# A DISCRETE FRACTIONAL EVOLUTIONARY TRANSFORM

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## ABSTRACT

*In this paper, we present a discrete fractional evolutionary transform (DFrET) for the time-frequency (TF) representation of non-stationary, wide band signals. The time–varying kernel of this transform is used to calculate the evolutionary spectrum. The DFrET kernel are obtained from the coefficients of a discrete fractional Gabor expansion. The proposed DFrET provides a tool for high-resolution representation of multicomponent signals with linear instantaneous frequencies. Performance of the proposed algorithm is illustrated by means of simulations and compared with existing TF methods.*

## 1. INTRODUCTION

Although the majority of signals encountered in applications have time–varying spectral content, estimation of these time-frequency (TF) spectra displaying acceptable resolution remains a challenging problem. This is mainly due to the estimation methods having difficulty adapting to the time–varying frequency of the signal components [1, 2]. Constant–bandwidth methods such as the short-time Fourier transform and the traditional Gabor expansion [4, 5] provide estimates with poor TF resolution [8]. Several approaches have been proposed to improve the resolution of the estimation: averaging of the estimates obtained using different windows [7], maximizing energy concentration measures [8, 9], and adapting the Gabor basis functions to the instantaneous frequency of the signal components [6, 11, 12, 13].

In [15], we present a Discrete Evolutionary Transform (DET) that provides a TF representation of the signal and an evolutionary spectrum simultaneously. It is shown that the time–varying kernel of this transform may be obtained through either the multi–window Gabor expansion that uses non–orthogonal bases, or the Malvar expansion using orthogonal bases. Computation of the evolutionary spectrum with the sinusoidal expansion provides estimates with poor time–frequency resolution for signals with wide-band components. We will show that the method can be improved by combining the advantages of the Fractional Fourier Transform (FrFT) and the DET. Hence we present a Discrete Fractional Evolutionary Transform (DFrET) and a method to obtain its kernel by using the recently introduced fractional Gabor expansion [10]. This will allow us to obtain a high resolution TF spectrum as well as a compact representation for signals with linear chirp components.

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## 2. DISCRETE EVOLUTIONARY TRANSFORM BY GABOR EXPANSION

The evolutionary spectral theory for the analysis of non-stationary random processes [14] has been extended to consider discrete-time, finite-support signals [9, 13, 7]. Hence for a discrete-time signal $x(n), n = 0, 1, \cdots, N-1$, its Discrete Evolutionary Transform (DET) is defined in terms of sinusoids with time–varying amplitudes as:

$$x(n) = \sum_{k=0}^{K-1} X(n,k) e^{j\omega_k n} \tag{1}$$

where $\omega_k = 2k\pi/K$, $K$ is the number of frequency samples and $X(n,k)$ the time-varying kernel of the DET. The above equation is analogous to the Wold–Cramer representation used to model the non–stationary processes as a combination of sinusoids with time–varying and random amplitudes [14]. The evolutionary spectrum of $x(n)$ is then given by,

$$S(n,k) = \frac{1}{K} |X(n,k)|^2 \tag{2}$$

[9]. It is shown in [15] that the kernel $X(n,k)$ may be calculated using conventional signal representations such as the Gabor expansion, that uses non-orthogonal basis, or the Malvar expansion that uses orthogonal basis.

Traditional discrete Gabor expansion [4] represents a signal as a combination of basis functions that are obtained by translating a single window uniformly in time and frequency. Hence Gabor basis functions allow a sinusoidal and constant-bandwidth analysis. However, if the signal to be analyzed does not satisfy the constant-bandwidth condition, i.e., if the frequency components change with time, its TF representation will not be parsimonious [9]. A multi–window Gabor expansion is presented in [13], using basis functions $\tilde{h}_{i,m,k}(n)$, that are obtained by scaling and translating in time and frequency a mother window:

$$\tilde{h}_{i,m,k}(n) = \tilde{h}_i(n - mL) e^{j\omega_k n} \tag{3}$$

Then the multi-window Gabor representation of $x(n)$,

$$x(n) = \frac{1}{I} \sum_{i=0}^{I-1} \sum_{m=0}^{M-1} \sum_{k=0}^{K-1} a_{i,m,k} \tilde{h}_i(n - mL) e^{j\omega_k n} \tag{4}$$

