MAXIMUM LIKELIHOOD SCALE PARAMETER ESTIMATION: AN APPLICATION TO GAIN ESTIMATION FOR QAM CONSTELLATIONS

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ABSTRACT

In this paper we address the problem of scale parameter estimation, introducing a reduced complexity Maximum Likelihood (ML) estimation procedure. The estimator stems from the observation that, when the *estimandum* acts as a shift parameter on a multinomially distributed statistic, direct maximization of the likelihood function can be conducted by an efficient DFT based procedure. A suitable exponential warping of the observation's domain is known to transform a scale parameter problem into a shift estimation problem, thus allowing the afore mentioned reduced complexity ML estimation for shift parameter to be applied also in scale parameter estimation problems. As a case study, we analyze a gain estimator for general QAM constellations. Simulation results and theoretical performance analysis show that the herein presented estimator outperforms selected state of the art high order moments estimator, approaching the Cramér-Rao Lower Bound (CRLB) for a wide range of SNR.

1. INTRODUCTION

Given a parametric family of probability density functions (pdf), a parameter is said to be a scale parameter if its value determines the scale or, equivalently, the statistical dispersion of the pdf itself along one or more of its components. Scale parameter estimation problems are often encountered in lots of applications. To give few examples, the Gamma distribution, and most of its related pdfs, are parameterized by a scale parameter. Of particular relevance is the Nakagami distribution, often used to model attenuation in multipath environment.

In this paper we propose a Maximum Likelihood estimation technique for scale parameters. We will first derive the ML estimator for the case in which the observed statistic is a multinomial distributed random variable. Then, we will show how, thanks to a link established between scale and location parameters, the direct maximization of the log-likelihood function is straightforwardly achieved via a computational efficient DFT based approach. Namely, a logarithmic transformation of the observed random variable modifies the pdf so that the scale parameter relation is mapped into a location parameter problem, that can be efficiently solved in the DFT domain.

As a case study, here we apply the ML estimation criterion to the problem of blind estimation of the gain factor for a general Quadrature Amplitude Modulated (QAM) signal. Gain estimation is often required, for instance, to drive blind estimation techniques for phase and frequency offset estimation, as for instance in [1], as well as decision directed estimations procedures. Simulation results show that the herein introduced estimator outperforms High Order Moments state of the art technique for gain estimation, as in [2], approaching the Cramér-Rao Lower Bound (CRLB) for a wide range of SNR. We also carried out theoretical performance analysis, whose results are confirmed by numerical simulations.

This paper is organized as follows. In Sect.2 we introduce the concept of scale parameter and determine the form of the ML estimation for scale parameters under multinomially distributed observations. In Sect.3 we explicitate the relation between a scale and a location parameter, and we show how this allows to devise a reduced complexity ML estimator. In Sect.4 we analyze the case of gain estimation for general QAM constellations. Finally Sect.5 reports simulation results and the related discussion.

2. ON THE ML ESTIMATION OF A SCALE PARAMETER

In this Section we will briefly define the concepts of scale parameter for a pdf family and we will determine the form of the ML parameter estimator when the observation statistic is multinomially distributed. Let us consider a parameter α[1] to be estimated after a finite number N of realizations of a related n-dimensional random variable x, *i.e.* $x \in \mathbb{K}^n$, gathered in the vector $\mathbf{x} = [x_0, \ldots, x_{N-1}]^{\mathrm{T}}$ is observed. Then, α is said to be a scale parameter for the pdf family $p_{\mathrm{x}|\alpha}\left(x^{(1)}, \ldots, x^{(n-1)}, x^{(n)}|\alpha\right)$ when this latter depends on α only through the scale relation $x^{(n)}/\alpha$:

$$p_{\mathrm{x}|\alpha}\left(x^{(1)}, \ldots, x^{(n-1)}, x^{(n)}|\alpha\right)$$
$$= \frac{1}{|\alpha|} p_{\mathrm{x}|\alpha}\left(x^{(1)}, \ldots, x^{(n-1)}, \frac{x^{(n)}}{\alpha}|1\right) \quad (1)$$

