SIGNAL SAMPLING ACCORDING TO TIME-VARYING BANDWIDTH

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ABSTRACT

The paper addresses the problem of signal-dependent sampling of analogue signals according to local bandwidth. An extended sampling theorem is given which states that signals can be sampled non-uniformly and then perfectly reconstructed if spectrum obtained by an extended Fourier transform (EFT) is bandlimited. Since, according to the theorem, the sampling instants are determined by the function used in EFT, the aim is to find such function which reflects the time-varying spectral content of the signal. This, in comparison to uniform sampling, allows reducing the number of samples required to represent the signal. The results have been demonstrated by numerical simulations on two signals.

***Index Terms*—** Signal-dependent sampling, extended sampling theorem, maximum instantaneous frequency, signal decomposition.

1. INTRODUCTION

The Nyquist-Shannon sampling theorem states that every bandlimited signal is uniquely determined by its samples taken uniformly at a rate of at least twice the bandwidth of the signal. The bandwidth follows from analyzing the signal by Fourier transform (FT) in the whole duration of the signal and thus can be considered as the global bandwidth. Such analyze, however, does not provide information about time-varying frequency content of the signal, which in turn could be used to sample the signal in signal-dependent way with more samples taken at high frequency regions and less samples – in low frequency regions.

In this paper, we address the problem, also discussed in [1], [2] and [3], of adapting the sampling rate to local bandwidth of the signal. In Section 2 we propose an extended sampling theorem which states that signals can be sampled non-uniformly and then perfectly reconstructed. Then, in Section 3, follows an estimation of maximum instantaneous frequency of the signal, which determines the sampling instants of signal-dependent sampling. And in the last Section 4, numerical simulations on two signals with time-varying spectral content are provided.

2. EXTENDED SAMPLING THEOREM

The extended sampling theorem follows from definitions of direct and inverse extended Fourier transforms in [3].

Given a positive function $g(t)$, the extended Fourier transform (EFT) of the signal $s(t)$ is defined as

$$S(\omega_g) = \tilde{F}[s(t), g(t)] = \int_{-\infty}^{\infty} \frac{s(t)}{g(t)} e^{-j\omega_g m(t)} dt, \quad (1)$$

where

$$m(t) = \int_0^t \frac{1}{g(\tau)} d\tau \quad (2)$$

The signal $s(t)$ can be reconstructed from $S(\omega_g)$ by the inverse EFT

$$s(t) = \tilde{F}^{-1}[S(\omega_g), g(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega_g) e^{j\omega_g m(t)} d\omega_g \quad (3)$$

The classical FT and inverse FT follow from (1) and (3) if $g(t) = 1$.

In [3] the authors have also defined an extended convolution in time domain

$$x(t) \otimes s(t) = \int_{-\infty}^{\infty} \frac{x(\tau)}{g(\tau)} s(m^{-1}(m(t) - m(\tau))) d\tau \quad (4)$$

with $m^{-1}(.)$ representing the inverse function of $m(t)$. This definition is useful since EFT of the convolution signal $y(t) = x(t) \otimes s(t)$ is

$$Y(\omega_g) = X(\omega_g) S(\omega_g) \quad (5)$$

Proposition of the extended sampling theorem: every bandlimited to $[-\Omega_g, \Omega_g]$ signal $s(t)$ is uniquely determined by its samples $s(t_n)$ taken at instants $t_n = m^{-1}(n\Upsilon)$ with a sampling step $\Upsilon \le \frac{\pi}{\Omega_g}$. The reconstruction formula is

$$s(t) = \sum_{n=-\infty}^{\infty} s(t_n) \mathrm{sinc}(\frac{\pi}{\Upsilon}(m(t) - m(t_n))) \quad (6)$$

In this case the signal is bandlimited if the spectrum $\tilde{F}[s(t), g(t)]$ is zero outside the band $[-\Omega_g, \Omega_g]$.

