TIME-DELAY AND DOPPLER-SHIFT BASED GEOLOCATION BY SEMI-DEFINITE PROGRAMMING

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ABSTRACT

Semi-Definite Programming (SDP) has been widely used for geolocation based on time-delay data. It offers lower computational costs at the expense of slight accuracy decreases. In this work, we consider the case of Doppler data, that is overlooked in existing works, since convex relaxation for Doppler data seems less obvious than for time-delay data. We fill this gap and provide SDP solutions for Doppler-based geolocation. We also show that geolocation based on both Doppler and time-delay data requires the same relaxation as geolocation based on time-delay data only or Doppler data only.

Index Terms— Geolocation, Doppler, Time-delay, Semi-Definite programming, Convex relaxation.

1. INTRODUCTION

We address the geolocation problem, i.e. the estimation of an emitter location using signals collected by receivers at known locations. Geolocation is usually cast as non-linear, non-convex, non-smooth optimization problems. Solving these problems may require excessive computational costs, mostly because of the absence of convexity.

The optimal scheme for geolocation consists of one-step methods in which the signals intercepted by all the sensors are processed simultaneously [1]. However, the scheme implemented in most of existing systems is based on two steps [2]: In a first step, receivers intercepting a signal provide measurements related to the location of the emitter. These measurements can be Angle-of-Arrival (AOA), Received-Signal-Strength (RSS), Time-of-Arrival (TOA), Time-Difference-of-Arrival (TDOA) and Doppler measurements. Then, in a second step, these location-dependent measurements are exploited in order to estimate the emitter location. Two-step methods are not optimal, since each measurement is realized at a sensor independently of the signals intercepted by other sensors.

This research was supported by the Israel Science Foundation (grant 218/08), by the Institute for Future Technologies Research named for the Medvedi, Shwartzman and Gensler Families and by the Israeli Center for Absorption in Science.

1.1. Convex Methods in Geolocation

The subject of non-convex optimization is also known as global optimization [3]. Many deterministic, stochastic and heuristic algorithms have been proposed in that field. Their high complexity, as well as recent advances in convex optimization justify attempts to solve non-convex problems by convex methods [4]. In particular, some advanced softwares have been developed to significantly reduce the computational load of Semidefinite Programming (SDP) and Second-Order Cone Programming (SOCP).

These recent advances in convex optimization have motivated many research works addressing the two-step geolocation problem. The approach consists of a convex relaxation of the initially non-convex problem, so that geolocation can be now easily solved using SDP or SOCP solvers at the expense of slight losses in positioning accuracy. In some works, the convex relaxation is so loose that an algebraic solution can be obtained and there is even no need for SDP or SOCP solvers [5]- [7]. However, in order to avoid superfluous accuracy losses, it is recommended to consider only tighter convex relaxation. In that case, the geolocation problem is relaxed into SDP or SOCP problems with tighter constraints.

1.2. Contribution

Geolocation by SDP or SOCP has already been widely treated in the case of RSS, TOA and TDOA measurements [4], [8], including for Wireless Sensor Networks [9]. Surprisingly, Doppler-based geolocation by SDP and SOCP has not yet been explored. The reason for this omission is that RSS, TOA and TDOA measurements are straightforwardly related to distance, or equivalently to quadratic *equality* constraints. Convex relaxation is easily achieved by changing the latter into quadratic *inequality* constraints. Conversely, the Doppler data structure is significantly more sophisticated, thus convex relaxation for Doppler data is not as obvious as in the case of RSS, TOA and TDOA data. This paper comes to fill this gap, and its contribution is a set of convex relaxation procedures enabling SDP and SOCP for Doppler-based geolocation.

2. SDP AND SOCP IN GEOLOCATION PROBLEMS

For better insight, we recall in this section some properties of SDP and SOCP, with a view toward geolocation applications.

