AN AMPLITUDE SPECTRAL CAPON ESTIMATOR WITH A VARIABLE FILTER LENGTH

Jesper Kjær Nielsen[††], *Paris Smaragdis*[†], *Mads Græsbøll Christensen*[†††], *and Søren Holdt Jensen*[††]

[†]University of Illinois at Urbana-Champaign
Dept. of Computer Science
paris@illinois.edu

[††]Aalborg University
Dept. of Electronic Systems
{jkn,shj}@es.aau.dk

[†††]Aalborg University
Dept. of Architecture, Design & Media Technology
mgc@create.aau.dk

ABSTRACT

The filter bank methods have been a popular non-parametric way of computing the complex amplitude spectrum. So far, the length of the filters in these filter banks has been set to some constant value independently of the data. In this paper, we take the first step towards considering the filter length as an unknown parameter. Specifically, we derive a very simple and approximate way of determining the optimal filter length in a data-adaptive way. Based on this analysis, we also derive a model averaged version of the forward and the forward-backward amplitude spectral Capon estimators. Through simulations, we show that these estimators significantly improve the estimation accuracy compared to the traditional Capon estimators.

Index Terms— Filter Bank Methods, Capon, Spectral Estimation

1. INTRODUCTION

The estimation of the complex amplitude spectrum is an important problem in several applications such as audio coding and radar imaging (see [1] and the references therein). Several solutions have been proposed in the literature ranging from simple estimators based on various windowed Fourier transforms to more complex estimators based on a parametric model of the observed data. A popular example of such a parametric model is the sinusoidal model

$$x(n) = \sum_{i=1}^{l} \alpha_i \exp(j\omega_i n) + w(n), \quad n = 1, 2, \cdots, N \quad (1)$$

where α_i and ω_i are the complex amplitude[1] and the frequency of the i'th complex sinusoid, respectively, and $w(n)$ is a stationary random process with a possibly non-flat power spectral density (psd). Unfortunately, the parametric methods are usually very sensitive to modelling errors such as the noise statistics [2] and may also suffer from a high computational complexity if the model has non-linear parameters such as the frequency. Therefore, non-parametric methods

may yield much better estimation results or lower the computational complexity significantly. For the estimation of the complex amplitude spectrum, the filter bank methods such as Capon [3] and APES [4] are examples of such non-parametric estimators which are fast and very robust to modelling errors. Although the APES method was originally proposed as an approximate maximum likelihood method, it can also be interpreted as a matched filter bank method like the Capon method [5]. Under this interpretation, the observed data is passed through an m-tap FIR-filter which is designed to maximise the signal-to-noise ratio (SNR) of the filter output subject to the constraint that the filter has a gain of one at some known frequency ω. The complex amplitude α at this frequency is then estimated from the filter output. By designing a filter for all desired frequency points, we get a filter bank whose outputs are used to estimate the complex amplitudes at all these frequency points. The statistical properties of the Capon and APES methods have been studied extensively in, e.g., [5–7]. These studies have shown that the APES amplitude estimates are unbiased for all filter lengths, but that the Capon estimator gives amplitude estimates which are biased towards zero. For long filter lengths, this bias increases significantly. On the other hand, the Capon method has in general a better resolution than APES and is therefore more practical for estimating frequencies [1]. In [4], the maximum filter length of $m = \lfloor N/2 \rfloor$ is recommended for the APES method since this maximises the resolution. For the Capon method, however, there is a trade-off between resolution and bias, and in [7] a filter length in the interval $N/8 < m < N/4$ is recommended. To the best of our knowledge, the filter length m is always selected to be the same for all frequency points and does not depend on the observed data. In this paper, we take a first step towards determining the optimal filter length for the Capon method in a data-adaptive way. Specifically, we give a simple and approximate solution which is based on Djuric's asymptotic MAP approach [8].