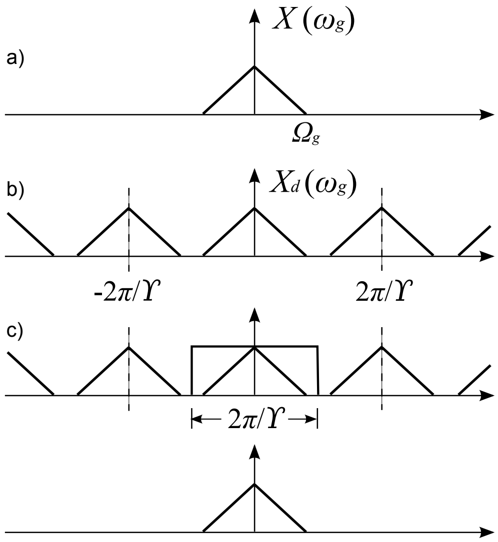


Fig. 1. EFT spectrums of the analog (a) and discrete (b) signals and recovery of the original spectrum by analogue filtering (c).

The proof of the theorem. The discrete signal can be represented as

$$s_d(t) = s(t) \sum_{n=-\infty}^{\infty} \delta(t - t_n) \tag{7}$$

If the sampling instants are chosen according to equation $m(t_n) = n\Upsilon$, i.e.,

$$t_n = m^{-1}(n\Upsilon) \tag{8}$$

with $\Upsilon > 0$, then the sequence of delta pulses can be equally written as

$$\sum_{n=-\infty}^{\infty} \delta(t - t_n) = \sum_{n=-\infty}^{\infty} \delta(m(t) - n\Upsilon) \tag{9}$$

By denoting $u = m(t)$, the obtained sequence $\sum_{n=-\infty}^{\infty} \delta(u - n\Upsilon)$ is periodic and can be expanded in Fourier series as

$$\sum_{n=-\infty}^{\infty} \delta(u - n\Upsilon) = \frac{1}{\Upsilon} \sum_{n=-\infty}^{\infty} e^{jn\frac{2\pi}{\Upsilon}u} \tag{10}$$

From (7), (9) and (10) follows

$$s_d(t) = \frac{1}{\Upsilon} \sum_{n=-\infty}^{\infty} s(t) e^{jn\frac{2\pi}{\Upsilon}m(t)} \tag{11}$$

Considering the EFT property: if $X(\omega_g) = \tilde{F}[x(t), g(t)]$, then $\tilde{F}[x(t)e^{\pm j\alpha m(t)}, g(t)] = X(\omega_g \mp \alpha)$, the EFT of $s_d(t)$ is

$$S_d(\omega_g) = \frac{1}{\Upsilon} \sum_{n=-\infty}^{\infty} S(\omega_g - n\frac{2\pi}{\Upsilon}) \tag{12}$$

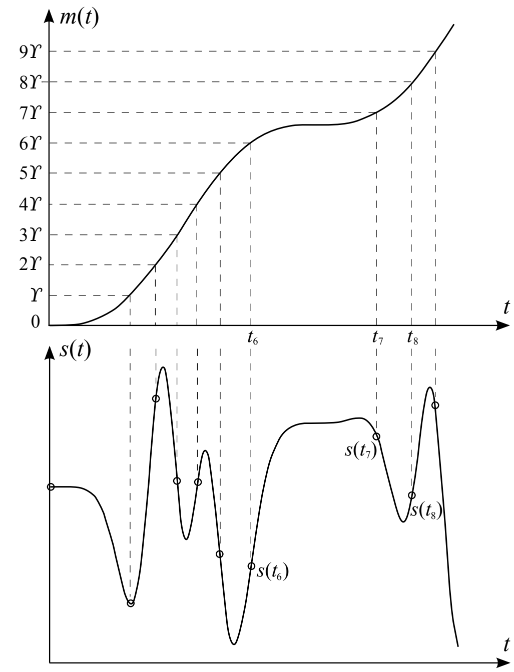


Fig. 2. Sampling according to $m(t)$: samples $s(t_n)$ ar taken at $t_n = m^{-1}(n\Upsilon)$.