Property 1 *SOCP problems are a subset of SDP problems, i.e. any SOCP problem can be reformulated as a SDP problem.*

To exploit Property 1, one just has to rearrange SOCP cone constraints into SDP inequality constraints on appropriate matrices [10]: Given a $N \times 1$ vector $\mathbf{u}$ and a scalar t, the cone constraint $\|\mathbf{u}\|_2 \leq t$ is equivalent to the constraint $\begin{bmatrix} t\mathbf{I} & u \\ \mathbf{u}^T & t \end{bmatrix} \succeq \mathbf{0}$. Note that the cone constraint involves $(N+1)$ variables, and the SDP constraint involves a matrix with $(N+1)N/2$ variables. Therefore, in practice, reformulating SOCP as SDP is not recommended since it just increases the number of variables, and therefore the computational complexity, without improving the performance. As a rule-of-thumb, SOCP solvers require significantly less computational resources than SDP solvers when addressing the same problem.

Property 2 *SOCP problems should be solved using specific SOCP solvers instead of generic SDP solvers.*

Furthermore, geolocation by SOCP relaxation frequently yields ill-posed problems, with solutions that systematically lie within the convex hull of the receivers [9]. In that case, the resulting location estimates are accurate as long as the emitter actually lies within this convex hull. Conversely, significant positioning errors are systematically observed when the emitter lies outside the convex hull since the location estimate is "trapped" inside.

Property 3 *The performance of geolocation by SOCP relaxation strongly depends on the actual location of the emitter.*

To circumvent this issue, some geolocation methods invoke some SDP relaxations that are tighter than SOCP relaxations. In that case, geolocation by SDP relaxation provides reliable location estimates, at the expense of increased computational costs [9].

In this paper, we derive in Section 3 a SOCP relaxation for Doppler based geolocation. For systems with Doppler data obtained from a complex ambiguity function, time-delay data is frequently available too. Thus, we address in Section 4 the exploitation of time-delay data by the SOCP problem initially derived for Doppler data. Since the reliability of the SOCP relaxation depends on the emitter location (Prop. 3), we propose in Section 5 a SDP relaxation method that provides reliable location estimates for any actual emitter location.

3. SOCP FOR DOPPLER BASED GEOLOCATION

3.1. Doppler Data Model

Consider a positioning system consisting of N receivers embedded in moving platforms with known trajectories, i.e. both their locations and velocities are known. The observed frequency at the n-th receiver is denoted ν_n. Let $\mathbf{s}_n$ be the vector of the n-th receiver location coordinates and $\mathbf{u}$ the vector of the emitter location coordinates. We have:

$$\nu_n = \frac{\nu_0}{c} \frac{(\dot{\mathbf{u}} - \dot{\mathbf{s}}_n)^T (\mathbf{u} - \mathbf{s}_n)}{\|\mathbf{u} - \mathbf{s}_n\|_2} + \nu_0 \tag{1}$$

where c is the propagation speed, ν_0 is the carrier frequency, and the dot superscript on a symbol refers to its time derivative. In this section, it is assumed that ν_0 is known precisely. For better insight, we focus on the case of a static emitter, i.e. its velocity is null thus $\dot{\mathbf{u}} = \mathbf{0}_D$ where $D = 2$ or $D = 3$ depending whether the problem is two-dimensional or three-dimensional. Multiplying $(\nu_n - \nu_0)$ by c/ν_0 yields the standardized data,

$$f_n = -\frac{\dot{\mathbf{s}}_n^T (\mathbf{u} - \mathbf{s}_n)}{\|\mathbf{u} - \mathbf{s}_n\|_2} \tag{2}$$

where f_n is expressed in speed units. Denote by $\mathbf{f}$ the vector obtained by concatenating f_n for any $n = 1, \ldots, N$. Denote by $\hat{\mathbf{f}}$ its noisy version, i.e. $\hat{\mathbf{f}} = \mathbf{f} + \mathbf{e_f}$ where $\mathbf{e_f}$ is composed of the measurement errors. The measurement error vector $\mathbf{e_f}$ has a zero-mean Gaussian distributions with covariance matrix $\sigma_f^2 \mathbf{I}_M$. The Maximum Likelihood (ML) estimator for $\mathbf{u}$ using the Doppler data is the solution of:

$$\mathcal{Q}_{ML}^{(1)} \quad \min_{\mathbf{u}} \quad \frac{1}{\sigma_f^2} \|\hat{\mathbf{f}} - \mathbf{f}\|_2^2 \quad s.t. \begin{cases} f_n = -\frac{\dot{\mathbf{s}}_n^T (\mathbf{u} - \mathbf{s}_n)}{r_n} \\ r_n = \|\mathbf{u} - \mathbf{s}_n\|_2 \end{cases} \tag{3}$$

The cost function to minimize is nonlinear, nonconvex and exhibits numerous local minima. Identifying the global minimum requires a D-dimensional fine grid search. In this work, we propose a SDP approach with lower computational costs. Define the vector $\mathbf{s}$ whose n-th entry is the square range of the emitter to the n-th stations, i.e. $q_n = r_n^2$. Note that

$$\begin{aligned} \|\hat{\mathbf{f}} - \mathbf{f}\|_2^2 &= \sum_{n=1}^{N} \left(\hat{f}_n - f_n\right)^2 = \sum_{n=1}^{N} \left(\hat{f}_n + \frac{\dot{\mathbf{s}}_n^T (\mathbf{u} - \mathbf{s}_n)}{r_n}\right)^2 \\ &= \sum_{n=1}^{N} \left(\frac{\hat{f}_n r_n + \dot{\mathbf{s}}_n^T (\mathbf{u} - \mathbf{s}_n)}{r_n}\right)^2 \\ &= \sum_{n=1}^{N} \frac{\left(\hat{f}_n r_n + \dot{\mathbf{s}}_n^T (\mathbf{u} - \mathbf{s}_n)\right)^2}{q_n} \end{aligned} \tag{4}$$

Thus, the problem $\mathcal{Q}_{ML}^{(1)}$ can be written as

$$\min_{\mathbf{u}} \quad \frac{1}{\sigma_f^2} \sum_{n=1}^{N} t_n \quad s.t. \begin{cases} t_n \geq \frac{\left(\hat{f}_n r_n + \dot{\mathbf{s}}_n^T (\mathbf{u} - \mathbf{s}_n)\right)^2}{q_n} \\ r_n = \|\mathbf{u} - \mathbf{s}_n\|_2 \\ q_n = r_n^2 \end{cases} \tag{5}$$

3.2. SOCP Relaxation for Doppler Data

Note first that the cost function is convex w.r.t. $\mathbf{t}$, and that the constraint on t_n is convex too, since it can be written as the cone constraint ([10], section 2.3)

$$\left\| \begin{bmatrix} 2\left(\hat{f}_n r_n + \dot{\mathbf{s}}_n^T(\mathbf{u}-\mathbf{s}_n)\right) \\ t_n - q_n \end{bmatrix} \right\|_2 \leq t_n + q_n \quad (6)$$

A first possibility to achieve convex relaxation consists of "convexifying" the two last constraints into second-order cone constraints. To that end, the constraint $r_n = \|\mathbf{u}-\mathbf{s}_n\|_2$ is relaxed into $r_n \geq \|\mathbf{u}-\mathbf{s}_n\|_2$, and the constraint $q_n = r_n^2$ into $q_n \geq r_n^2$ (the latter is equivalent to the convex cone constraint $\| \frac{2r_n}{q_n-1} \|_2 \leq q_n + 1$). As a result, the convex problem obtained by second-order cone relaxation of $\mathcal{Q}_{ML}^{(1)}$ is obtained from (6) by changing equality constraints into inequality constraints,

$$\min_{\mathbf{u}} \quad \frac{\mathbf{1}_N^T \mathbf{t}}{\sigma_f^2} \quad s.t. \begin{cases} t_n \geq \frac{\left(\hat{f}_n r_n + \dot{\mathbf{s}}_n^T(\mathbf{u}-\mathbf{s}_n)\right)^2}{q_n} \\ r_n \geq \|\mathbf{u}-\mathbf{s}_n\|_2 \\ q_n \geq r_n^2 \end{cases} \quad (7)$$