2. THE AMPLITUDE SPECTRAL CAPON ESTIMATOR

The filter bank methods are a way of bypassing some of the difficulties associated with the parametric methods. This is

---

[1]Note that α_i is not an amplitude in the usual sense since it is not a real, positive scaler. However, in the lack of better words, we refer to it as the complex amplitude.

achieved by first rewriting (1) as

$$x(n) = \alpha \exp(j\omega n) + z(n) , \quad n = 1, 2, \cdots, N \quad (2)$$

where α is the complex amplitude at the known (angular) frequency ω. In practice, we rarely know the true frequency parameters $\{\omega_i\}_{i=1}^{l}$ or the number l of them in (1), and these quantities are typically hard to estimate. In the filter bank methods, this problem is bypassed by selecting a set Ω of R candidate frequencies for which we wish to estimate the complex amplitude α. Comparing (2) to (1), we see that $z(n)$ models all the l sinusoids and the coloured noise $w(n)$, except for the complex sinusoid with an unknown complex amplitude α at the known frequency ω. We then pass this signal through an m-tap FIR filter and obtain

$$y(n) = \alpha \exp(j\omega n)\boldsymbol{h}^H \boldsymbol{a} + \boldsymbol{h}^H \boldsymbol{z}(n) . \quad (3)$$

for $n = m, \cdots, N$ where we have defined

$$\boldsymbol{a} \triangleq \begin{bmatrix} 1 & \exp(-j\omega) & \cdots & \exp(-j\omega(m-1)) \end{bmatrix}^T \quad (4)$$

$$\boldsymbol{h} \triangleq \begin{bmatrix} h_0 & h_1 & \cdots & h_{m-1} \end{bmatrix}^H \quad (5)$$

$$\boldsymbol{z}(n) \triangleq \begin{bmatrix} z(n) & z(n-1) & \cdots & z(n-m+1) \end{bmatrix}^T . \quad (6)$$

The notation $(\cdot)^T$ and $(\cdot)^H$ denote transposition and complex transposition, respectively. Maximising the SNR of the filter output is equivalent to solving [5]

$$\underset{\boldsymbol{h} \in \mathbb{C}^m}{\arg\min} \, \boldsymbol{h}^H \boldsymbol{Q} \boldsymbol{h} \quad \text{subject to} \quad \boldsymbol{h}^H \boldsymbol{a} = 1 \quad (7)$$

whose solution is

$$\boldsymbol{h} = (\boldsymbol{a}^H \boldsymbol{Q}^{-1} \boldsymbol{a})^{-1} \boldsymbol{Q}^{-1} \boldsymbol{a} . \quad (8)$$

The covariance matrix $\boldsymbol{Q} \triangleq E\{\boldsymbol{z}(n)\boldsymbol{z}^H(n)\}$ is unknown and must be estimated in some way. For $\boldsymbol{C} \triangleq E\{\boldsymbol{x}(n)\boldsymbol{x}^H(n)\}$ where $\boldsymbol{x}(n)$ is defined analogously to $\boldsymbol{z}(n)$, we have that

$$\boldsymbol{C} = |\alpha|^2 \boldsymbol{a}\boldsymbol{a}^H + \boldsymbol{Q} , \quad (9)$$

and a simple estimate of $\boldsymbol{Q}$ is therefore $\hat{\boldsymbol{Q}} = \hat{\boldsymbol{C}} - |\hat{\alpha}|^2 \boldsymbol{a}\boldsymbol{a}^H$. Inserting this estimate in (7) yields

$$\underset{\boldsymbol{h} \in \mathbb{C}^m}{\arg\min} \, \boldsymbol{h}^H \hat{\boldsymbol{C}} \boldsymbol{h} \quad \text{subject to} \quad \boldsymbol{h}^H \boldsymbol{a} = 1 \quad (10)$$

so that the Capon filter is

$$\boldsymbol{h}^{\text{Capon}} = (\boldsymbol{a}^H \hat{\boldsymbol{C}}^{-1} \boldsymbol{a})^{-1} \hat{\boldsymbol{C}}^{-1} \boldsymbol{a} . \quad (11)$$

The covariance matrix $\boldsymbol{C}$ of the input vector $\boldsymbol{x}(n)$ is typically estimated in one of two different ways. If we define $K \triangleq N - m + 1$ and

$$\boldsymbol{X} \triangleq \begin{bmatrix} \boldsymbol{x}(m) & \cdots & \boldsymbol{x}(N) \end{bmatrix} , \quad (12)$$

the forward estimate is given by

$$\hat{\boldsymbol{C}}_{\text{f}} = K^{-1} \boldsymbol{X}\boldsymbol{X}^H , \quad (13)$$