Let us now suppose that we observe the histogram of the marginal variable $x^{(n)}$ on a set of K uniform intervals of width Δ centered around the points ξ_k, $k = 0, \ldots, K-1$. In other words, the observations are given by the number v_k, $k = 0, \ldots, K-1$ of occurrences of the event $\mathscr{E}_k$, being $\mathscr{E}_k = \{x^{(n)} \in [\xi_k - \Delta/2, \xi_k + \Delta/2)\}$, in N statistically independent trials. The K random variables $v_0, \ldots, v_{K-1}$ are then multinomially distributed:

$$P(v_0, \ldots, v_{K-1}) = \frac{N!}{\prod\limits_{k=0}^{K-1} v_k!} \cdot \prod_{k=0}^{K-1} \pi_k^{v_k}$$

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[1]From now on we will always assume, without loss of generality, the parameter α to be monodimensional. Extension to multidimensional parameters is straightforward.

being π_k the probability of the event $\mathscr{E}_k$. The multinomial model applies in several frameworks in which histograms are evaluated for measurement, calibration, characterization purposes (see for instance [3]). The probabilities π_k of the r.v. v_k, $k = 0, \ldots, K-1$ are nothing else that the area of the marginal conditional probability $p_{x|\alpha}\left(x^{(n)}|\alpha\right)$ over an interval of width Δ centered in ξ_k:

$$
\pi_k = f\left(\frac{\xi_k}{\alpha}\right) = \frac{1}{\Delta} \int_{\xi_k - \Delta/2}^{\xi_k + \Delta/2} dx^{(n)}
$$
$$
\cdot \left[\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} p_{x|\alpha}(x|\alpha)\ dx^{(1)}\cdots dx^{(n-1)}\right]
$$

(2)

Let us introduce for notation purposes the following vectors, collecting the values of $f\left(\frac{\xi_k}{\alpha}\right)$ and $\hat{f}_k \overset{\text{def}}{=} v_k/N$:

$$
\mathbf{f}(\alpha) = \left[f\left(\frac{\xi_k}{\alpha}\right)\right]_{k=0}^{K-1}
$$
$$
\hat{\mathbf{f}} = \left[\hat{f}_k\right]_{k=0}^{K-1}
$$

The log-likelihood of $\hat{\mathbf{f}}$ can be written as:

$$
l\left(\hat{\mathbf{f}}; \mathbf{f}(\alpha)\right) = S + N \sum_{k=0}^{K-1} \hat{f}_k \ln\left(f\left(\frac{\xi_k}{\alpha}\right)\right)
$$
$$
S = \ln N! - \ln \prod_{k=0}^{K-1} \left(N\hat{f}_k\right)!
$$

(3)

The maximum likelihood estimate of α is then attained by maximizing (3) with respect to α. Neglecting all the terms in (3) that do not explicitly depend on α, we come up with the following estimation rule

$$
\hat{\alpha} = \arg\max_\alpha \hat{\mathbf{f}}^T \cdot \tilde{\mathbf{f}}(\alpha)
$$

(4)

where we have compactly denoted $\tilde{\mathbf{f}}(\alpha) \overset{\text{def}}{=} \ln \mathbf{f}(\alpha)$. In this form, the above relation is highly nonlinear and must be solved by exhaustive-search or suitably initialized gradient-search techniques. A reduced complexity ML procedure is presented in the following Section.

3. REDUCED COMPLEXITY ML ESTIMATION FOR SCALE PARAMETER

In this Section, we explicitate the relation between a scale parameter and a location one, and we show how this allows to devise a fast, FFT based, computational procedure that obtains $\hat{\alpha}$ in a two-stage, coarse-to-fine, estimation steps.

