigma_d^2 + \theta^T \hat{\theta}}{\sqrt{\sigma_d^2 + \theta^T \theta} \sqrt{\sigma_d^2 + \hat{\theta}^T \hat{\theta}}}. \tag{2}$$

In addition, it has been shown that:

$$J_0^N = 0 \Longleftrightarrow J_p^N = 0, \forall p \geq 1, \tag{3}$$

i.e. that J_0^N and J_p^N are equivalent in the noise-free case [2] in the sense that all the $\hat{\theta}$ that minimize J_0^N also minimize J_p^N and vice versa. Fig. 2 shows criterion J_0^N comparing with J_1^N for $N = 500$ which can illustrate (3). As it's seen in this figure, all the $\hat{\theta}$ that minimize J_0^N also minimize J_p^N ($p = 1$ in this example). Based on this equivalence between the two criteria, we establish in the next section some non-asymptotical properties of J_0^N and see how they apply to J_p^N in order to investigate the estimation quality and performance efficiency of our WLS criteria.

3. CONSEQUENCE OF A FINITE NUMBER OF SAMPLES

In order to investigate the quality of our WLS approach for parameter estimation based on binary observation introduced in 2.2 [2], the relation between accuracy (in the sense of correlation coefficient between z and $\hat{z}$) and the number of necessary samples for identifying a system is figured out.

The purpose of this section is to determine how many samples N are necessary to estimate a given system with length L by a given "accuracy". To ensure the quality of the estimation, a sufficient condition is that:

$$\forall \hat{\theta}, J_p^N(\hat{\theta}) \approx J_p^\infty(\hat{\theta}).$$

Or, in other words, a sufficient condition is that we are "close" to the limiting case while N goes to infinity ($N \longrightarrow \infty$). Regarding $J_p^N(\hat{\theta})$ with a fixed $\hat{\theta}$ as a random variable (the value of which changes from one experiment to the other), one can consider that the number of necessary samples (N) is large enough when:

$$\frac{\text{var}\left(J_p^N(\hat{\theta})\right)}{\text{E}\left(J_p^N(\hat{\theta})\right)^2} < 1. \tag{4}$$

We have not been able to obtain a satisfactory expression for any of these quantities except in the case $p = 0$. However, because of the "equivalence" between J_p^N and J_0^N (3), reasonably good results can be expected if J_0^N is used instead of J_p^N in (4) . This is motivated by the fact that is mentioned in the previous section (3). Therefore, from (3) and (4), we can consider that N is large enough when:

$$\frac{\text{var}\left(J_0^N(\hat{\theta})\right)}{\text{E}\left(J_0^N(\hat{\theta})\right)^2} < 1, \tag{5}$$

As it is illustrated in Fig. 3, when N is small, the probability that $J_0^N = 0$ for $\hat{\theta} \neq \theta$ is non zero. Increasing N reduces the variance of J_0^N which reduces the misestimating probability of θ.

Since $1/4(s_l - \hat{s}_l)^2$ takes only two values (0 or 1), it can

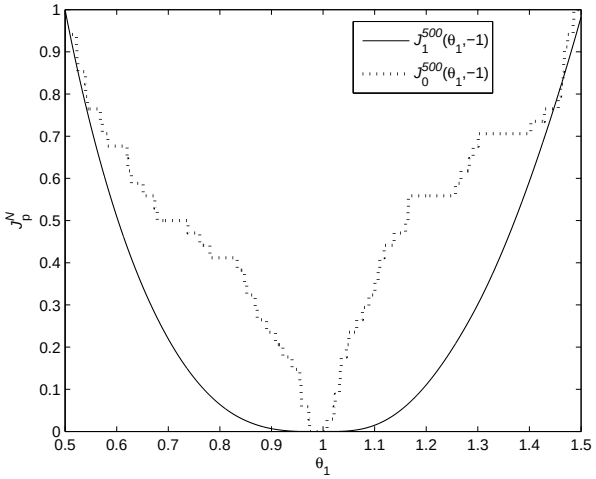

Figure 2: Comparison of J_0^N with J_1^N in order to verify (3) for $N = 500$ and $\theta = [1,-1]$.