and the forward-backward (FB) estimate is given by

$$\hat{\boldsymbol{C}}_{\text{fb}} = (\hat{\boldsymbol{C}}_{\text{f}} + \boldsymbol{J}_m \hat{\boldsymbol{C}}_{\text{f}}^T \boldsymbol{J}_m)/2 \quad (14)$$

where $\boldsymbol{J}_m$ is the $m \times m$ exchange matrix. Like the true covariance matrix, $\hat{\boldsymbol{C}}_{\text{fb}}$ is persymmetric and simulation results have shown that it reduces the bias of the complex amplitude estimate significantly compared to $\hat{\boldsymbol{C}}_{\text{f}}$ [6]. However, the resolution is slightly better for $\hat{\boldsymbol{C}}_{\text{f}}$ [9]. The APES filter can be derived by using a different estimate of $\boldsymbol{Q}$, which can be found in [5], and it also exists in a forward and a FB version [6].

Due to the constraint $\boldsymbol{h}^H \boldsymbol{a} = 1$, we can write the filter output in (3) in vector form as

$$\boldsymbol{y} = \boldsymbol{X}^T \boldsymbol{h}^* = \alpha \boldsymbol{b} + \boldsymbol{e} \quad (15)$$

where $(\cdot)^*$ denotes complex conjugation and

$$\boldsymbol{y} \triangleq \begin{bmatrix} y(m) & \cdots & y(N) \end{bmatrix}^T \quad (16)$$

$$e(n) \triangleq \boldsymbol{h}^H \boldsymbol{z}(n) \quad (17)$$

$$\boldsymbol{e} \triangleq \begin{bmatrix} e(m) & \cdots & e(N) \end{bmatrix}^T \quad (18)$$

$$\boldsymbol{b} \triangleq \begin{bmatrix} \exp(j\omega m) & \cdots & \exp(j\omega N) \end{bmatrix}^T . \quad (19)$$

The least squares estimate of the complex amplitude is then

$$\hat{\alpha}_m = K^{-1} \boldsymbol{h}^H \boldsymbol{X} \boldsymbol{b}^* \quad (20)$$

where we have used the subscript m to indicate the length of the filter. When the Capon filter is used in (20), we term the resulting estimator as the amplitude spectral Capon (ASC) estimator.

3. THE MODEL AVERAGED ASC ESTIMATOR

To estimate the optimal filter length, we first briefly review the asymptotic MAP approach by Djuric [8]. In his framework, we wish to find the posterior distribution $p(m|\boldsymbol{x})$ on the filter length m given the K data points in the vector $\boldsymbol{x}$. This distribution is by Bayes' theorem given by

$$p(m|\boldsymbol{x}) \propto p(\boldsymbol{x}|m)p(m) \quad (21)$$

where $\propto$ denotes 'proportional to', and $p(\boldsymbol{x}|m)$ is referred to as the model likelihood or evidence which is given by

$$p(\boldsymbol{x}|m) = \int_{\Theta_m} p(\boldsymbol{x}|\boldsymbol{\theta}_m, m)p(\boldsymbol{\theta}_m|m)d\boldsymbol{\theta}_m \quad (22)$$

where $\boldsymbol{\theta}_m$ denotes the d_m model parameters with support Θ_m, $p(\boldsymbol{x}|\boldsymbol{\theta}_m, m)$ is the likelihood, and $p(\boldsymbol{\theta}_m|m)$ is the prior on the model parameters under model index m. Unfortunately, the integral in (22) can in general not be evaluated analytically, so Djuric suggests that the integral is approximately evaluated using the Laplace approximation. Provided that $\boldsymbol{\theta}$ is purely complex, the Laplace approximation gives

$$p(\boldsymbol{x}|m) \approx f(\hat{\boldsymbol{\theta}}_m)\pi^{d_m}|-\boldsymbol{H}(\hat{\boldsymbol{\theta}}_m)|^{-1} \quad (23)$$

where $f(\boldsymbol{\theta}_m) \triangleq p(\boldsymbol{x}|\boldsymbol{\theta}_m, m)p(\boldsymbol{\theta}_m|m)$ is the integrand of (22), $\hat{\boldsymbol{\theta}}_m$ is the MAP estimate of $\boldsymbol{\theta}_m$, and

$$\boldsymbol{H}(\boldsymbol{\theta}_m) = \frac{\partial^2 \ln f(\boldsymbol{\theta}_m)}{\partial \boldsymbol{\theta}_m^* \partial \boldsymbol{\theta}_m^T} \tag{24}$$