That means the spectrum of the discrete signal is periodic with period $2\pi/\Upsilon$ (Figure 1). It also follows, that if the analogue signal is bandlimited to $[-\Omega_g, \Omega_g]$, then the sampling step must be chosen

$$\Upsilon \le \frac{\pi}{\Omega_g} \tag{13}$$

to ensure the periodic spectrum of the discrete signal does not overlap. If (13) holds, then the original spectrum $S(\omega_g)$ can be reconstructed by multiplying $S_d(\omega_g)$ and

$$H(\omega_g) = \begin{cases} \Upsilon, & \text{if} \quad |\omega_g| \le \pi/\Upsilon \\ 0, & \text{if} \quad |\omega_g| > \pi/\Upsilon \end{cases} \tag{14}$$

According to (4) and (5) multiplication in frequency domain

$$S(\omega_g) = S_d(\omega_g)H(\omega_g) \tag{15}$$

corresponds to extended convolution in time domain

$$s(t) = \int_{-\infty}^{\infty} \frac{s_d(\tau)}{g(\tau)} h(m^{-1}(m(t) - m(\tau)))d\tau, \tag{16}$$

where

$$h(t) = \tilde{F}^{-1}[H(\omega_g), g(t)] = \text{sinc}(\frac{\pi}{\Upsilon}m(t)) \tag{17}$$

From (7), (9), (16) and (17) follows the reconstruction formula (6). The special case when $g(t) = 1$ leads to classical uniform sampling.

Example. If we have a signal composed of frequency modulated oscillations

$$s_1(t) = \sum_{l=1}^{L} A_l \cos(k_l m(t)) \quad (18)$$

with coefficients $0 < k_1 < k_2 < \cdots < k_L$, then EFT of $s_1(t)$ is

$$S_1(\omega_g) = \sum_{l=1}^{L} A_l \pi(\delta(\omega_g + k_l) + \delta(\omega_g - k_l)) \quad (19)$$

It follows that the signal is bandlimited to $[-k_L, k_L]$ and can be represented by samples $s_1(t_n)$ taken at $t_n = m^{-1}(n\pi/k_L)$. The sampling case when $\Upsilon = \pi/k_L$ and $k_L = 1$ is illustrated in Figure 2.

3. ESTIMATION OF THE MAXIMUM INSTANTANEOUS FREQUENCY

The question is: according to the extended sampling theorem, how should we sample the signal with sampling rate being adapted to local bandwidth? The answer is to find such function $g(t)$ which reflects the time-varying spectral content of the signal and gives the spectrum $S(\omega_g) = \tilde{F}[s(t), g(t)]$ with zero values outside some limited band $[-\Omega_g, \Omega_g]$. However, an analytical solution for many real signals is difficult or even impossible to obtain, thus the following proposition is made.

Given a signal

$$s(t) = \sum_{l=1}^{L} A_l \cos(\Phi_l(t)) \quad (20)$$

with constant A_l and monotonically increasing $\Phi_l(t)$, a maximum instantaneous frequency $f_{max}(t)$ of $s(t)$ is defined as having values

$$f_{max}(\tau) = \max(f_1(\tau), f_2(\tau), \ldots, f_L(\tau)) \quad (21)$$

at any given $t = \tau$ and $f_l(t)$ being instantaneous frequencies of cosines

$$f_l(t) = \frac{1}{2\pi} \frac{d\Phi_l(t)}{dt} \quad (22)$$

From $f_{max}(t)$ the functions

$$g(t) = 1/(2\pi f_{max}(t)) \quad (23)$$

and $m(t)$ according to (2) are found.

By such definition of $g(t)$ we can not in general expect that $S(\omega_g)$ will be located within some limited band, however, we can find the frequency Ω_g with corresponding band $[-\Omega_g, \Omega_g]$ that contains the major part of signal energy. Given this frequency, the samples $s(t_n)$ ar taken at $t_n = m^{-1}(n\pi/\Omega_g)$ and for reconstruction the interpolation formula (6) is used

$$\hat{s}(t) = \sum_{n=-\infty}^{\infty} s(t_n) \mathrm{sinc}(\Omega_g(m(t) - m(t_n))) \quad (24)$$

Because of sampling, frequency aliasing occurs and $\hat{s}(t)$ does not fit the original $s(t)$ exactly. The precision improves as the bandwidth Ω_g increases.