Note that the above SOCP relaxation is ill-posed. Indeed, since $t_n \geq 0$, the minimum of the objective function is 0. The objective function can reach 0 simply by letting q_n approach infinity, for any bounded $\mathbf{u}$. Thus, in order to pose the problem correctly, it is necessary to insert a penalty term aimed to avoid infinitely large q_n. A common proposal for this penalty term is simply $\sum_{n=1}^{N} q_n = \mathbf{1}_N^T \mathbf{q}$, and the proposed relaxation for $\mathcal{Q}_{ML}^{(1)}$ is finally

$$\min_{\mathbf{u}} \quad \frac{\mathbf{1}_N^T \mathbf{t}}{\sigma_f^2} + \mathbf{1}_N^T \mathbf{q} \quad s.t. \begin{cases} t_n \geq \frac{\left(\hat{f}_n r_n + \dot{\mathbf{s}}_n^T(\mathbf{u}-\mathbf{s}_n)\right)^2}{q_n} \\ r_n \geq \|\mathbf{u}-\mathbf{s}_n\|_2 \\ q_n \geq r_n^2 \end{cases} \quad (8)$$

This optimization problem can be solved by SOCP. The standard SOCP formulation for (8) is

$$\min_{\mathbf{u}} \quad \frac{\mathbf{1}_N^T \mathbf{t}}{\sigma_f^2} + \mathbf{1}_N^T \mathbf{q} \quad (9)$$

$$s.t. \begin{cases} t_n + q_n \geq \left\| \begin{bmatrix} 2\left(\hat{f}_n r_n + \dot{\mathbf{s}}_n^T(\mathbf{u}-\mathbf{s}_n)\right) \\ t_n - q_n \end{bmatrix} \right\|_2 \\ r_n \geq \|\mathbf{u}-\mathbf{s}_n\|_2 \\ q_n + 1 \geq \left\| \begin{bmatrix} 2r_n \\ (q_n - 1) \end{bmatrix} \right\|_2 \end{cases} \quad (10)$$

3.3. Discussion

The SOCP problem (8) for Doppler-based geolocation is obtained from $\mathcal{Q}_{ML}^{(1)}$ simply by changing equality constraints into inequality constraints (a penalty term is also inserted to avoid ill-posed formulations). Therefore, the obtained estimator is strongly related to the ML estimator.

4. INTEGRATING TIME-DELAY INFORMATION

4.1. Time-Delay Data Model

In many positioning systems, the Doppler data is obtained together with time-delay data. For example, consider a positioning system built of N receivers embedded in platforms, where the transmit signal waveform is known at the receivers but the transmit time t_0 is unknown. Using the ambiguity function evaluated independently at each receiver, we obtain N pairs of measurements composed of a time-delay measurement and a Doppler-shift measurement. The time-delay measurement at the n-th receiver is denoted by τ_n. We have:

$$\tau_n = \frac{1}{c}\|\mathbf{u}-\mathbf{s}_n\|_2 + t_0 = \frac{1}{c} r_n + t_0 \quad (11)$$

where $r_n = \|\mathbf{u}-\mathbf{s}_n\|_2$. Multiplying τ_n by c yields the standardized data,

$$d_n = \|\mathbf{u}-\mathbf{s}_n\|_2 + d_0 = r_n + d_0 \quad (12)$$

where d_n is expressed in distance units and $d_0 \triangleq ct_0$. Note that $f_n = \dot{d}_n$. Denote by $\mathbf{r}$ the vector whose n-th entry is $\|\mathbf{u}-\mathbf{s}_n\|_2$. Denote by $\mathbf{d} = \mathbf{r} + d_0 \mathbf{1}_N$ the vector obtained by concatenating d_n for any $n = 1, \ldots, N$. Denote by $\hat{\mathbf{d}} = \mathbf{r} + d_0\mathbf{1}_N + \mathbf{e_d}$ its noisy version where $\mathbf{e_d}$ is the vector of measurement errors. The time-delay measurement error $\mathbf{e_d}$ has a zero-mean Gaussian distribution with covariance matrix $\sigma_d^2 \mathbf{I}_M$, and $\mathbf{e_d}$ is independent of the Doppler-shift error $\mathbf{e_f}$. The Maximum Likelihood estimator for $\mathbf{u}$ is the solution of:

$$\mathcal{Q}_{ML}^{(2)} \qquad \min_{\mathbf{u}} \frac{1}{\sigma_f^2}\|\hat{\mathbf{f}} - \mathbf{f}\|_2^2 + \frac{1}{\sigma_d^2}\|\hat{\mathbf{d}} - \mathbf{r} - d_0\mathbf{1}_N\|_2^2 \quad (13)$$

$$s.t. \quad \begin{cases} f_n &= -\frac{\dot{\mathbf{s}}_n^T(\mathbf{u}-\mathbf{s}_n)}{r_n} \\ r_n &= \|\mathbf{u}-\mathbf{s}_n\|_2 \end{cases} \quad (14)$$

In other words, the optimization problem $\mathcal{Q}_{ML}^{(2)}$ is actually obtained by adding a time-delay based objective to the Doppler-shift based objective function of $\mathcal{Q}_{ML}^{(1)}$. One can easily check that $\mathcal{Q}_{ML}^{(2)}$ is equivalent to

$$\mathcal{Q}_{ML}^{(2)} \qquad \min_{\mathbf{u}} \frac{1}{\sigma_f^2}\|\hat{\mathbf{f}} - \mathbf{f}\|_2^2 + \frac{1}{\sigma_d^2}\|\mathbf{P}(\hat{\mathbf{d}} - \mathbf{r})\|_2^2 \quad (15)$$

$$s.t. \quad \begin{cases} f_n &= -\frac{\dot{\mathbf{s}}_n^T(\mathbf{u}-\mathbf{s}_n)}{\|\mathbf{u}-\mathbf{s}_n\|_2} \\ r_n &= \|\mathbf{u}-\mathbf{s}_n\|_2 \end{cases} \quad (16)$$

where $\mathbf{P} = \mathbf{I}_M - \frac{1}{M}\mathbf{1}_M\mathbf{1}_M^T$. Again, this Maximum Likelihood cost function is nonlinear, nonconvex with numerous local minima, making its minimization a sensitive task.

4.2. SOCP Relaxation for Doppler & Time-Delay Data

Following the same steps as in (4)-(5), we rewrite $\mathcal{Q}_{ML}^{(2)}$ as:

$$\mathcal{Q}_{ML}^{(2)} \qquad \min_{\mathbf{u}} \frac{1}{\sigma_f^2}\sum_{n=1}^{N} t_n + \frac{1}{\sigma_d^2}\|\mathbf{P}(\hat{\mathbf{d}}-\mathbf{r})\|_2^2 \tag{17}$$

$$s.t. \quad \begin{cases} t_n \geq \frac{\left(\hat{f}_n r_n + \dot{\mathbf{s}}_n^T(\mathbf{u}-\mathbf{s}_n)\right)^2}{q_n} \\ r_n = \|\mathbf{u}-\mathbf{s}_n\|_2 \\ q_n = r_n^2 \end{cases} \tag{18}$$

Following the same steps as in (7)-(8), second-order cone relaxations of $\mathcal{Q}_{ML}^{(2)}$ yields the SOCP $\mathcal{Q}^{(2)}$ defined by

$$\mathcal{Q}^{(2)} \qquad \min_{\mathbf{u}} \frac{1}{\sigma_f^2}\mathbf{1}_N^T\mathbf{t} + \mathbf{1}_N^T\mathbf{q} + \frac{1}{\sigma_d^2}\|\mathbf{P}(\hat{\mathbf{d}}-\mathbf{r})\|_2^2 \tag{19}$$

$$s.t. \quad \begin{cases} t_n \geq \frac{\left(\hat{f}_n r_n + \dot{\mathbf{s}}_n^T(\mathbf{u}-\mathbf{s}_n)\right)^2}{q_n} \\ r_n \geq \|\mathbf{u}-\mathbf{s}_n\|_2 \\ q_n \geq r_n^2 \end{cases} \tag{20}$$

This convex problem can be solved using SOCP solvers.