Synthesis windows $\tilde{h}_i(n)$ are obtained from a unit-energy mother window $g(n)$ by scaling in time $h_i(n) = 2^{i/2} g(2^i n)$, $i = 0, 1, \cdots, I-1$, and periodically extending by $N$. Here $I$

denotes the number of scales used and $L, M, L', K$ positive integers satisfy the condition $LM = L'K = N$. $L$ and $L'$ are the sampling steps in time and frequency, $M$ and $K$ are the number of samples in time and frequency respectively. The Gabor coefficients $a_{i,m,k}$, may be calculated by the analysis windows $\tilde{\gamma}_i(n)$ that are bi-orthogonal to $\tilde{h}_i(n)$ [4]:

$$a_{i,m,k} = \sum_{n=0}^{N-1} x(n)\tilde{\gamma}_i^*(n-mL)e^{-j\omega_k n} \tag{5}$$

Now, by considering the representations of $x(n)$ in equations (1) and (4), the DET kernel $X(n,k)$ is

$$\begin{aligned} X(n,k) &= \frac{1}{I}\sum_{i=0}^{I-1}\sum_{m=0}^{M-1} a_{i,m,k}\,\tilde{h}_i(n-mL) \\ &= \frac{1}{I}\sum_{i=0}^{I-1} X_i(n,k) \end{aligned} \tag{6}$$

where $X_i(n,k)$ show the kernels calculated for different scales. They may be combined using arithmetic average or other averaging techniques [9]. Substituting $a_{i,m,k}$ in (5) into (6), we get

$$X(n,k) = \sum_{\ell=0}^{N-1} x(\ell)\,\mathrm{w}(n,\ell)\,e^{-j\omega_k \ell} \tag{7}$$

where $\mathrm{w}(n,\ell)$ is a time-dependent window function given by

$$\mathrm{w}(n,\ell) = \frac{1}{I}\sum_{i=0}^{I-1}\sum_{m=0}^{M-1} \tilde{\gamma}_i^*\,(\ell-mL)\,\tilde{h}_i(n-mL). \tag{8}$$

By considering all possible scales, a high-resolution representation for the signal may be obtained by combining the kernel set $\{X_i(n,k)\}$ [13]. However, this is not sufficient in general; because signals with wide-band components may require non-sinusoidal basis functions for a compact representation. In such cases, a fractional time–frequency representation will be more appropriate for the spectral signals.

## 3. DISCRETE FRACTIONAL EVOLUTIONARY TRANSFORM

In the following, we briefly review the closed-form fractional Gabor expansion that is especially useful for the representation of chirp-type signals and then connect it to the proposed DFrET.

### 3.1 Discrete Fractional Gabor Expansion

A discrete fractional Gabor expansion for a signal $x(n)$, $n = 0,1,\cdots,N-1$ ($N$ odd) is defined in [10] as:

$$x(n) = \sum_{m=0}^{M-1}\sum_{k=0}^{K-1} a_{m,k,\alpha}\,\tilde{h}_{m,k,\alpha}(n) \tag{9}$$

where $\alpha$ is the fraction order and the fractional Gabor basis functions are

$$\tilde{h}_{m,k,\alpha}(n) = \tilde{h}(n-mL)\,K_\alpha(n,k). \tag{10}$$

Here $M$ and $K$ are the number of samples in time and in $\alpha$ fractional domain $u$ (combination of time and frequency) respectively; $L$ and $L'$ are the sampling steps in time and in $u$ domain, and the previous condition holds: $ML = KL' = N$. Furthermore $K_\alpha(n,k)$ is the fractional kernel and it replaces the sinusoidal kernel $\{e^{j\omega_k n}\}$ of the traditional Gabor expansion where the term $e^{j\omega_k n}$ modulates synthesis windows and shifts them in the frequency domain by $L'$. Similarly, in the fractional case, the kernel $K_\alpha(n,k)$ will shift the window in the $u$ domain by the same step. A kernel which will provide such a shift can be obtained from the kernel of the closed-form discrete FrFT [16]:

$$\begin{aligned} K_\alpha(n,k) &= \sqrt{\frac{\sin\alpha - j\cos\alpha}{N}}\,e^{\frac{j}{2}[n^2\Delta t^2+(kL')^2\Delta u^2]\cot\alpha} \\ &\times\; e^{-j\frac{2\pi kL'}{N}n} \end{aligned} \tag{11}$$

where $\Delta t\Delta u = \frac{2\pi|\sin\alpha|}{N}$. Then the fractional Gabor coefficients are calculated as,

$$a_{m,k,\alpha} = \sum_{n=-(N-1)/2}^{(N-1)/2} x(n)\;\tilde{\gamma}^*_{m,k,\alpha}(n) \tag{12}$$

where the analysis basis functions are periodic versions of

$$\gamma_{m,k,\alpha}(n) = \gamma(n-mL)\,K_\alpha(n,k). \tag{13}$$

The bi-orthogonality condition of this basis system is derived in [10] and given by

$$\frac{1}{LL'}\sum_{n=-(N-1)/2}^{(N-1)/2} \tilde{h}(n)\tilde{\gamma}^*(n+mK)\,e^{-j\frac{2\pi k}{N}n} = \delta_m\delta_k \tag{14}$$

$0 \le k \le L-1$, $0 \le m \le L'-1$ and $-(N-1)/2 \le n \le (N-1)/2$. For a given Gauss synthesis window, the analysis window can be solved from the above equation system and used to calculate the fractional Gabor coefficients. Above fractional expansion is a generalization of the sinusoidal expansion, such that it reduces to the traditional Gabor for $\alpha = \pi/2$.

### 3.2 Discrete Fractional Evolutionary Transform

A more general TF representation for a discrete-time signal $x(n), n = 0,1,\cdots,N-1$ may be obtained by generalizing the sinusoidal DET in Section 2. Then a Discrete Fractional Evolutionary Transform (DFrET) is defined to represent a signal as a combination of linear chirps with time–dependent amplitudes:

$$x(n) = \sum_{k=0}^{K-1} X(n,k,\alpha)\,K_\alpha(n,k) \tag{15}$$

where $K_\alpha(n,k)$, is the kernel of the FrFT given in (11)and $X(n,k,\alpha)$ is the time and fraction order $\alpha$ dependent DFrET kernel. Thus the evolutionary spectrum of $x(n)$ in the fractional domain is calculated by

$$S(n,k,\alpha) = \frac{1}{K}|X(n,k,\alpha)|^2 \tag{16}$$

$X(n,k,\alpha)$ kernel may be obtained by considering the representations of the signal in (15) and (9);

$$X(n,k,\alpha) = \sum_{m=0}^{M-1} a_{m,k,\alpha}\,\tilde{h}(n-mL) \tag{17}$$

Finally, substituting for $a_{m,k,\alpha}$ from (12) into (17), we get

$$X(n,k,\alpha) = \sum_{m=0}^{M-1} \sum_{\ell=-(N-1)/2}^{(N-1)/2} x(\ell)\tilde{\gamma}^*_{m,k,\alpha}(\ell)\tilde{h}(n-mL) \quad (18)$$

Defining a time-dependent window function $\mathrm{w}(n,\ell)$ as before by,

$$\mathrm{w}(n,\ell) = \sum_{m=0}^{M-1} \tilde{\gamma}^*(\ell - mL)\tilde{h}(n-mL) \quad (19)$$

the DFrET in (18) becomes

$$X(n,k,\alpha) = \sum_{\ell=-(N-1)/2}^{(N-1)/2} x(\ell)\mathrm{w}(n,\ell)K^*_\alpha(\ell,k) \quad (20)$$

The proposed DFrET will provide a suitable approach for the TF representation and spectral estimation of non-stationary signals especially signals containing linear chirps. In the following, we demonstrate the performance of the proposed transform by means of computer simulations. Note that without apriori information about the frequency content of the signal, determining the value of $\alpha$ is a question. There are signal-adaptive methods in the literature to obtain the best possible analysis parameters for a given signal [6, 8]. In our simulations we use the local energy concentration measure presented in [8] and used in [10] to obtain the best matched value of $\alpha$ from a set of values.