If no frequencies $f_l(t)$ are given or the signal differs from (20), then the idea for estimation of $f_{max}(t)$ is to decompose the signal into a finite number of Intrinsic Mode Functions (IMFs) using the Empirical Mode Decomposition (EMD) [4], [5]. According to this technique, the IMFs are acquired by sifting process, which sequentially gives several IMFs subjected to certain conditions and a low frequency residue component $r(t)$. In result the signal can be written as

$$s(t) = \sum_{j=1}^{J} c_j(t) + r(t), \quad (25)$$

where the first IMF $c_1(t)$ contains the highest frequency component of the signal, and thus we are interested in finding only this component. The procedure is following:

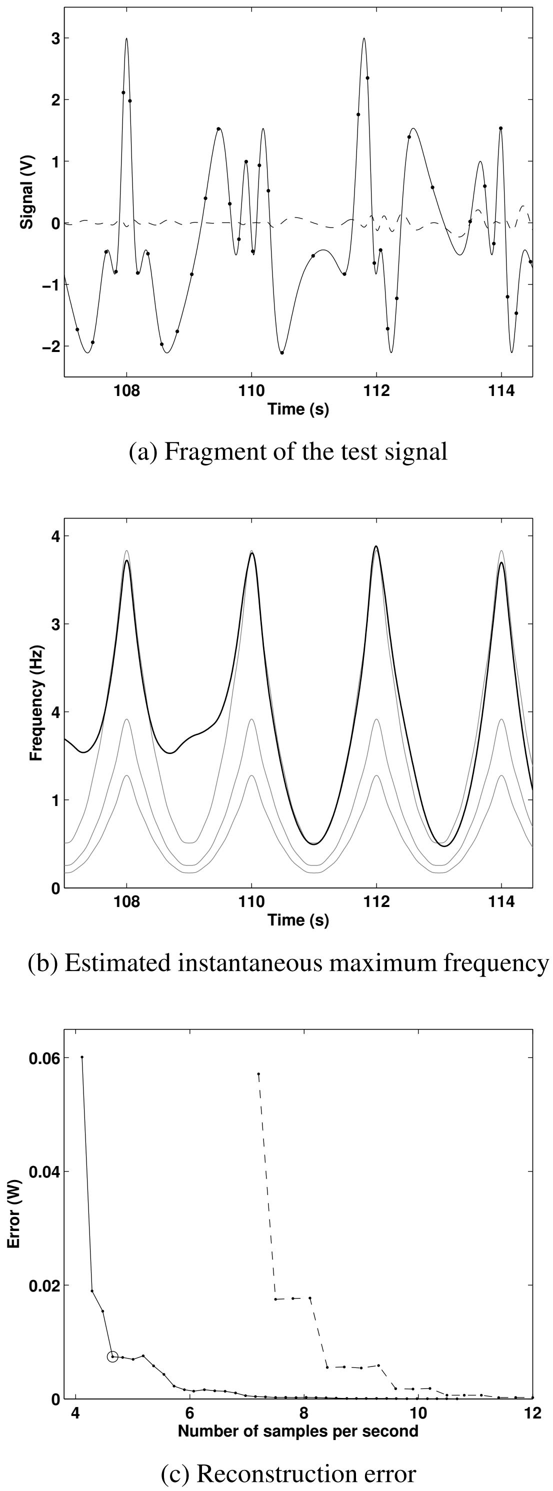
1. All the local extrema of the signal $s(t)$ are found and then connected using smooth cubic splines to get the top and bottom envelopes $s_{up}(t)$ and $s_{low}(t)$ of the signal. To make connections at the end points, either signal values or the mean values of local maxima and minima are used depending on their relations to each other. If the mean value of all local maxima is less than the signal value, then the signal value is used for connection to get the top envelope. Similarly, if the mean value of all local minima is less than the signal value, then the mean value is used to get the bottom envelope [5].
2. The next step is finding the mean envelope $\bar{s}_1(t) = (s_{up}(t) + s_{low}(t))/2$, which is then subtracted from the signal to get the difference $x_1(t) = s(t) - \bar{s}_1(t)$.
3. Regarding $x_1(t)$ as the new data and repeating steps (1) and (2) until the resulting signal meets the criteria of an IMF, the first component $c_1(t)$ is found.