is the Hessian matrix. Djuric also suggests that we simplify (23) by neglecting the terms of order $\mathcal{O}(1)$ and by evaluating the determinant of the observed information matrix $-\boldsymbol{H}(\hat{\boldsymbol{\theta}}_m)$ using asymptotic considerations. Specifically, for the linear Gaussian model $\boldsymbol{x} = \boldsymbol{F}_m \boldsymbol{\theta}_m + \boldsymbol{e}$ and for a uniform prior on the model index and the model parameters, Djuric approximates $p(m|\boldsymbol{x})$ on the model index by [8]

$$p(m|\boldsymbol{x}) \propto (\hat{\sigma}^2)^{-K} |\boldsymbol{F}_m^H \boldsymbol{F}_m|^{-1} \tag{25}$$

where $\hat{\sigma}^2$ is the maximum likelihood estimate of the noise variance. The asymptotic approximation of $|\boldsymbol{F}_m^H \boldsymbol{F}_m|$ depends on the particular structure of $\boldsymbol{F}_m$.

3.1. Derivation

To construct a simple way of selecting the optimal filter length, we first integrate the constraint $\boldsymbol{h}^H \boldsymbol{a} = 1$ into the filter vector. Specifically, we write $h_0 = 1 - \tilde{\boldsymbol{a}}^H \tilde{\boldsymbol{h}}$ with

$$\tilde{\boldsymbol{h}} \triangleq \begin{bmatrix} h_1 & h_2 & \cdots & h_{m-1} \end{bmatrix}^T \tag{26}$$

$$\tilde{\boldsymbol{a}} \triangleq \begin{bmatrix} \exp(j\omega) & \exp(j\omega 2) & \cdots & \exp(j\omega m) \end{bmatrix}^T \tag{27}$$

so that the constraint $\boldsymbol{h}^H \boldsymbol{a} = 1$ is satisfied for all $\tilde{\boldsymbol{h}}$. This leads to that

$$\boldsymbol{y} = \boldsymbol{X}^T \boldsymbol{h}^* = \boldsymbol{x}_1 - \boldsymbol{F}_m \tilde{\boldsymbol{h}} \tag{28}$$

where we have defined

$$\boldsymbol{x}_1 \triangleq \begin{bmatrix} x(m) & \cdots & x(N) \end{bmatrix}^T \tag{29}$$

$$\boldsymbol{F}_m \triangleq \boldsymbol{X}^T \begin{bmatrix} \tilde{\boldsymbol{a}}^H \\ -\boldsymbol{I}_{m-1} \end{bmatrix} . \tag{30}$$

where $\boldsymbol{I}_{m-1}$ is the $(m-1)$-dimensional identity matrix. We now assume that $\boldsymbol{x}_1$ is independent of $\boldsymbol{F}_m$ and that $\boldsymbol{y}$ is a white Gaussian noise vector, so that $\boldsymbol{x}_1 = \boldsymbol{F}_m \tilde{\boldsymbol{h}} + \boldsymbol{y}$ is our simple linear model. Although this is in direct contradiction with the model of the filter output in (15), it is a necessary assumption to derive the ASC estimator in our framework. The APES estimator would be obtained instead if $\boldsymbol{y}$ was replaced by (15) and $\boldsymbol{e}$ was assumed to be a white Gaussian noise. Under the assumption that $\boldsymbol{y} \sim \mathcal{CN}(\boldsymbol{y}; \boldsymbol{0}, \sigma^2 \boldsymbol{I}_K)$, where σ^2 is the noise variance, the likelihood is approximately given by

$$p(\boldsymbol{x}_1|\boldsymbol{x}_0, \boldsymbol{h}, \sigma^2) \approx (\pi\sigma^2)^{-K} \exp\left\{\frac{-K}{\sigma^2} S(\boldsymbol{h})\right\} \tag{31}$$

where $\boldsymbol{x}_0 = \begin{bmatrix} x(1) & \cdots & x(m-1) \end{bmatrix}^T$ and

$$S(\boldsymbol{h}) \triangleq \boldsymbol{h}^H \hat{\boldsymbol{C}}_\mathrm{f} \boldsymbol{h} = K^{-1}(\boldsymbol{x}_1 - \boldsymbol{F}_m \tilde{\boldsymbol{h}})^H (\boldsymbol{x}_1 - \boldsymbol{F}_m \tilde{\boldsymbol{h}}) . \tag{32}$$