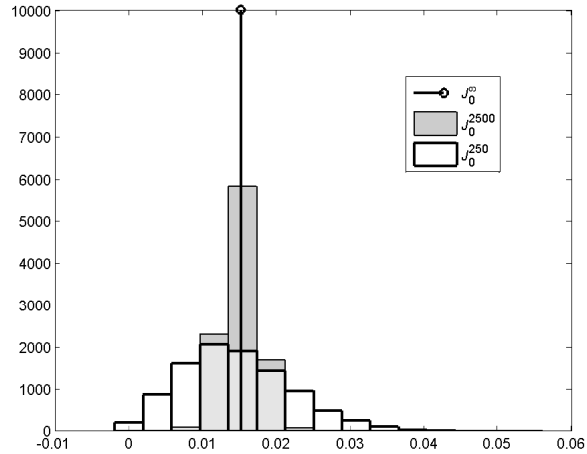

Figure 3: Histogram of J_0^∞, J_0^{2500} and J_0^{250} for $\theta = [1,-1]$ and $\hat{\theta} = [1,1,-1]$.

be considered as a Bernoulli random variable with parameter q. The value of q is equal to:

$$\begin{aligned} q = \mathrm{E}\left(\frac{1}{4}\left(\hat{s}_l\left(\hat{\theta}\right) - s_l\right)^2\right) &= \mathrm{E}\left(\frac{1}{2} - \frac{1}{2}\hat{s}_l s_l\right) \\ &= \frac{1}{2} - \frac{1}{2}\mathrm{cov}\left(\hat{s}_l, s_l\right). \end{aligned} \tag{6}$$

From [12], (6) reduces to:

$$\begin{aligned} q &= \frac{1}{2}\left(1 - \frac{2}{\pi}\arcsin\left(\mathrm{cov}\left(\hat{z}_k, z_k\right)\right)\right) \\ &= \frac{1}{\pi}\arccos(r) = J_0^\infty(r), \end{aligned}$$

thus,

$$\mathrm{E}\left(J_0^N\left(\hat{\theta}\right)\right) = \frac{1}{N}\sum_{l=1}^{N}\mathrm{E}\left(\frac{1}{4}\left(\hat{s}_l\left(\hat{\theta}\right) - s_l\right)^2\right) = J_0^\infty(r).$$

The numerator of (5) is also given by:

$$\begin{aligned} \mathrm{var}\left(J_0^N\left(\hat{\theta}\right)\right) &= \frac{1}{N^2}\mathrm{var}\left(\sum_{l=1}^{N}\frac{\left(\hat{s}_l\left(\hat{\theta}\right) - s_l\right)^2}{4}\right) \\ &= \frac{1}{N^2}\mathrm{var}\left(\sum_{l=1}^{N}\Delta(l)\right). \end{aligned} \tag{7}$$

Expanding the right-hand side of (7) leads to:

$$\begin{aligned} \mathrm{var}\left(J_0^N\left(\hat{\theta}\right)\right) &= \frac{1}{N^2}\sum_{k=1}^{N}\mathrm{var}\left(\Delta(k)\right) \\ &\quad + \frac{1}{N^2}\sum_{k=1}^{N}\sum_{l=1, l\neq k}^{N}\mathrm{cov}\left(\Delta(k), \Delta(l)\right). \end{aligned}$$

Because of the stationarity hypothesis, this can be further transformed into:

$$\begin{aligned} \mathrm{var}\left(J_0^N\left(\hat{\theta}\right)\right) &= \frac{1}{N^2}\left(N\mathrm{var}\left(\Delta(t)\right)\right) \\ &\quad + \frac{2}{N^2}\sum_{k=1}^{N-1}(N-k)\,\mathrm{cov}\left(\Delta(t), \Delta(t+k)\right). \end{aligned} \tag{8}$$

The second term on the right-hand side is split in two parts:

$$\begin{aligned} &\sum_{k=1}^{N-1}(N-k)\mathrm{cov}\left(\Delta(t), \Delta(t+k)\right) = \\ &\sum_{k=1}^{L}(N-k)\mathrm{cov}\left(\Delta(t), \Delta(t+k)\right) + \\ &\sum_{k=L+1}^{N-1}(N-k)\mathrm{cov}\left(\Delta(t), \Delta(t+k)\right). \end{aligned} \tag{9}$$

Since H has length L and the input signal u is white, the second term of the right-hand side of (9) equals 0. The Cauchy-Schwartz inequality is then used to yield:

$$\begin{aligned} &\sum_{k=1}^{L}(N-k)\mathrm{cov}\left(\Delta(t), \Delta(t+k)\right) \leq \\ &\sum_{k=1}^{L}(N-k)\mathrm{var}\left(\Delta(t)\right) = L\left(N - \frac{L+1}{2}\right)\mathrm{var}\left(\Delta(t)\right). \end{aligned}$$