Given $c_1(t)$, the instantaneous frequency of the first IMF is found by Hilbert transform and then lowpass filtered in order to remove spurious oscillations. The obtained frequency function $\hat{f}_{max}(t)$ is then assumed to be the estimate of the instantaneous maximum frequency of the signal $s(t)$.

4. SIMULATION RESULTS

The extended sampling with non-uniformly placed samples has been tested on two signals. The first signal $s_1(t)$ is artificial and consists of three frequency modulated oscillations

$$s_1(t) = \cos(\Phi(t)) + \cos(3\Phi(t)/2) + \cos(3\Phi(t)), \quad (26)$$

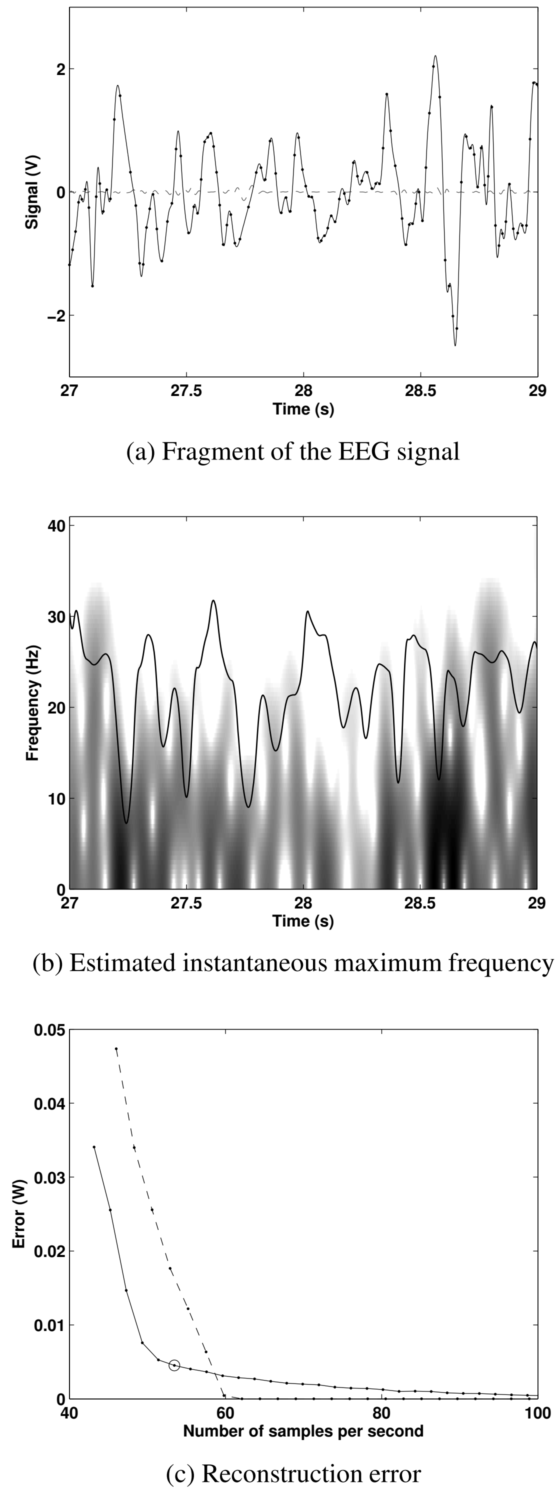


(a) Fragment of the test signal

(b) Estimated instantaneous maximum frequency

(c) Reconstruction error

Fig. 3. Reconstruction of the test signal.



