STATISTICALLY OPTIMAL SELF-CALIBRATION OF REGULAR IMAGING ARRAYS

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ABSTRACT

Many imaging arrays have a regular sensor configuration. This regularity can be exploited for self-calibration of the array. In this paper, we introduce a new self-calibration method for regular arrays based on weighted alternating least squares (WALS) optimization that appears to be statistically efficient and does not impose requirements on the source structure or on pre-calibration of the array. We show results from Monte Carlo simulations indicating that the proposed method already attains the Cramer-Rao bound (CRB) at very low SNR and produces unbiased results. Our simulations also indicate that the approach most commonly used in the literature does not attain the CRB at high SNR and produces biased results at low SNR.

Index Terms— autocalibration, self-calibration, uniform rectangular array, uniform linear array, redundancy calibration

1. INTRODUCTION

Calibration errors can significantly reduce the performance of high dynamic range imaging methods and high-resolution DOA estimation techniques. Fortunately, many imaging arrays used in, for example, SAR applications and radio astronomy exploit regularly spaced antennas. This regularity can be exploited for self-calibration of the array as originally discussed indepedently in [1,2]. Based on new developments in several fields, several groups recently started to study this problem again [3–5]. Since these methods exploit the fact that the regularity of the array causes specific spatial frequencies to be measured multiple times thus making the information contained in the array covariance matrix redundant, we refer to such methods as redundancy calibration.

Unfortunately, all methods proposed so far are either not statistically optimal (i.e., they produce biased results or results with a relatively high variance) [1, 2, 4, 5], or impose requirements on the source structure (e.g., a limited number of discrete sources) [6] or require some form of pre-calibration [3]. In this paper, we present a statistically optimal self-calibration scheme that should work for arbitrary source structures and only assumes that the array is composed of identical antennas, which is usually the case for imaging arrays. Although the algorithm is inspired by and exploits some of the results in [7], it deals with a different problem: the algorithm proposed here deals with self-calibration of a regular array without requiring prior knowledge of the source structure while [7] discusses self-calibration of an array with arbitrary geometry using prior knowledge of the source structure. We compare this algorithm with the commonly used method based on the logarithm of the elements of the array covariance matrix and with the Cramer-Rao bound (CRB), showing that the proposed algorithm is unbiased and attains the CRB.

The paper is organised as follows: in the next section, we introduce the data model and provide a description of both algorithms. In Sec. 3 we assess the performance of both algorithms using Monte Carlo simulations by checking for biases and comparison with the CRB before summarizing our conclusions in Sec. 4.

Notation: $\angle a$ denotes the phase of the complex value a, while $|a|$ gives its absolute value. The natural logarithm is denoted by $\log$, the matrix transpose by ^{T}, the Hermitian transpose by ^{H} and the pseudo-inverse by †. The Frobenius norm is denoted by $\|\cdot\|_F^2$, conjugation by $\bar{\cdot}$, an estimated value by $\hat{\cdot}$ and the Kronecker product by $\otimes$. $\mathbf{0}$ represents a vector of appropriate size filled with zeros, $\mathrm{diag}(\mathbf{a})$ forms a diagonal matrix $\mathbf{A}$ with the vector $\mathbf{a}$ on its main diagonal, $\mathrm{vec}(\mathbf{A})$ vectorizes the matrix $\mathbf{A}$ by stacking its columns and $\mathrm{unvec}(\cdot)$ undoes this operation.

2. THEORY

2.1. Data model

The $P \times P$ array covariance matrix $\mathbf{R}$ of a P-element sensor array with unknown receiver gains $\boldsymbol{\gamma} = [\gamma_1, \cdots, \gamma_P]^T$ and phases $\boldsymbol{\phi} = [\mathrm{e}^{\mathrm{j}\phi_1}, \mathrm{e}^{\mathrm{j}\phi_2}, \cdots, \mathrm{e}^{\mathrm{j}\phi_P}]^T$ and unknown receiver noise powers $\boldsymbol{\sigma}_\mathrm{n} = [\sigma_{\mathrm{n},1}, \sigma_{\mathrm{n},2}, \cdots, \sigma_{\mathrm{n},P}]^T$ can be

described as

$$\mathbf{R} = \boldsymbol{\Gamma}\boldsymbol{\Phi}\mathbf{R}_0\boldsymbol{\Phi}^H\boldsymbol{\Gamma}^H + \boldsymbol{\Sigma}_n = \mathbf{G}\mathbf{R}_0\mathbf{G}^H + \boldsymbol{\Sigma}_n. \quad (1)$$