## 4. SIMULATION RESULTS

We consider the combination of two linear chirp signals shown in Fig. 1. The signal is first represented using multi-window Gabor expansion based DET with parameters $N = 256, M = 64, K = 256, I = 2$. The evolutionary spectrum calculated by the sinusoidal DET is given in Fig. 2. Then we obtain a TF representation and hence an evolutionary spectral estimation by searching for fraction orders in $8 \times 8$ TF regions. We have a set of 30 $\alpha$ values from 0 to $\pi$ and search for the optimum one by maximizing the local energy concentration measure [10]. Fig. 3 shows the calculated well-localized evolutionary spectrum using the proposed DFrET method.

## 5. CONCLUSIONS

In this paper a discrete fractional evolutionary transform is introduced by combining the advantages of fractional signal expansions and evolutionary spectral theory. The kernel of this transform is calculated by the coefficients of a fractional Gabor expansion. A high–resolution TF representation and an evolutionary spectrum are obtained simultaneously. Simulation results show that the proposed DFrET provide a well localized TF spectra for signals containing linear chirp components.

## REFERENCES


[1] Cohen, L., *Time-Frequency Analysis.* Prentice Hall, Englewood Cliffs, NJ, 1995.

[2] S. Qian, and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*. Prentice Hall, Upper Saddle River, NJ, 1996.

[3] Gabor, D., "Theory of Communication," *J. IEE,* Vol. 93, pp. 429–459, 1946.

[4] Wexler, J., and Raz, S., "Discrete Gabor Expansions," *Signal Processing,* Vol. 21, No. 3, pp. 207–220, Nov. 1990.

[5] Qian, S., and Chen, D., "Discrete Gabor Transform," *IEEE Trans. on Signal Proc.,* Vol. 41, No. 7, pp. 2429–2439, July 1993.

[6] Baraniuk, R.G., and Jones, D.L., "Shear Madness: New Orthonormal Bases and Frames Using Chirp Functions," *IEEE Trans. on Signal Proc.,* Vol. 41, No. 12, pp. 3543–3549, Dec. 1993.

[7] Akan, A., and Chaparro, L.F., "Evolutionary Spectral Analysis and the Generalized Gabor Expansion," *Proc. ICASSP-95,* Vol. 3, pp. 1537–1540, May 1995.

[8] Jones, D.L., and Parks, T.W., "A High Resolution Data–Adaptive Time–Frequency Representation," *IEEE Trans. on Signal Proc.,* Vol. 38, No. 12, pp. 2127–2135, Dec. 1990.

[9] Akan, A., and Chaparro, L.F., "Evolutionary Spectral Analysis Using a Warped Gabor Expansion," *IEEE Proc. ICASSP-96,* Atlanta, GA, May 1996.

[10] Akan, A., and Onen, E., "A Discrete Fractional Gabor Expansion for Multi-component Signals," *AEÜ-Int J of Electronics and Comm,* Vol. 61, No. 5 pp. 279-285, May 2007.

[11] Mallat, S., and, Zhang, Z., "Matching Pursuit with Time–Frequency Dictionaries," *IEEE Trans. on Signal Proc.,* Vol. 41, No. 12, pp. 3397–3415, Dec. 1993.

[12] Brown, M., Williams, W., and Hero, A., "Non–Orthogonal Gabor Representation of Biological Signals," *IEEE Proc. ICASSP–94,* Adelaide, Australia, Apr. 1994.

[13] Akan, A., and Chaparro, L.F., "Multi–window Gabor Expansion for Evolutionary Spectral Analysis," *Signal Processing*, Vol. 63, pp. 249–262, Dec. 1997.

[14] Priestley, M.B., "Evolutionary Spectra and Non–stationary Processes," *J. of Royal Statistical Society,* B, Vol.27, No. 2, pp. 204–237, 1965.