(a) Fragment of the EEG signal

(b) Estimated instantaneous maximum frequency

(c) Reconstruction error

Fig. 4. Reconstruction of the EEG signal.

while the second signal $s_2(t)$ is a real up to 30 Hz low-pass filtered EEG signal. The fragments of the signals are shown in Figures 3a and 4a by the solid lines.

In order to perform the sampling the first step is to estimate the instantaneous maximum frequency of the signal. The proposition is to use EMD method to find the first IMF of the signal and then obtain its lowpass filtered instantaneous frequency, which is assumed to be the instantaneous maximum frequency $\hat{f}_{max}(t)$ of the signal.

The obtained result for $s_1(t)$ is shown in Figure 3b by the bold line, while the 3 thin lines correspond to frequencies of the true components of the signal. From the figure follows that estimated frequency is either greater or very close to the frequency of the third component. The desirable result would be that these two frequencies fit.

The estimated frequency for EEG signal is shown in Figure 4b (bold line). The colors in background correspond to time-frequency representation of the signal obtained by short-time Fourier transform.

After estimation of $\hat{f}_{max}(t)$, the sampling follows. The functions $g(t)$ and $m(t)$ are found from (23) and (2) and samples $s(t_n)$ are taken at $t_n = m^{-1}(n\Upsilon)$. The sampling step Υ must be chosen according to (13) if the spectrum $S(\omega_g) = \tilde{F}[s(t), g(t)]$ is zero outside the band $[-\Omega_g, \Omega_g]$. If the signal is not bandlimited, then the frequency Ω_g is chosen such that the bandwidth $[-\Omega_g, \Omega_g]$ contains the most part of signal energy.

In the case of the first signal $s_1(t)$ the spectrum $S_1(\omega_g)$ is not bandlimited. It would be bandlimited if the estimated frequency (bold line in Figure 3b) conformed to the frequency of the third component (upper thin line in Figure 3b). Since this is not the case here, then the frequency Ω_g is chosen to be $\Omega_g = 1.3$ and the sampling step is $\Upsilon = \pi/\Omega_g$. Sampling causes frequency aliasing and the reconstructed signal $\hat{s}_1(t)$ differs from the original by $e_1(t) = s_1(t) - \hat{s}_1(t)$. The error signal $e_1(t)$ is shown by the dashed line in Figure 3a.

The precision of reconstruction improves as the the sampling step π/Ω_g decreases, i.e., the average number of samples per second increases. This is shown in Figure 3c by the solid line, where the values on y-axis correspond to the average power of the error signal $e_1(t)$. The result is quite obvious since decrease in sampling step follows from increase in Ω_g and thus the spectral overlap decreases. An empty circle in Figure 3c corresponds to reconstruction example when $\Omega_g = 1.3$.

To compare with the classical sampling approach, the signal was also reconstructed from uniform samples. The reconstruction error depending on the sampling rate is shown by the dashed line in Figure 3c. From figure follows, that equally good reconstruction in signal-dependent sampling case is obtained with considerably less number of samples than in uniform sampling case.

In the case of the EEG signal the spectrum $S_2(\omega_g)$ is also not bandlimited and thus the frequency Ω_g is chosen to be $\Omega_g = 1.3$. Due to spectral overlap the reconstructed signal $\hat{s}_2(t)$ differs from the original by $e_2(t) = s_2(t) - \hat{s}_2(t)$, as shown in Figure 4a by the dashed line. The reconstruction error reduces with the increase of Ω_g, which in turn increases the average number of samples per second. This is shown in Figure 4c by the solid line, while the thin line corresponds to the uniform sampling case. From the figure follows that better reconstruction from less samples is achieved in signal-dependent sampling case.

5. CONCLUSIONS

The results show that adaptation of sampling rate to time-varying spectral content of the signal in the form of estimated maximum instantaneous frequency of the first EMD component allows obtaining better reconstruction from less samples in comparison to uniform sampling. Besides EMD method there may be other techniques like spectrogram analysis or time-varying filtering approach for finding more optimal frequency functions providing more compact EFT spectrums of the signal. The presence of noise in the signal should also be considered. These are the topics for further investigation.

6. ACKNOWLEDGEMENT

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