3.1 On the Relation between Scale and Location Parameter

Let us now briefly recall the concept of location parameter for a pdf family and prove how a suitable exponential warping constitutes a bind between a scale parameter and a location parameter. As far as a location parameter is concerned the pdf family $p_{x|\alpha}\left(x^{(1)}, \ldots, x^{(n-1)}, x^{(n)}|\alpha\right)$ must satisfy the

following property:

$$
p_{x|\alpha}\left(x^{(1)}, \ldots, x^{(n-1)}, x^{(n)}|\alpha\right)
$$
$$
= p_{x|\alpha}\left(x^{(1)}, \ldots, x^{(n-1)}, x^{(n)} - \alpha|0\right)
$$

(5)

Properties of location parameters have been exploited by the authors in [4] to devise a gain control free near efficient phase offset estimator for QAM constellations.

Now let us consider a transformation $\mathscr{Z} : \mathbb{K}^n \to \mathbb{K}^n$ and the corresponding transformed random variable $z = \mathscr{Z}(x)$. When the transformation $\mathscr{Z}(\cdot)$ assumes the following remarkable form:

$$
z^{(1)}(x^{(1)}) = x^{(1)}, \cdots, z^{(n-1)}(x^{(n-1)}) = x^{(n-1)}
$$
$$
z^{(n)}(x^{(n)}) = \log x^{(n)}
$$

(6)

it can be easily proved that relation (5) holds, with the care of substituting α with $\alpha_z = \log\alpha$. Thus, whenever a scale parameter estimation problem is encountered, it is always possible to map it into a location parameter one, by means of a preliminary transformation of the observation as the one appearing in (6). The estimandum α, scale parameter for the r.v x, has then correctly the meaning of a location parameter for the transformed r.v. z, in the form of $\alpha_z = \log\alpha$.

3.2 Reduced Complexity ML Estimation for Location Parameter

In analogy to the case of scale parameters, the maximum likelihood estimate of α_z is attained by maximizing

$$
\hat{\alpha}_z = \arg\max_{\alpha_z} \hat{\mathbf{f}}^T \cdot \tilde{\mathbf{f}}(\alpha_z)
$$

(7)

with respect to α_z where we have compactly denoted $\tilde{\mathbf{f}}(\alpha_z) \overset{\text{def}}{=} \ln \mathbf{f}(\alpha_z)$, with $\mathbf{f}(\alpha) = [f(\zeta_k - \alpha_z)]_{k=0}^{K-1}$, and where

$$
f(\zeta_k - \alpha_z) = \frac{1}{\Delta} \int_{\zeta_k - \Delta/2}^{\zeta_k + \Delta/2} dz^{(n)}
$$
$$
\cdot \left[\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} p_{z|\alpha_z}(z|\alpha_z)\ dz^{(1)}\cdots dz^{(n-1)}\right]
$$

(8)

Since α_z is a location parameter for the statistic $f(\zeta_k - \alpha_z)$, we can expand the inner product in (7) having:

$$
\hat{\alpha}_z = \arg\max_{\alpha_z} \mathscr{C}(\alpha_z)
$$
$$
\mathscr{C}(\alpha_z) = \sum_{k=0}^{K-1} \hat{f}_k \ln f(\zeta_k - \alpha_z)
$$

(9)

where we have discarded all the terms that do not explicitly depend on α_z. We recognize that in (9) we have the cross correlation between the sequences collected in the vector $\hat{\mathbf{f}}$ and $\mathbf{f}(0)$.

Therefore, the relation between a scale parameter and a location one allows to devise a fast, FFT based, computational procedure that obtains $\hat{\alpha}_z$ in a two-stage, coarse-to-fine, estimation steps. Specifically, first the lag of the maximum of the cross correlation obtains a coarse estimate. The resolution in this step is limited by the values of K. The

employment of an interpolation technique, as for instance in [8], obtains a finer estimate. The overall computational complexity is significantly reduced by choosing the value of K according to selected FFT algorithms.