where $\mathbf{G} = \boldsymbol{\Gamma}\boldsymbol{\Phi} = \mathrm{diag}\,(\mathbf{g})$ with $\boldsymbol{\Gamma} = \mathrm{diag}\,(\boldsymbol{\gamma})$ end $\boldsymbol{\Phi} = \mathrm{diag}\,(\boldsymbol{\phi})$, $\mathbf{R}_0$ represents the array covariance matrix that would be measured by an ideal array without receiver noise and calibration errors and $\boldsymbol{\Sigma}_n = \mathrm{diag}\,(\boldsymbol{\sigma}_n)$. Note that $\mathbf{R}_0$ has a Toeplitz-structure for a uniform linear array (ULA) and a Toeplitz-block Toeplitz structure for a uniform rectangular array (URA) [4]. This property is exploited in redundancy calibration by correcting any deviation from this structure in $\mathbf{R}$ by compensating the complex valued receiver gains $\mathbf{G}$.

2.2. Conventional redundancy calibration

The conventional approach [1, 2, 4, 5] is based on the observation that the off-diagonal elements of $\mathbf{R}$ are described by

$$R_{ij} = \gamma_i \mathrm{e}^{\mathrm{j}\phi_i} \gamma_j \mathrm{e}^{-\mathrm{j}\phi_j} R_{0,ij}. \quad (2)$$

The problem of finding amplitudes and phases is then split into two separate problems by taking the logarithm, such that

$$\log|R_{ij}| = \log\gamma_i + \log\gamma_j + \log|R_{0,ij}| \quad (3)$$
$$\angle R_{ij} = \phi_i - \phi_j + \angle R_{0,ij}. \quad (4)$$

Equation (4) may suffer from a 2π phase ambiguity. Possible solutions to this problem have been proposed (see, e.g., [3]), so we will not elaborate on this issue, which is outside the scope of this paper.

Due to the Toeplitz or Toeplitz-block Toeplitz structure of $\mathbf{R}_0$, we can identify sets $\mathcal{S}_q$ with $q = 1, \cdots, Q$ of pairs of indices such that $R_{0,ij} = R_{0,q} \forall\, (i,j) \in \mathcal{S}_q$. For each of these sets of redundantly measured spatial frequencies, we can build a system of equations cf. (3) and (4) with $R_{0,ij} = R_{0,q}$. For the gain phases, we can therefore define a parameter vector $\boldsymbol{\theta}_\phi = [\phi_1, \phi_2, \cdots, \phi_P, \angle R_{0,1}, \angle R_{0,2}, \cdots, \angle R_{0,Q}]^T$ and define the data vector as

$$\boldsymbol{\psi}_\phi = \begin{bmatrix} \mathrm{vec}\left(\angle\widehat{\mathbf{R}}_{ij}\right) \\ \mathbf{0} \end{bmatrix}. \quad (5)$$

Since there is no prior knowledge on the source structure, there is an identifiability problem: a phase gradient over the array is unidentifiable from a pointing error. Furthermore, we can only observe phase differences between the elements. The zeros at the end of the data vector are added to describe the phase constraints needed to resolve these ambiguities. In the literature, they are usually resolved by imposing the following constraints (see, e.g., [4, 6]):

$$\sum_{p=1}^{P}\phi_p = 0, \quad \sum_{p=1}^{P}x_p\phi_p = 0, \quad \sum_{p=1}^{P}y_p\phi_p = 0, \quad (6)$$

where (x_p, y_p) denotes the position of the pth element in a Cartesian coordinate frame. With these definitions, we can formulate the phase estimation problem as

$$\boldsymbol{\psi}_\phi = \mathbf{M}_\phi\boldsymbol{\theta}_\phi, \quad (7)$$

where the last three rows of $\mathbf{M}$ describe the constraints formulated in (6) and the other rows describe the set of equations defined by (4). This provides a straightforward solution for $\boldsymbol{\theta}_\phi$.

With a similar definition for $\boldsymbol{\theta}_\gamma$, $\boldsymbol{\psi}_\gamma$ and $\mathbf{M}_\gamma$, we can obtain a solution for the amplitudes as well. Since we are estimating both $\mathbf{g}$ and $\mathbf{R}_0$, the gain vector can only be determined up to a scaling factor. In this paper, we will normalize $\mathbf{g}$ such that it has norm 1.