This can be injected into (8):

$$\mathrm{var}\left(J_0^N\left(\hat{\theta}\right)\right) \leq \frac{N(2L+1) - L(L+1)}{N^2}\mathrm{var}\left(\Delta(t)\right). \tag{10}$$

Now $\mathrm{var}\left(\Delta(t)\right)$ can be split into:

$$\mathrm{var}\left(\Delta(t)\right) = \mathrm{var}\left(\frac{1}{4}\left(\hat{s}_l\left(\hat{\theta}\right) - s_l\right)^2\right).$$

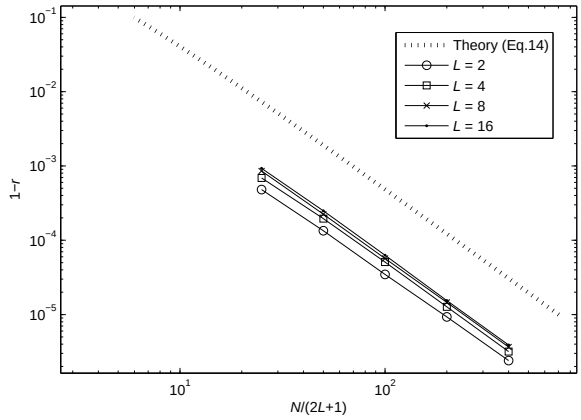

Figure 4: Plot $1-r$ vs. $\frac{N}{2L+1}$ for different values of L and comparison with the theoretical prediction derived from (14).

As $1/4\left(\hat{s}_l\left(\hat{\theta}\right)-s_l\right)^2$ is a Bernoulli random variable with parameter $q=J_0^{\infty}(r)$, thus, we obtain:

$$\mathrm{var}(\Delta(t))=J_0^{\infty}(r)-J_0^{\infty 2}(r). \tag{11}$$

As a consequence, (11) can be injected into (10):

$$\begin{aligned}\mathrm{var}\left(J_0^N\left(\hat{\theta}\right)\right)\leq&\frac{N(2L+1)-L(L+1)}{N^2}\times\\&J_0^{\infty}(r)\left(1-J_0^{\infty}(r)\right).\end{aligned} \tag{12}$$

When N is large with respect to L, (12) reduces to:

$$\mathrm{var}\left(J_0^N\left(\hat{\theta}\right)\right)\leq\frac{2L+1}{N}J_0^{\infty}(r)\left(1-J_0^{\infty}(r)\right).$$

Thus, a sufficient condition for (5) to hold is (in the limit of large N):

$$\frac{N}{2L+1}>\frac{1}{J_0^{\infty}(r)}-1. \tag{13}$$

When r is close to 1, assuming $\frac{N}{2L+1}>>1$, a Taylor series expansion can be applied to (13), which yields:

$$e=1-r\approx\frac{\pi^2}{2}\left(\frac{2L+1}{N}\right)^2, \tag{14}$$

where $e=1-r$ is the error on the correlation coefficient, i.e. the accuracy of the method.

Suppose, for example, that we want to make sure that the error on the correlation coefficient between the nominal and estimated system is about 0.01. Letting $e=0.01$ in (14) yields:

$$N\approx\frac{10\pi}{\sqrt{2}}(2L+1)\approx 22(2L+1).$$

Choosing N according to (14) thus ensures that $\hat{\theta}$, the parameter vector resulting from the optimization, is "close" to θ in the sense that their correlation coefficient (given by (2)) is about $r=0.99$. The simulation results are shown in the next section to confirm the theoretical results.

4. SIMULATION RESULTS AND DISCUSSION

In this section, the validity of the results established in previous section is put to the test. Four impulse responses of lengths 2, 4, 8 and 16 are analyzed: they consist of repetitions of the sequence [1,-1] (oscillatory behaviour is commonplace in MEMS devices). A sequence of N samples of a white Gaussian noise with zero mean and unit variance is applied at the system input. A dithering signal (of the same nature as the input signal) is applied at the input of the comparator. A parameter vector is estimated thanks to the gradient algorithm proposed in [2] and its correlation coefficient with the nominal parameter vector is calculated and stored. The algorithm is stopped when $J_1^N=0$, which is always achievable in the noise-free case. This experiment is repeated a large number of times (typically 10^4) in order to precisely determine the average value of r for a given number of samples and thus, the accuracy of the method $(e=1-r)$.