After the estimation of $\hat{\alpha}_z$ has been performed, the estimate of the original parameter α is simply obtained by inverting the transformation $\mathscr{Z}(\cdot)$, having thus:

$$\hat{\alpha} = e^{\hat{\alpha}_z} \qquad (10)$$

Elaborating over the above described estimation procedure, it turns out that it is constituted by an exponential warping over one component of the observed variable (the remaining being saturated while performing the nonlinear moment in (2)), and then by a DFT based cross-correlation between the warped versions of the moment in (2) and its unbiased estimate.

Interestingly enough, the cascade of an exponential warping and a DFT is in turn implicitly realized by the discrete Mellin Transform [5]-[6]. Hence, the estimation procedure, can be implemented by means of the discrete Mellin transform, which can be efficiently implemented as described in [7].

4. A CASE STUDY: GAIN ESTIMATION FOR QAM CONSTELLATIONS

In this Section, as a case study, we apply the scale parameter estimation technique described in the previous sections, to the problem of gain factor estimation for general QAM constellations.

Let us consider a digital transmission system where the information is carried on by M-ary QAM symbols drawn from a, power normalized, constellation $\mathscr{A} = \{s_0, .., s_{M-1}\}$. At the receiver side, a complex low-pass version of the received signal is extracted by means of front-end processing. Let x_n be the samples of the complex low-pass received signal extracted at symbol rate. We assume the following analytical model of the observations:

$$x_n = G s_n e^{j\theta} + w_n \qquad (11)$$

where s_n is the n-th transmitted symbol, G is the unknown overall gain, θ is the unknown phase-offset, and w_n is a realization of a circularly complex Gaussian stationary noise process, statistically independent of s_n, with variance $\sigma_W^2 \overset{\text{def}}{=} \mathrm{E}\{|w_n|^2\}$. The signal-to-noise ratio (SNR) is $\eta \overset{\text{def}}{=} G^2/\sigma_W^2$. Here we address the estimation of the unknown gain factor G after the observation of N consecutive received signal samples $x_n, n = 0, \cdots N-1$.

Let us then represent the received samples in polar coordinates i.e. $x_n = r_n e^{j \cdot \varphi_n}$ with $r_n = |x_n|$, $\varphi_n = \arg x_n$. We recognize that G is a scale parameter for the pdf family $p_{R,\Phi}(r_n, \varphi_n|G)$. More specifically let us consider the nonlinear moment as in (2):

$$f\left(\frac{\rho}{G}\right) = \frac{1}{\Delta} \int\limits_{\rho-\Delta/2}^{\rho+\Delta/2} dr \cdot \left[\int_{-\pi}^{\pi} p_{R,\Phi}(r, \varphi|G) d\varphi\right] \qquad (12)$$

where we dropped the subscript n for the sake of simplicity. The nonlinear moment in (12) is proved to exhibit the fol-

lowing remarkable form, for equiprobable constellation symbols:

$$
\begin{aligned}
&f\left(\frac{\rho}{G}\right) = \\
&\sum_{m=0}^{M-1} \frac{2\eta}{MG} \frac{\rho}{G} \cdot \exp\left(-\eta\left(\frac{\rho^2}{G^2} + \rho_m^2\right)\right) I_0\left(2\rho_m \eta \frac{\rho}{G}\right)
\end{aligned} \qquad (13)
$$

where $I_n(\cdot)$ is the n-th order modified Bessel function of the first kind and ρ_m is the magnitude of the m-th constellation symbol s_m.