For non-informative and flat priors, the MAP estimates of the noise variance and the filter coefficient are equal to the maximum likelihood estimates which are

$$\hat{\sigma}^2 = S(\hat{\boldsymbol{h}}) = (\boldsymbol{a}^H \hat{\boldsymbol{C}}_\mathrm{f}^{-1} \boldsymbol{a})^{-1} \tag{33}$$

$$\hat{\boldsymbol{h}} = \boldsymbol{h}^{\mathrm{Capon}} = \hat{\sigma}^2 \hat{\boldsymbol{C}}_\mathrm{f}^{-1} \boldsymbol{a} . \tag{34}$$

The determinant $|\boldsymbol{F}_m^H \boldsymbol{F}_m|$ can be written as

$$|\boldsymbol{F}_m^H \boldsymbol{F}_m| = K^{m-1} \left| \begin{bmatrix} \tilde{\boldsymbol{a}} & -\boldsymbol{I}_{m-1} \end{bmatrix} \hat{\boldsymbol{C}}_\mathrm{f}^* \begin{bmatrix} \tilde{\boldsymbol{a}}^H \\ -\boldsymbol{I}_{m-1} \end{bmatrix} \right| . \tag{35}$$

Since the last factor does not grow with K, the asymptotic approximation is $|\boldsymbol{F}_m^H \boldsymbol{F}_m| \approx K^{m-1}$. Thus, (25) gives

$$p(m|\boldsymbol{x}) \propto S(\hat{\boldsymbol{h}})^{-K} K^{-(m-1)} \tag{36}$$

which is the same as the Bayesian information criterion (BIC) or Schwarz criterion [10].

We use the approximate expression for $p(m|\boldsymbol{x})$ to compute the amplitude estimate averaged over all models. We call this estimator for the model averaged amplitude spectral Capon (MAASC) estimator, and it is given by

$$\hat{\alpha} = \sum_{m=1}^{\lfloor N/2 \rfloor} p(m|\boldsymbol{x}) E\{p(\alpha|\boldsymbol{x}, m)\} = \sum_{m=1}^{\lfloor N/2 \rfloor} p(m|\boldsymbol{x}) \hat{\alpha}_m . \tag{37}$$

Thus, the MAASC estimate is a weighted sum of the ASC estimates for each filter lengths with the weight determined by the probability of each filter length. As with the ASC estimate, the MAASC estimate can be computed using either the forward or the forward-backward covariance matrix estimate.

4. ITERATIVE COMPUTATION OF THE INVERSE COVARIANCE MATRIX

The major contribution to the computational complexity in the ASC estimator is the inversion of the covariance matrix estimate. For the MAASC estimator this contribution is even more pronounced as we have to do the inversion for every filter length. In this section, we derive an iterative algorithm for computing the inverse of the forward estimate of the covariance matrix.

For any filter length m, it follows from (13) that the forward estimate of the covariance matrix scaled by a factor of K is $\boldsymbol{X}_m \boldsymbol{X}_m^H$ where $\boldsymbol{X}_m$ is defined in (12). This scaled estimate is related to $\boldsymbol{X}_{m+1} \boldsymbol{X}_{m+1}^H$ by

$$\boldsymbol{X}_{m+1} \boldsymbol{X}_{m+1}^H = \begin{bmatrix} q_{m+1} & \boldsymbol{r}_{m+1}^H \\ \boldsymbol{r}_{m+1} & \boldsymbol{D}_{m+1} \end{bmatrix} \tag{38}$$

where we have defined

$$q_{m+1} \triangleq \boldsymbol{x}_1^H \boldsymbol{x}_1 - |x(m)|^2 \tag{39}$$

$$\boldsymbol{r}_{m+1} \triangleq \begin{bmatrix} \boldsymbol{0} & \boldsymbol{X}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^* \\ 0 \end{bmatrix} \tag{40}$$

$$\boldsymbol{D}_{m+1} \triangleq \boldsymbol{X}_m \boldsymbol{X}_m^H - \boldsymbol{x}(N)\boldsymbol{x}^H(N) . \tag{41}$$