[15] R. Suleesathira, A. Akan, and L.F. Chaparro, "Discrete Evolutionary Transform for Time-Frequency Signal Analysis," *J. Franklin Institute, Special Issue on Time-Frequency Signal Analysis and its Applications*, pp. 347-364, Vol. 337, No. 4, Jul. 2000.

[16] S.C. Pei, and J.J. Ding, "Closed-Form Discrete Fractional and Affine Fourier Transform," *IEEE Trans. on Signal Proc.*, Vol. 48 pp. 1338-1353, 2000.

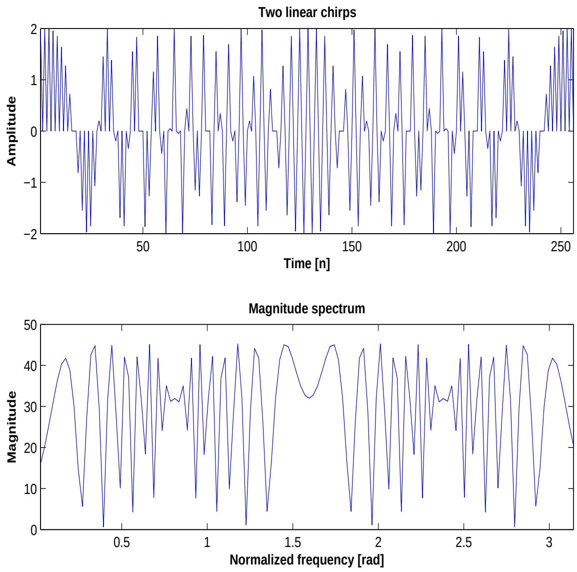


Figure 1: Combination of two linear chirps.

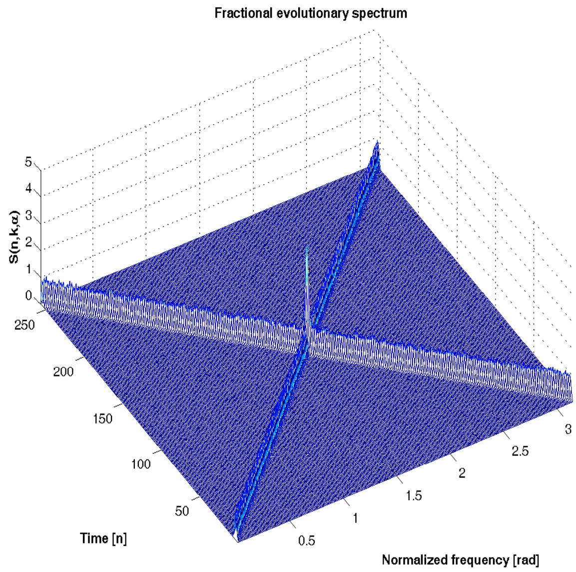


Figure 3: The proposed DFrET based evolutionary spectrum.

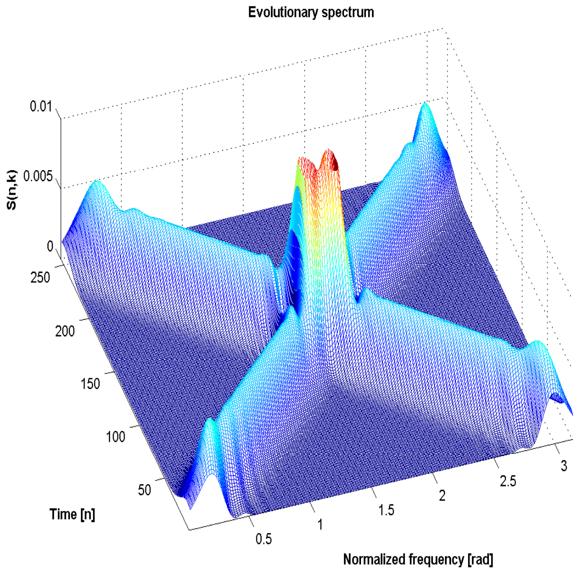


Figure 2: Sinusoidal DET based evolutionary spectrum of the chirps.