A sample estimate of the nonlinear moment in (13) is calculated by evaluating the histogram of the magnitude ρ of the received signal samples in K intervals of width $\Delta = \rho_{\max}/K$:[2]

$$\hat{f}_k = \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{rect}_\Delta\left(|x_n| - \frac{(2k+1)\Delta}{2}\right) \qquad (14)$$

We remark that the values $\hat{f}_k$ in (14), being histogram estimates, are multinomially distributed. According to the estimation criterion exposed in Sect.2, the ML estimate of G is given by:

$$
\begin{aligned}
\hat{G} &= \arg\max_G \mathscr{C}_g(G) \\
\mathscr{C}_g(G) &= \hat{\mathbf{f}}^{\mathrm{T}} \cdot \tilde{\mathbf{f}}(G)
\end{aligned} \qquad (15)
$$

The maximization problem in (15) is non-convex, and its solution would require the employment of computationally onerous numerical algorithms, like exhaustive-search or suitably initialized gradient-search techniques.

A reduced complexity solution is obtained, following the guidelines in Sect.3.2, by applying the transformation $\mathscr{Z}(\cdot)$ as in (6) to the observations in (11), and then performing the estimation of the location parameter $G_z = \log G$.

The employment of the Mellin transform allows to implicitly perform the cascade of the exponential warping and of the DFT to solve the maximization problem in (15).

Hence, being $\mathscr{M}\{\cdot\}$ and $\mathscr{M}^{-1}\{\cdot\}$ respectively the discrete Mellin transform and its inverse we can write:

$$
\begin{aligned}
\hat{G} &= \arg\max_G \mathscr{C}_g(G) \\
\mathscr{C}_g(G) &= \mathscr{M}^{-1}\left\{\mathscr{M}\{\hat{\mathbf{f}}\}^{\mathrm{T}} \cdot \mathscr{M}\{\tilde{\mathbf{f}}(G)\}^*\right\}
\end{aligned} \qquad (16)
$$

where the superscript $\{\cdot\}^*$ denotes complex conjugation.

The accuracy of $\hat{G}$ being limited by the value of K, a finer estimate is obtained by means of parabolical interpolation technique [8], being

$$\hat{G}^{(f)} = \hat{G} - \frac{\Delta}{2} \cdot \frac{\mathscr{C}_g(\hat{G}+\Delta) - \mathscr{C}_g(\hat{G}-\Delta)}{\mathscr{C}_g(\hat{G}+\Delta) - 2\mathscr{C}_g(\hat{G}) + \mathscr{C}_g(\hat{G}-\Delta)} \qquad (17)$$

The theoretical performance analysis can be conducted resorting to the parabolical interpolation formulas as in [8], where the objective function is approximated around its maximum with the second order Taylor expansion. Following the guidelines of [8] the asymptotic variance of the fine estimate

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[2]Albeit in principle the value of the magnitude is unbounded, we can approximate its maximum value to be $\rho_{\max} = \max_m\{|s_m|\} + 4\sigma_W$.

is written in terms of the first and second order moment of the objective function. Let us set:

$$x \overset{\text{def}}{=} \mathscr{C}_g\left(\hat{G}+\Delta\right), \quad X \overset{\text{def}}{=} \text{E}\{x\},$$
$$y \overset{\text{def}}{=} \mathscr{C}_g\left(\hat{G}-\Delta\right), \quad Y \overset{\text{def}}{=} \text{E}\{y\},$$
$$z \overset{\text{def}}{=} \mathscr{C}_g\left(\hat{G}\right), \qquad Z \overset{\text{def}}{=} \text{E}\{z\}$$
$$c = X - Y, \;\; d = X - 2Z + Y.$$

Then, within a first-order approximation of (17), the variance of $\hat{G}^{(f)}$ is given by:

$$\begin{aligned}
\text{Var}\{\hat{G}^{(f)}\} = \frac{\Delta^2}{2} \Big[&\left(\frac{d-c}{d^2}\right)^2 \text{Var}\{x\} \\
&+ \left(\frac{d+c}{d^2}\right)^2 \text{Var}\{y\} + \left(\frac{2c}{d^2}\right)^2 \text{Var}\{z\} \\
&- \left(\frac{d^2-c^2}{d^4}\right) \text{Cov}\{x,y\} + \left(\frac{2dc+2c^2}{d^4}\right) \text{Cov}\{z,y\} \\
&+ \left(\frac{2dc-2c^2}{d^4}\right) \text{Cov}\{x,z\} \Big]
\end{aligned}$$

(18)

The mean values X, Y, Z and the covariances of x, y, z of the objective function are reported in Appendix I.