4.3. Discussion

The Maximum Likelihood cost function associated with time-delay data is added to the SOCP problem designed for Doppler data *with no need for further convex relaxations*.

5. SDP FOR ROBUST GEOLOCATION

We have mentioned earlier the drawbacks of the SOCP relaxation, and the motivation for SDP relaxation to circumvent them. We describe here a SDP relaxation approach. Define $\mathbf{Q} = \mathbf{r}\mathbf{r}^T$ and $z = \|\mathbf{u}\|_2^2$. Then, the following identities hold:

$$\|\mathbf{P}(\hat{\mathbf{r}}-\mathbf{r})\|_2^2 = tr\left\{\begin{bmatrix} \mathbf{Q} & \mathbf{r} \\ \mathbf{r}^T & 1 \end{bmatrix}\begin{bmatrix} \mathbf{P} & -\mathbf{P}\hat{\mathbf{r}} \\ -\hat{\mathbf{r}}^T\mathbf{P} & \|\mathbf{P}\hat{\mathbf{r}}\|_2^2 \end{bmatrix}\right\} \tag{21}$$

$$\|\mathbf{u}-\mathbf{s}_n\|_2^2 = tr\left\{\begin{bmatrix} \mathbf{I} & \mathbf{u} \\ \mathbf{u}^T & z \end{bmatrix}\begin{bmatrix} \mathbf{s}_n\mathbf{s}_n^T & -\mathbf{s}_n \\ -\mathbf{s}_n^T & 1 \end{bmatrix}\right\} \tag{22}$$

$$q_n = \{\mathbf{Q}\}_{n,n} \tag{23}$$

Then, inserting (21)-(23) in (17), the ML problem $\mathcal{Q}_{ML}^{(2)}$ can be equivalently written as

$$\min_{\mathbf{u}} \frac{\mathbf{1}_N^T\mathbf{t}}{\sigma_f^2} + \frac{1}{\sigma_d^2}tr\left\{\begin{bmatrix} \mathbf{Q} & \mathbf{r} \\ \mathbf{r}^T & 1 \end{bmatrix}\begin{bmatrix} \mathbf{P} & -\mathbf{P}\hat{\mathbf{r}} \\ -\hat{\mathbf{r}}^T\mathbf{P} & \|\mathbf{P}\hat{\mathbf{r}}\|_2^2 \end{bmatrix}\right\} \tag{24}$$

$$s.t. \quad \begin{cases} t_n \geq \frac{\left(\hat{f}_n r_n + \dot{\mathbf{s}}_n^T(\mathbf{u}-\mathbf{s}_n)\right)^2}{\{\mathbf{Q}\}_{n,n}} \\ \{\mathbf{Q}\}_{n,n} = tr\left\{\begin{bmatrix} \mathbf{I} & \mathbf{u} \\ \mathbf{u}^T & z \end{bmatrix}\begin{bmatrix} \mathbf{s}_n\mathbf{s}_n^T & -\mathbf{s}_n \\ -\mathbf{s}_n^T & 1 \end{bmatrix}\right\} \\ r_n \geq 0 \\ \mathbf{Q} = \mathbf{r}\mathbf{r}^T \quad \text{and} \quad z = \|\mathbf{u}\|_2^2 \end{cases} \tag{25}$$