2.3. Weighted Alternating Least Squares

In the conventional approach, the logarithm is used to separate the problem into a phase estimation problem and an amplitude estimation problem. The disadvantage of applying a logarithm to the measured data is, that it disturbs the probability density function of the noise on the data, which may cause problems in cases with low SNR [3, 5]. We therefore propose to solve the problem using a weighted alternating least squares (WALS) approach that iterates between estimating the complex valued sensor gains and $\mathbf{R}_0$.

Based on (1), we can formulate our calibration problem as the weighted least squares estimation problem

$$\left\{\widehat{\mathbf{g}}, \widehat{\mathbf{R}}_0, \widehat{\boldsymbol{\sigma}}_n\right\} =$$
$$\underset{\mathbf{g}, \mathbf{R}_0, \boldsymbol{\sigma}_n}{\mathrm{argmin}} \left\| \mathbf{W}_c \left(\widehat{\mathbf{R}} - \mathbf{G}\mathbf{R}_0\mathbf{G}^H - \boldsymbol{\Sigma}_n\right) \mathbf{W}_c \right\|_F^2 (8)$$

Covariance matched weighting provides estimates that are asymptotically, for a large number of samples, equivalent to ML estimates [8]. We therefore choose $\mathbf{W}_c = \mathbf{R}^{-1/2}$. The knowledge of the structure of $\mathbf{R}_0$ can be included in the estimation problem by defining a real valued parameter vector $\boldsymbol{\theta}$ containing the real and imaginary parts of all unique values in $\mathbf{R}_0$ and a selection matrix $\mathbf{I}_s$ such that $\mathrm{vec}\,(\mathbf{R}_0) = \mathbf{I}_s\boldsymbol{\theta}$. With these substitutions, we can reformulate our estimation problem as

$$\left\{\widehat{\mathbf{g}}, \widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\sigma}}_n\right\} =$$
$$\underset{\mathbf{g}, \boldsymbol{\theta}, \boldsymbol{\sigma}_n}{\mathrm{argmin}} \left\| \mathbf{W} \left(\mathrm{vec}\left(\widehat{\mathbf{R}} - \boldsymbol{\Sigma}_n\right) - \left(\overline{\mathbf{G}} \otimes \mathbf{G}\right) \mathbf{I}_s\boldsymbol{\theta}\right) \right\|_F^2 (9)$$

where $\mathbf{W} = \left(\overline{\mathbf{R}}^{-1} \otimes \mathbf{R}^{-1}\right)$.

If $\boldsymbol{\theta}$ and $\boldsymbol{\sigma}_n$ are known, we can solve for $\mathbf{g}$. This problem has been extensively discussed in [7], so we simply

state the solution here. First, construct the column vectors

$$\mathbf{c}_{1,ij} = \left[\left\{\widehat{R}_{ik}R_{0,jk} : i,j \neq k\right\}\right] \qquad (10)$$

$$\mathbf{c}_{2,ij} = \left[\left\{R_{0,ik}\widehat{R}_{jk} : i,j \neq k\right\}\right] \qquad (11)$$

where the curly braces denote a set of values $\widehat{R}_{ik}R_{0,jk}$ whose indices satisfy the indicated condition. We use these vectors to construct a matrix $\mathbf{M}$ with entries $M_{ij} = \mathbf{c}_{2,ij}^{\dagger}\mathbf{c}_{1,ij}$. The gain vector $\mathbf{g}$ can be extracted from this matrix by an eigenvalue decomposition.

If $\mathbf{g}$ and $\boldsymbol{\sigma}_{\mathrm{n}}$ are known, $\boldsymbol{\theta}$ follows from

$$\widehat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}}{\arg\min} \left\| \mathbf{W}\mathrm{vec}\left(\widehat{\mathbf{R}} - \boldsymbol{\Sigma}_{\mathrm{n}}\right) - \mathbf{W}\left(\overline{\mathbf{G}} \otimes \mathbf{G}\right)\mathbf{I}_{\mathrm{s}}\boldsymbol{\theta} \right\|_F^2$$

$$= \left(\mathbf{W}\left(\overline{\mathbf{G}} \otimes \mathbf{G}\right)\mathbf{I}_{\mathrm{s}}\right)^{\dagger} \mathbf{W}\mathrm{vec}\left(\widehat{\mathbf{R}} - \boldsymbol{\Sigma}_{\mathrm{n}}\right). \qquad (12)$$