5. NUMERICAL EXPERIMENTS

In this Section we report simulation results concerning the analytical and numerical performance of the reduced complexity ML (RCML) gain factor estimator introduced in the previous Section. The simulations settings are fixed as follows. The signal samples are generated according to the model in (11); the sample size is set to $N = 512$ samples and the phase offset θ is chosen randomly with an uniform distribution in $(-\pi, \pi]$. The value of K has been set to $K = 512$. The value of G to be estimated has been set to 1dB. Each numerical experiment consist of 1000 Monte Carlo runs. For the sake of comparison we also reported the accuracy of a classical state of the art fourth order estimator [2] (M2M4). The performance are illustrated by plotting the results of the theoretical analysis in terms of the normalized standard deviation ($\sqrt{N} \cdot$ StdDev) of the estimation error and the results of the numerical simulation are reported in terms of the normalized Root Mean Square Error ($\sqrt{N} \cdot$ RMSE). For reference sake, we also report the Cramér-Rao lower bound, derived following the guidelines in [9]. Fig.1 shows both the theoretical and numerical performance of the herein presented estimator for 16 and 32 QAM constellations. We observe a good matching between the theoretical performance and numerical results; for both of the constellations, at medium to high SNR, the herein described estimator approaches the CRLB, outperforming the estimator in [2]. Since the evaluation of the nonlinear moment in (13) requires the knowledge of the SNR, that is, in turn, to be estimated from the received samples, we tested the performance of the estimator in presence of a SNR estimation mismatch. Fig.2 reports the degradation performance for 16 and 32 QAM constellations in presence of a ± 2 dB SNR estimation mismatch. The curves report also the worst measured performance. We observe that the accuracy preserves the CRLB slope, although the mismatch slightly affects the estimator performance.

Finally, we show in Fig.3 the Symbol Error Rate (SER) reduction achieved in correspondence of the gain estimation error variance reduction. We plot the SER obtained after 2000 Montecarlo runs over a sample size of $N = 512$ samples by the herein presented estimator and by the estimator in [2] for 256 and 512 QAM constellations. For the sake of reference we also report the corresponding SER for an AWGN channel. Result in 3 show that the herein presented estimator tightly approaches the AWGN performance, outperforming the M2M4 estimator.

Appendix I. FIRST AND SECOND ORDER MOMENTS OF THE OBJECTIVE FUNCTION

As far as the first order moments are concerned, since $\text{E}\{\hat{\mathbf{f}}\} = \mathbf{f}(G)$ we have:

$$\text{E}\left\{\mathscr{C}_g(G)\right\} = \mathbf{f}(G)^{\text{T}} \cdot \tilde{\mathbf{f}}(G)$$

The variances-covariances are evaluated as follows:

$$\begin{aligned}
N \cdot &\text{Cov}\left\{\mathscr{C}_g(G_1), \mathscr{C}_g(G_2)\right\} \\
&= N \cdot \tilde{\mathbf{f}}(G_1)^{\text{T}} \cdot \text{Cov}\left\{\hat{\mathbf{f}}, \hat{\mathbf{f}}^{\text{T}}\right\} \cdot \tilde{\mathbf{f}}(G_2)
\end{aligned}$$
(I.1)

For what the (k, l)-entry of the covariance matrix $\text{Cov}\left\{\hat{\mathbf{f}}, \hat{\mathbf{f}}^{\text{T}}\right\}$, we have:

$$N \cdot \text{Cov}\left\{\hat{f}_k, \hat{f}_l\right\} = f\left(\frac{k\Delta}{G}\right) \delta_{k,l} - f\left(\frac{k\Delta}{G}\right) \cdot f\left(\frac{l\Delta}{G}\right) \quad \text{(I.2)}$$

where $\delta_{k,l}$ is the Kronecker delta.