This optimization problem is not convex only because of the two last constraints. Thus, similarly to many recent publications addressing SDP-based geolocation in the absence of Doppler data, the non-convex constraints

$$\mathbf{Q} = \mathbf{r}\mathbf{r}^T \quad \text{and} \quad z = \|\mathbf{u}\|_2^2$$

are "convexified" by the mean of the convex relaxation:

$$\mathbf{Q} \succeq \mathbf{r}\mathbf{r}^T \quad \text{and} \quad z \geq \|\mathbf{u}\|_2^2$$

where $\mathbf{A} \succeq \mathbf{B}$ means that the matrix $\mathbf{A}-\mathbf{B}$ is positive semi-definite. These new constraints are equivalent to

$$\begin{bmatrix} \mathbf{Q} & \mathbf{r} \\ \mathbf{r}^T & 1 \end{bmatrix} \succeq \mathbf{0} \quad \text{and} \quad \begin{bmatrix} \mathbf{I} & \mathbf{u} \\ \mathbf{u}^T & z \end{bmatrix} \succeq \mathbf{0}$$

Therefore, the proposed SDP relaxation is

$$\min_{\mathbf{u}} \frac{\mathbf{1}_N^T\mathbf{t}}{\sigma_f^2} + \frac{1}{\sigma_d^2}tr\left\{\begin{bmatrix} \mathbf{Q} & \mathbf{r} \\ \mathbf{r}^T & 1 \end{bmatrix}\begin{bmatrix} \mathbf{P} & -\mathbf{P}\hat{\mathbf{r}} \\ -\hat{\mathbf{r}}^T\mathbf{P} & \|\mathbf{P}\hat{\mathbf{r}}\|_2^2 \end{bmatrix}\right\} \tag{26}$$

$$s.t. \quad \begin{cases} t_n \geq \frac{\left(\hat{f}_n r_n + \dot{\mathbf{s}}_n^T(\mathbf{u}-\mathbf{s}_n)\right)^2}{\{\mathbf{Q}\}_{n,n}} \\ \{\mathbf{Q}\}_{n,n} = tr\left\{\begin{bmatrix} \mathbf{I} & \mathbf{u} \\ \mathbf{u}^T & z \end{bmatrix}\begin{bmatrix} \mathbf{s}_n\mathbf{s}_n^T & -\mathbf{s}_n \\ -\mathbf{s}_n^T & 1 \end{bmatrix}\right\} \\ r_n \geq 0 \\ \begin{bmatrix} \mathbf{Q} & \mathbf{r} \\ \mathbf{r}^T & 1 \end{bmatrix} \succeq \mathbf{0} \quad \text{and} \quad \begin{bmatrix} \mathbf{I} & \mathbf{u} \\ \mathbf{u}^T & z \end{bmatrix} \succeq \mathbf{0} \end{cases} \tag{27}$$

This convex problem can be solved using SDP solvers.

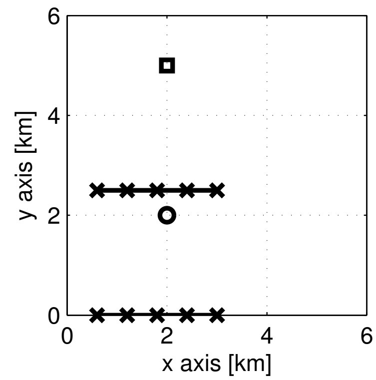
6. NUMERICAL EXAMPLES

Two receivers are embedded in mobile platforms with velocity equal to 300 [m/s] and trajectories parallel to the x-axis, at $y = 0$ and $y = 2500$ [m] as shown in Fig. 1 (thick lines). Pairs of time-delay and Doppler-shift measurements are realized every 2 seconds (thick crosses), and each receiver performs 5 pairs of measurements, so that there are all together 10 pairs of measurements. The standard deviation of small errors in time-delay is 25 [m]. The standard deviation of Doppler errors is increased from 1 to 20 [m/s]. A set of 200 experiments is realized. In each experiment, the errors are drawn from a zero mean Gaussian distribution.