The diagonal elements of $\mathbf{R}$ can be modeled to arbitrary precision by compensating the difference between $\mathbf{R}_0$ perturbed by the sensor gains with the receiver noise powers. To avoid the interplay between the sets of parameters, we replace $\widehat{\mathbf{R}} - \boldsymbol{\Sigma}_{\mathrm{n}}$ by $\widehat{\mathbf{R}}_-$, which is equal to $\widehat{\mathbf{R}}$ with the entries on its main diagonal set to zero. Replacing the pseudo-inverse with the Moore-Penrose inverse, we obtain

$$\widehat{\boldsymbol{\theta}} = \left(\mathbf{I}_{\mathrm{s}}^H\left(\overline{\mathbf{G}}^H\overline{\widehat{\mathbf{R}}}^{-1}\overline{\mathbf{G}} \otimes \mathbf{G}^H\widehat{\mathbf{R}}^{-1}\mathbf{G}\right)\mathbf{I}_{\mathrm{s}}\right)^{-1} \times$$

$$\mathbf{I}_{\mathrm{s}}^H\left(\overline{\mathbf{G}}^H\overline{\widehat{\mathbf{R}}}^{-1} \otimes \mathbf{G}^H\widehat{\mathbf{R}}^{-1}\right)\mathrm{vec}\left(\mathbf{R}_-\right). \quad (13)$$

Note that we have used the measured covariance matrix $\widehat{\mathbf{R}}$ instead of the true covariance matrix $\mathbf{R}$, which is unknown in actual measurements, to provide the covariance matched weighting. Also note, that although $\mathbf{R}^{-1/2}$ provides optimal weighting in the original formulation of the problem in (8), the closed form solution only requires $\widehat{\mathbf{R}}$ and does not require computation of its matrix square root.

Based on these results, we propose the following WALS algorithm:

1. *Initialization* Set the iteration counter $i = 1$, and initialize the gains $(\widehat{\mathbf{g}}^{[0]})$ by setting all of them to unity (or another equal value).
2. *Estimate* $\widehat{\boldsymbol{\theta}}^{[i]}$ using (13) with $\mathbf{G} = \mathrm{diag}\left(\mathbf{g}^{[i-1]}\right)$ and compute $\mathbf{R}_0^{[i]} = \mathrm{unvec}\left(\mathbf{I}_{\mathrm{s}}\boldsymbol{\theta}^{[i]}\right)$
3. *Estimate* $\mathbf{g}^{[i]}$ using the procedure in [7] summarized above with $\mathbf{R}_0 = \mathbf{R}_0^{[i]}$ and apply the gain constraint and phase constraints to resolve the gain and phase ambiguities.
4. *Check for convergence or maximum number of iterations* If $\frac{1}{P}\sum_{p=1}^{P}\left|g_p^{[i-1]} - g_p^{[i]}\right| / \left|g_p^{[i]}\right| < \delta$ or $i >$

$i_{\max}$, stop, otherwise increase i by 1 and continue with step 2. In our simulations, we used $\delta = 10^{-6}$ and $i_{\max} = 15$.
5. *Estimate* $\boldsymbol{\sigma}_{\mathrm{n}}$ by setting them equal to the diagonal elements of $\widehat{\mathbf{R}}$.

2.4. Improving computational efficiency

Even if (13) is implemented efficiently by exploiting the structure of $\mathbf{I}_s$, it still requires order P^4 multiplications due to the Kronecker product. This makes this part of the algorithm the computationally most demanding part.

The operation described in (13) should, in principle, represent a weighted average over all baselines in each set $\mathcal{S}_q$ of redundantly measured spatial frequencies. If all sensors have approximately the same gain and receiver noise, all covariances in each set $\mathcal{S}_q$ are measured with the same SNR and the covariance matched weighting, which introduces the Kronecker product, is not expected to give a big improvement. This holds in particular for the low-SNR regime, where we have $\mathbf{R} \approx \sigma_n\mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix. If we omit $\mathbf{W}$ and calibrate the measured covariance matrix using the complex valued sensor gains obtained in the previous iteration of the algorithm, we can formulate the problem of estimating $\boldsymbol{\theta}$ as

$$\widehat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}}{\arg\min} \left\| \left(\overline{\mathbf{G}} \otimes \mathbf{G}\right)^{-1}\mathrm{vec}\left(\mathbf{R}_-\right) - \mathbf{I}_s\boldsymbol{\theta} \right\|_F^2. \quad (14)$$