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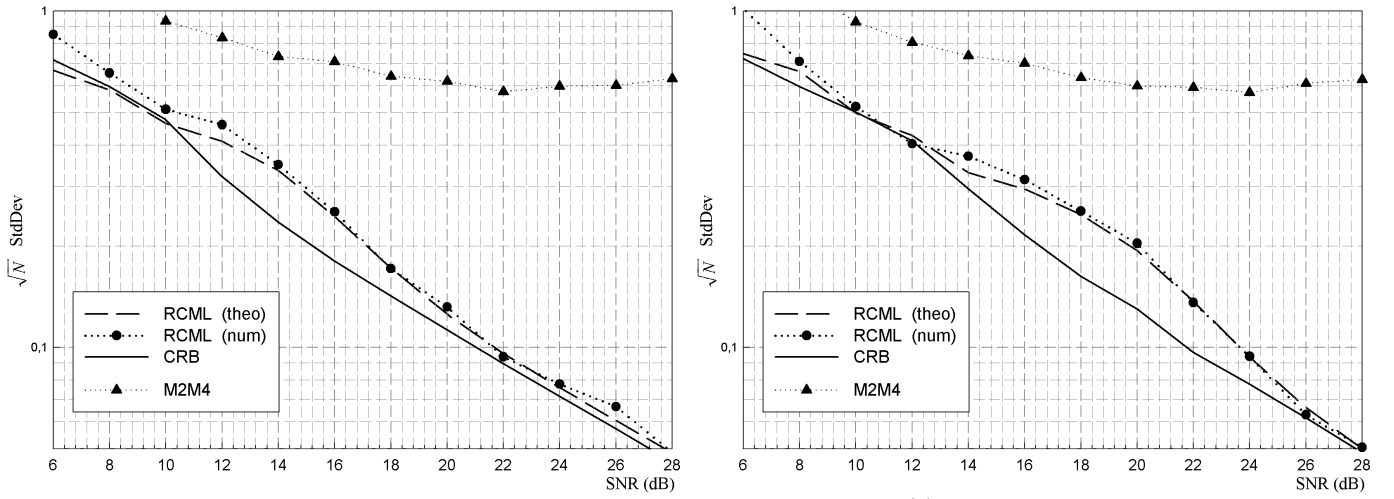
Figure 1: *Normalized standard deviation of the gain estimation error* $\sqrt{N} \cdot StdDev\{\hat{G}^{(f)}\}$ *vs. SNR for 16-QAM and 32-QAM constellations: RCML estimator (numerical: circles, theoretical: dashed line) and M2M4 estimator (triangles). The solid line represents the CRB .*
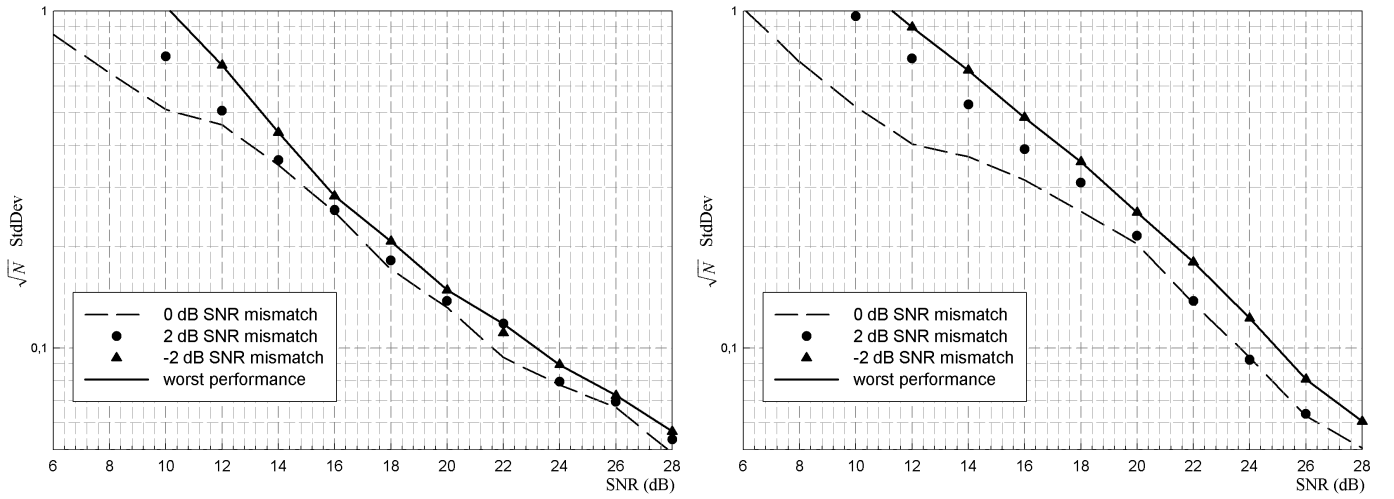