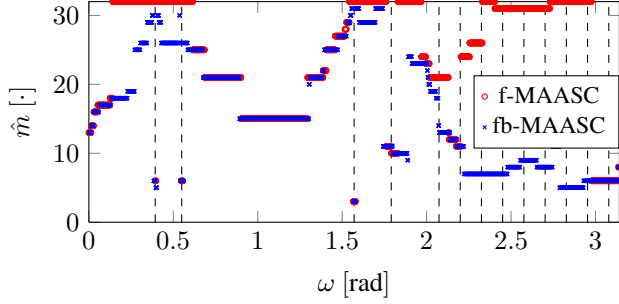**Fig. 1**. The optimal filter length as a function of the frequency.
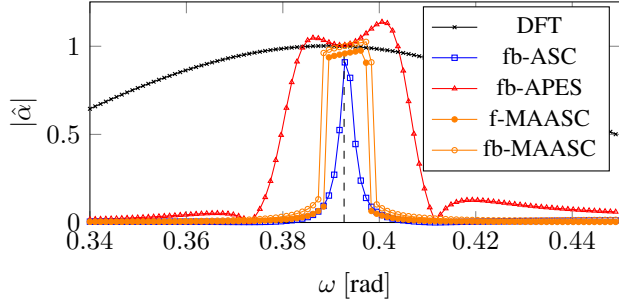

Fig. 2. The amplitude estimate for various estimators. The filter length for the fb-ASC and the fb-APES was $K = 32$.

Using blockwise matrix inversion, $(\boldsymbol{X}_{m+1}\boldsymbol{X}_{m+1}^H)^{-1}$ is

$$(\boldsymbol{X}_{m+1}\boldsymbol{X}_{m+1}^H)^{-1} = \begin{bmatrix} \lambda_{m+1} & \boldsymbol{\phi}_{m+1}^H \\ \boldsymbol{\phi}_{m+1} & \boldsymbol{\Psi}_{m+1} \end{bmatrix} \qquad (42)$$

where we have defined

$$\lambda_{m+1} \triangleq (q_{m+1} - \boldsymbol{r}_{m+1}^H \boldsymbol{D}_{m+1}^{-1} \boldsymbol{r}_{m+1})^{-1} \qquad (43)$$

$$\boldsymbol{\phi}_{m+1} \triangleq -\lambda_{m+1} \boldsymbol{D}_{m+1}^{-1} \boldsymbol{r}_{m+1} \qquad (44)$$

$$\boldsymbol{\Psi}_{m+1} \triangleq \boldsymbol{D}_{m+1}^{-1} + \lambda_{m+1}^{-1} \boldsymbol{\phi}_{m+1} \boldsymbol{\phi}_{m+1}^H . \qquad (45)$$

The inverse of $\boldsymbol{X}_{m+1}\boldsymbol{X}_{m+1}^H$ is related the inverse of $\boldsymbol{X}_m\boldsymbol{X}_m^H$ through $\boldsymbol{D}_{m+1}^{-1}$ which can be written as

$$\boldsymbol{D}_{m+1}^{-1} = [\boldsymbol{X}_m\boldsymbol{X}_m^H - \boldsymbol{x}(N)\boldsymbol{x}^H(N)]^{-1} \qquad (46)$$

$$= (\boldsymbol{X}_m\boldsymbol{X}_m^H)^{-1} \qquad (47)$$

$$+ \frac{(\boldsymbol{X}_m\boldsymbol{X}_m^H)^{-1}\boldsymbol{x}(N)\boldsymbol{x}^H(N)(\boldsymbol{X}_m\boldsymbol{X}_m^H)^{-1}}{1 - \boldsymbol{x}^H(N)(\boldsymbol{X}_m\boldsymbol{X}_m^H)^{-1}\boldsymbol{x}(N)}$$

by using the matrix inversion lemma. Thus, we can iteratively compute the inverse of the forward estimate of the covariance matrices for all filter lengths without doing any matrix inversions. Whether a similar algorithm for the FB estimate of the covariance matrix exists or not is still an open issue.