Solving for $\boldsymbol{\theta}$ gives

$$\widehat{\boldsymbol{\theta}} = \left(\mathbf{I}_s^H\mathbf{I}_s\right)^{-1}\mathbf{I}_s^H\left(\mathrm{vec}\left(\mathbf{R}_-\right) \oslash \left(\overline{\mathbf{g}} \otimes \mathbf{g}\right)\right), \qquad (15)$$

where $\oslash$ denotes element-wise division. Since the structure of $\mathbf{I}_s$ makes $\mathbf{I}_s^H\mathbf{I}_s$ a diagonal matrix with the multiplicity of the respective parameters on the main diagonal, the gain correction of the measured values and the multiplication of the result with $\mathbf{I}_s^H$ has now become the most demanding part of the calculation. However, both operations require only order P^2 multiplications, making the gain estimation the most demanding part of the algorithm with order P^3 operations [7]. We have tested this idea using Monte Carlo simulations for the same array and source model as described in the next section and found that it produced similar results. This simplified algorithm has the same order P^3 computational complexity as the conventional approach.

3. MONTE CARLO SIMULATIONS

We have conducted Monte Carlo simulations to compare the statistical performance of the WALS based redundancy calibration with conventional redundancy calibration and the Cramer-Rao bound (CRB). These simulations were done for an 8×8 uniform rectangular
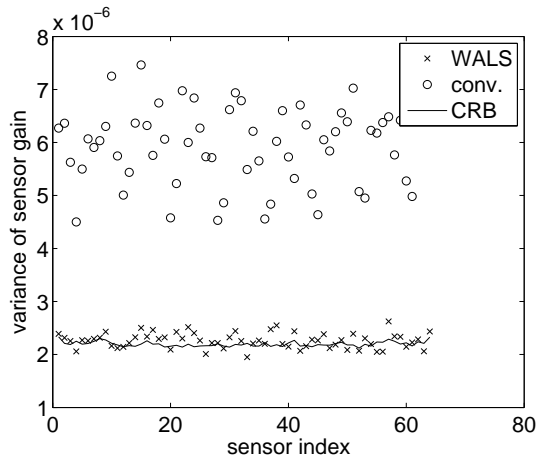
Fig. 1. Comparison of the variance on the gain amplitude estimates obtained using the WALS approach and the conventional (conv.) method with the CRB for $N = 3 \cdot 10^5$ samples.

Fig. 2. Comparison of the variance on the gain phase estimates obtained using the WALS approach and the conventional (conv.) method with the CRB for $N = 3 \cdot 10^5$ samples.

array (URA) with half wavelength spacing between its elements. We defined a source model consisting of three sources at (l, m)-positions $(-0.3, 0.4)$, $(0.2, -0.7)$ and $(0.8, 0)$, where l and m are 2-D directional cosines, and source powers 1, 0.7 and 0.1 respectively. Although both methods can handle far more complex source models including diffuse sources, we have opted for this simple model for reproducibility of the results.

The complex valued receiver gains were generated by adding i.i.d. complex Gaussian noise with zero mean and standard deviation 0.1 to unit gains. Before applying these gains to the modeled signal, we applied the same phase and amplitude constraints as required by the calibration algorithms to avoid identifiability problems. This step facilitates comparison between the estimated gains and the true gains. Although the algorithms can handle even larger gain differences between the elements, we did not want the probability density function of the estimated gains and phases to differ significantly from the Gaussian distribution, since that would only complicate the analysis. Finally, receiver noise powers with unit amplitude were added to all autocorrelations. This gave an SNR varying between 2.1% and 3.9% per receiving element due to the gain differences between them.