Figure 2: *Normalized standard deviation of the gain estimation error* $\sqrt{N} \cdot StdDev\{\hat{G}^{(f)}\}$ *vs. SNR for 16-QAM and 32-QAM constellations: in presence of* ± 2 *dB SNR estimation mismatch. No mismatch (dashed line) 2dB mismatch (circles),* $-2dB$ *mismatch (triangles), worst case performance (solid line) .*
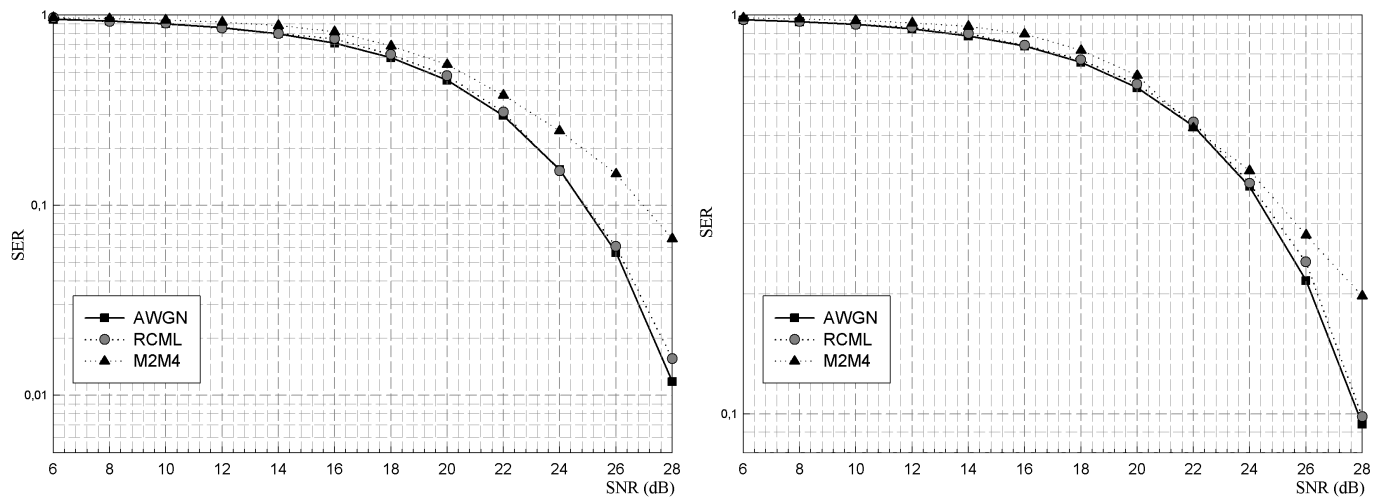

Figure 3: *SER vs. SNR for 256-QAM and 512-QAM constellations RCML estimator (circles gray), and M2M4 estimator (triangles). The black squares represents the AWGN channel .*