In a first experiment, the emitter is placed at (2000, 2000) [m] (thick circle in Fig. 1). Geolocation is achieved by SOCP relaxation for Doppler data only (8), by SOCP relaxation for Doppler and time-delay data (20), and by SDP relaxation for Doppler and time-delay data (27). The performance of these three algorithms is plotted in Fig. 2, together with the Cramer-Rao bounds. As expected, SOCP relaxation with time-delay data achieves better results than without. But the main result is that when using both Doppler and time-delay data, the SOCP relaxation provides slightly better results than the estimator based on SDP relaxation. A further drawback of SDP over SOCP is its increased computational costs. From this experiment, one may erroneously conclude that SDP offers no benefits over SOCP. Note that in this experiment, the emitter location lies in the convex hull of the successive receivers locations, i.e. is encompassed within the geographical area defined by the receivers trajectories.

In a second experiment, the emitter is placed out of this area at (2000, 5000) [m] (thick square in Fig. 1). The performance of the considered algorithms is plotted in Fig. 3. In this setting, the SOCP algorithms (Doppler data with and without time-delay data) both fail in providing location estimates since the positioning error is about 3000 [m]. Conversely, the SDP algorithm still provides accurate location estimates, as in the first experiment. Therefore, this experiment confirms that SDP relaxation ensures robustness against emitter lying at locations that make SOCP-based geolocation fail.

Fig. 1: The successive locations of the receivers (cross markers) and the emitter location (circle marker for the 1st experiment, square marker for the 2nd experiment)



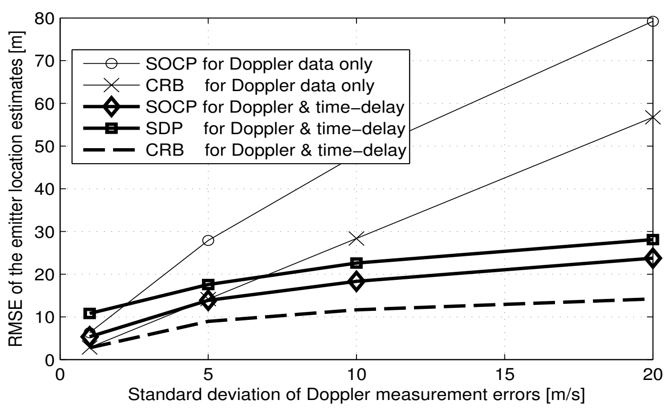


Fig. 2: Performance of SOCP and SDP relaxations for an emitter location encompassed by receivers trajectories.

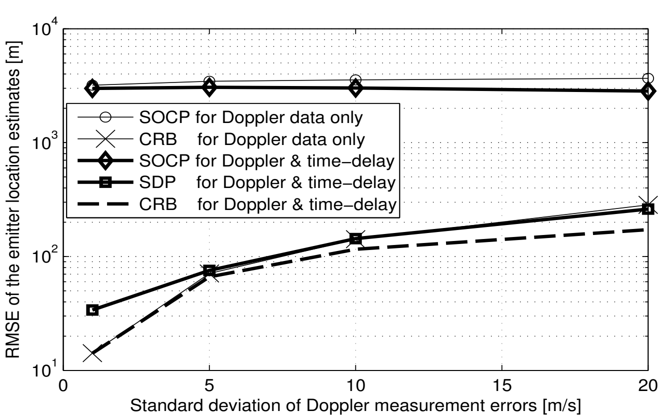


Fig. 3: Performance of SOCP and SDP relaxations for an emitter location *not* encompassed by receivers trajectories.

7. CONCLUSION

In this work we have exploited convex relaxation methods to solve the geolocation problem using Doppler data and Doppler with time-delay data. Two convex relaxation approaches are considered, enabling positioning by Second-Order Cone Programming (SOCP) or Semi-Definite Programming (SDP). The former requires lower computational resources, but accurate positioning is not guaranteed for any emitter location. Conversely, the SDP approach systematically provides accurate estimates, at the expense of increased computational costs. Furthermore, we have shown that adding time-delay data to the Doppler data set does not require further relaxations to get SOCP and SDP estimators.

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