We did Monte Carlo simulations for $N = 300, 1000, 3000, 10^4, 3 \cdot 10^4, 10^5, 3 \cdot 10^5$ samples assuming that all signals were i.i.d. complex Gaussian noise. Each simulation was run 500 times. Figures 1 and 2 show the variance on the estimated gains and phases respectively for both methods for $N = 3 \cdot 10^5$ and compare this with the CRB. This result indicates that for a large number of samples, the WALS method attains the CRB, while the conventional method does not. The latter can be explained by
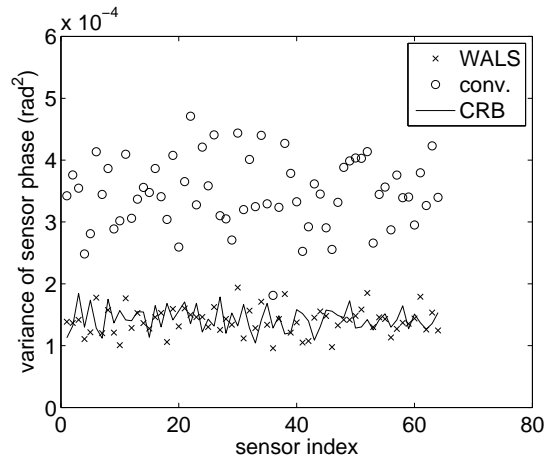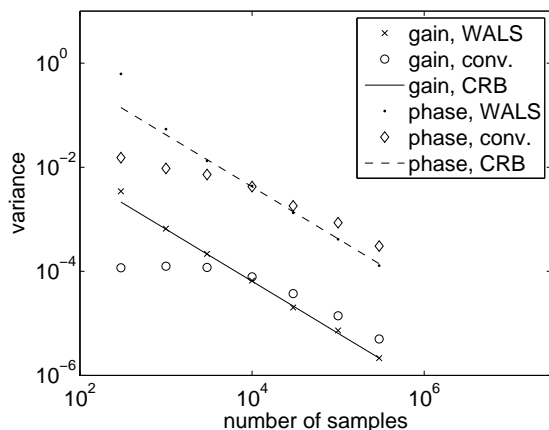

Fig. 3. Comparison of the variance on the gain and phase estimates for a representative element ($p = 12$) obtained using the WALS algorithm and the conventional (conv.) approach with the CRB for different numbers of samples.

the sub-optimal weighting that is implicitly applied to the data by taking the logarithm.

If we compare the variance of the estimated parameters with the CRB as function of the number of samples, as done in Fig. 3 for two representative parameters, we see that about 1000 samples are already sufficient for the WALS method to attain the CRB. With 1000 samples, the SNR per element varies between 0.66 and 1.22 per element. Fortunately, the covariance matrix provides about 10 times as many data points as there are free parameters, allowing us to find a meaningful solution despite the low SNR.
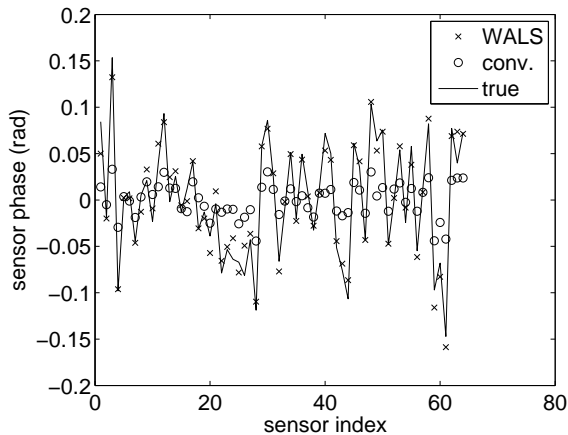
Fig. 4. Comparison of the average gain phase estimate for each element obtained using the WALS approach and the conventional (conv.) method with the true value for $N = 1000$ samples.

From random matrix theory, it is known that matrix-wise convergence of the array covariance matrix requires $P/N < 0.1$ [9]. Since covariance matched weighting requires inversion of the array covariance matrix, matrix-wise convergnece is required. With $P = 64$ and $N = 1000$, this requirement is just met, while $N = 300$ gives $P/N \approx 0.21$. The proposed method thus attains the CRB already with a fairly minimal number of time samples.

It is interesting to note that the conventional method seems to perform better than the CRB when the number of samples drops below 10^4. Figure 4 indicate how this is possible by showing the mean value of the estimated phases of all runs in the Monte Carlo simulation for 1000 samples and comparing them with the true values of the parameters. The true values and the estimates found by the WALS method vary significantly between the receiving elements and converge to the same values if the number of runs in the Monte Carlo simulations is increased. The mean of the estimates produced by the conventional method only span a limited range of values causing them to deviate from the true values. A similar effect was seen for the gain parameters. This indicates that the conventional method has a tendency to equalize the gain and phase estimates in the low-SNR regime, thus causing a bias in these estimates.

4. CONCLUSIONS

In this paper, we presented a new algorithm based on WALS optimization to exploit the regularity in the sensor configuration of many imaging arrays for self-calibration. Our Monte Carlo simulations indicate that this method is statistically efficient, even for very low SNR and a fairly

minimal number of time samples. The method commonly used in the literature does not attain the CRB at high SNR and produces a bias in the estimated parameters at low SNR.

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