5. SIMULATIONS

We evaluate the f-MAASC and the fb-MAASC estimators on the same synthetic signal as used in [5]. This signal is the sum of 13 sinusoids at the angular frequencies $2\pi(0.0625, 0.0875,$

0.25, 0.285, 0.33, 0.35, 0.37, 0.39, 0.41, 0.43, 0.45, 0.47, 0.49), and these frequencies are marked with at dashed line in Fig. 1. All the sinusoidal components have a phase of $\pi/4$, and the amplitudes of the sinusoids are 1 for the first three, 0.3 for the fourth, and 0.1 for the rest. Our data set consists of $N = 64$ observations from a noise corrupted version of the sinusoidal signal where the noise is complex, white, and Gaussian with variance σ^2. As in [5], we define the signal-to-noise ratio (SNR) of the signal as the local SNR of the first sinusoid

$$\eta \triangleq 10 \log_{10}(|\alpha_1|^2 \sigma^{-2}) . \qquad (48)$$

Fig. 1 shows the MAP estimate $\hat{m}$ of the model index as a function of the frequency ω. We see that $\hat{m}$ is large when ω is close to one of the sinusoidal components whereas $\hat{m}$ is small when ω is either at a sinusoidal component or far away from one of the sinusoidal components. This is reasonable as a very long filter is necessary to filter out a sinusoidal component close to ω. On the other hand, a short filter length at a sinusoidal component means that the output vector $\boldsymbol{y}$ has more elements so that we may estimate the complex amplitude with a higher accuracy.

In Fig. 2, an example of the estimated amplitude spectrum is shown at the first sinusoidal component for an SNR of 20 dB. Despite the fact that the f-MAASC and fb-MAASC estimators are model averaged estimators, their resolution is only slightly worse than the resolution of the fb-ASC estimator with a maximum filter length of $K = 32$. However, the resolution was better than for the fb-APES estimator.

Based on 500 Monte Carlo iterations, the mean squared errors (MSE) at the first sinusoidal component for the estimates of the real and complex part of the complex amplitude, the amplitude, and the phase are shown and compared to the asymptotic Cramer-Rao lower bound (CRLB) in Fig. 3. Clearly, the f-MAASC and fb-MAASC estimators significantly reduce the MSE of the corresponding f-ASC and fb-ASC estimators. Other simulations have shown that this reduction comes from a reduction in both the bias and the variance. Although the maximum filter length of $K = 32$ is not a recommended length of the ASC-filters (see Sec. 1), we used this length in our simulations to see how much the bias was lowered by the MAASC-estimators while still having nearly the same resolution as demonstrated in Fig. 2. If the filter length is reduced, the MSE-performance of the ASC-filters also improves significantly, but at the expense of a coarser resolution.

Note that the f-APES and the fb-APES can also be cast in a model averaged framework. However, we observed only minor improvements in the MSE at the cost of a slightly worse resolution and a higher computational complexity. On the other hand, the model averaging attenuates the line-splitting (see Fig. 2) in the APES estimate of the amplitude spectrum.
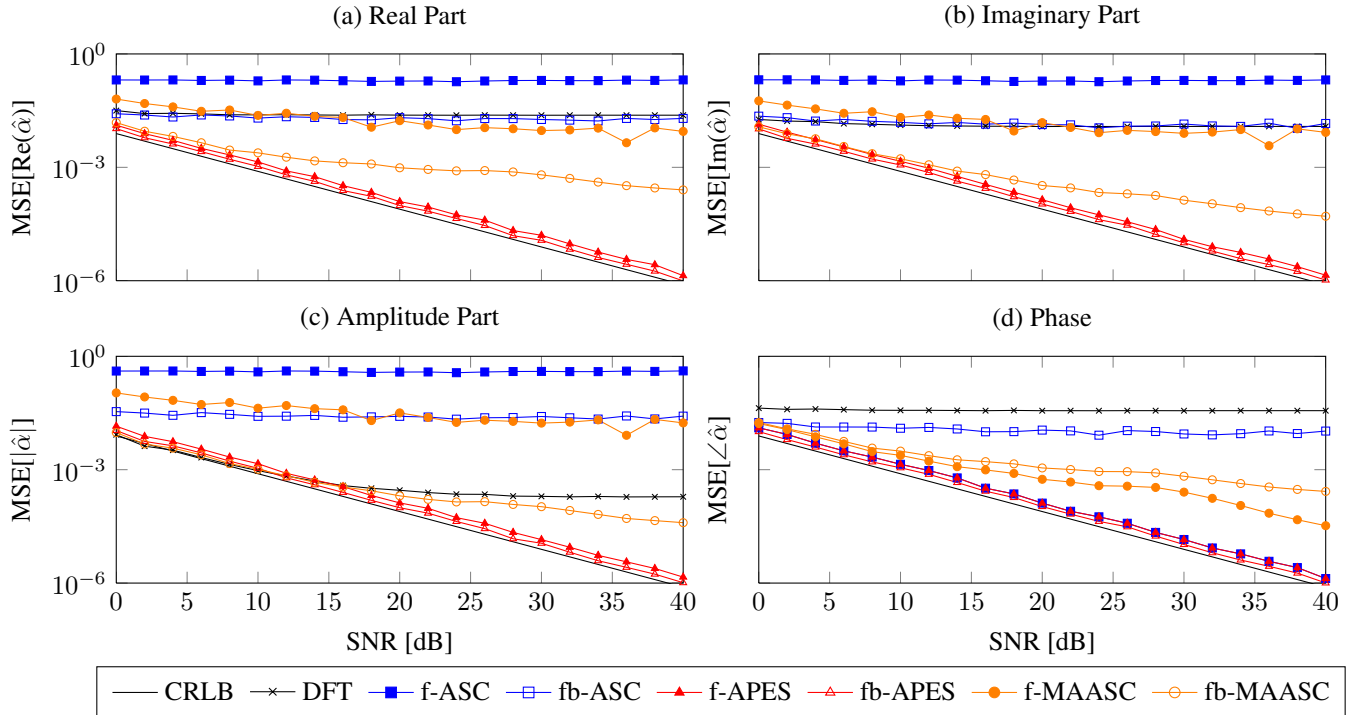
Fig. 3. The mean squared error (MSE) for various estimators based on 500 Monte Carlo iterations. The filter length for the f-ASC, the fb-ASC, the f-APES, and the fb-APES was $K = 32$.