Fig. 4 illustrates the accuracy of the estimation (e) versus $N/(2L+1)$ obtained for $\sigma_d=0$. These simulation results for different lengths of impulse response (L) are obtained from J_1^N. Note that the same simulation results can be also obtained with J_p^N, $p\geq 0$ because of (3). The simulations agree rather well with the theoretical results obtained in the previous section. It confirms that e is inversely proportional to $(N/2L+1)^2$. However, it should be noted that the experimental value of the accuracy is not only a function of $N/(2L+1)$ but also of L (Fig. 4), i.e. the error behaves as:

$$e=1-r=K(L)\left(\frac{2L+1}{N}\right)^2,$$

where $K(L)$ is a factor which depends on the filter that should be identified and its impulse response length. In the presence case, the number of necessary samples for reaching a given accuracy is overestimated by (14). This is a consequence of using the Cauchy-Schwartz inequality for going from (9) to (10).

When $\sigma_d\neq 0$, the same results as in Fig. 4 are obtained. However, one must keep in mind that in this case, r represents the correlation between z and $\hat{z}$, not θ and $\hat{\theta}$. Thus,

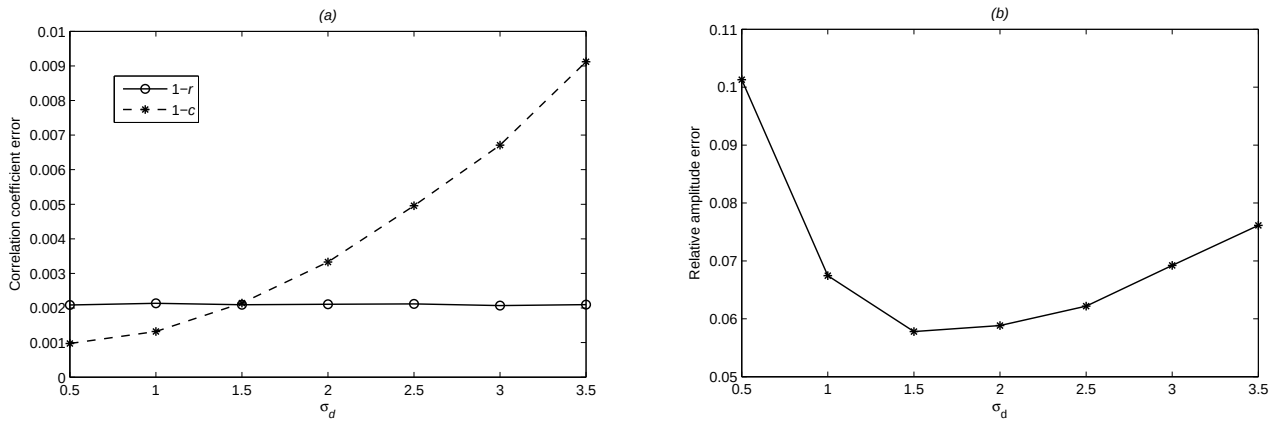

Figure 5: (a) Correlation coefficient error ($1-r$ and $1-c$) vs σ_d, (b) Relative amplitude error $\left(1-\frac{\|\hat{\theta}\|}{\|\theta\|}\right)$ vs σ_d for $L=2$ and $\frac{N}{2L+1}=20$.

supposing σ_d is large with respect to $\theta^T\theta$ and $\hat{\theta}^T\hat{\theta}$, r (2) can be close to 1 regardless of whether $\hat{\theta}$ is actually "close" to θ. Consequently, it is also interesting to plot the cosine of the angle made by $\hat{\theta}$ and θ:

$$c=\frac{\theta^T\hat{\theta}}{\sqrt{\theta^T\theta}\sqrt{\hat{\theta}^T\hat{\theta}}},$$

versus σ_d compared with r versus σ_d (Fig. 5-a). These results show that as σ_d decreases, thus the angle made by $\hat{\theta}$ and θ becomes smaller. On the other hand, too large or too small dither is detrimental to the quality of the identification as it is shown in Fig. 5.

5. CONCLUSION

In this paper, estimation quality of a weighted least-square (WLS) approach to parameter estimation problems based on binary observations is investigated in the case of existence of only a finite number of samples. The relation between accuracy and the number of samples for identifying a system is figured out. Furthermore, simulation results were compared with good agreement to the theoretical results. This work will be extended to the cases when measurement noise is present at the input of the comparator.

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