6. CONCLUSION

In this paper, we have proposed a data-adaptive way of determining the filter length of the Capon filter. The adaptation was based on the approximate MAP approach by Djuric, and it led to a very simple way of computing an approximate posterior distribution for the filter length. Based on this posterior distribution, we also derived a model averaged amplitude spectral Capon estimator for both the forward and the forward-backward estimate of the covariance matrix. For nearly the same resolution, simulations on a synthetic signal showed that the f-MAASC and fb-MAASC significantly lowered the mean squared error of the complex amplitude estimates as compared to the traditional forward and forward-backward amplitude spectral Capon estimators, respectively.

7. REFERENCES

[1] P. Stoica and R. L. Moses, *Spectral Analysis of Signals*, Prentice Hall, May 2005.

[2] M. G. Christensen and A. Jakobsson, *Multi-Pitch Estimation*, Morgan & Claypool, 2009.

[3] J. Capon, "High-resolution frequency-wavenumber spectrum analysis," *Proc. IEEE*, vol. 57, no. 8, pp. 1408–1418, Aug. 1969.

[4] J. Li and P. Stoica, "An adaptive filtering approach to spectral estimation and SAR imaging," *IEEE Trans. Signal Process.*, vol. 44, no. 6, pp. 1469–1484, June 1996.

[5] P. Stoica, A. Jakobsson, and J. Li, "Matched-filter bank interpretation of some spectral estimators," *Elsevier Signal Processing*, vol. 66, no. 1, pp. 45–59, 1998.

[6] H. Li, J. Li, and P. Stoica, "Performance analysis of forward-backward matched-filterbank spectral estimators," *IEEE Trans. Signal Process.*, vol. 46, no. 7, pp. 1954–1966, July 1998.

[7] P. Stoica, H. Li, and J. Li, "Amplitude estimation of sinusoidal signals: Survey, new results, and an application," *IEEE Trans. Signal Process.*, vol. 48, no. 2, pp. 338–352, Feb. 2000.

[8] P. M. Djuric, "Asymptotic MAP criteria for model selection," *IEEE Trans. Signal Process.*, vol. 46, no. 10, pp. 2726–2735, Oct. 1998.

[9] A. Jakobsson and P. Stoica, "Combining Capon and APES for estimation of spectral lines," *Circ. Syst. Signal Process.*, vol. 19, no. 2, pp. 159–169, 2000.

[10] G. Schwarz, "Estimating the dimension of a model," *Ann. Stat.*, vol. 6, no. 2, pp. 461–464, Mar. 1978.