# ANGULAR RESOLUTION LIMIT FOR ARRAY PROCESSING: ESTIMATION AND INFORMATION THEORY APPROACHES

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## ABSTRACT

In this paper, we study the behavior of the angular resolution limit (ARL) for two closely spaced sources in the context of array processing. Particularly, we derive new closed-form expressions of the ARL, denoted by $\delta$, for three methods: the first one, which is the main contribution of this work, is based on the Stein's lemma which links the Chernoff's distance and a given/fixed probability of error, $P_e$, associated to the binary hypothesis test: $\mathcal{H}_0 : \delta = 0$ versus $\mathcal{H}_1 : \delta \neq 0$. The two other methods are based on the well-known Lee and Smith's criteria using the Cramér-Rao Bound (CRB). We show that the proposed ARL based on the Stein's lemma and the one based on the Smith's criterion have a similar behavior and they are proportional by a factor which depends only on $P_{fa}$ and $P_d$ and not on the model parameters (number of snapshots, sensor, sources, ....). Another conclusion is that for orthogonal signals and/or for a large number of snapshots, it is possible to give an unified closed-form expression of the ARL for the three approaches.

***Index Terms***– Angular Resolution Limit, Information Theory, Estimation Theory

## 1. INTRODUCTION

The problem of resolution limit for two closely spaced sources in the context of array processing has attached many interests. Generally, in the literature, there are four different ways to describe the resolution limit. The first one is based on the mean null spectrum concerning a specific algorithm [1,2]. The second one is based on the estimation accuracy, *i.e.* an implicit equation based on the Cramér-Rao bound (CRB) [3,4]. It is important to highlight the strong link between the Smith's criterion [4] and the asymptotic performance of the generalized likelihood ratio test [5]. Another approach is based on the detection theory using the hypothesis test formulation [5–7]. Finally, a recent and promising way to determine the resolution limit is based on the Stein's lemma [8,9] which establishes the relation between measures (Chernoff distance) of the difference between two probability distributions and the probability of error for a hypothesis test. In the literature, this lemma has already been used to derive the relative entropy to study detection performance and for waveform design in the context of MIMO radar in [10,11] and multi-static radar in [12]. Besides, in [13], the angular resolution limit (ARL) on resolving two closely spaced polarized sources using array processing is also studied by using the relative entropy.

In this work, two approaches will be considered to determine the closed-form expression of the resolution limit in the context of array processing. First, we follow the approach based on the Stein's lemma to derive the Chernoff distance ($\mathcal{CD}$), and then, the ARL for a fixed probability of error. Next, we derive the ARL using the standard ways based on the Cramér-Rao bound by applying the Lee's criterion [3] and the Smith's criterion [4]. Finally, a comparison in terms of closed-form expressions and numerical results between the three approaches is introduced to show the relevance of the Stein's lemma criterion and the Smith's criterion.

## 2. PROBLEM SETUP

We consider a linear sensor array of $N$ elements in the case of two source signals. The two source signals, denoted by $\mathbf{s}_1 = [s_1(1) \dots s_1(L)]^T$ and $\mathbf{s}_2 = [s_2(1) \dots s_2(L)]^T$, are assumed to be deterministic and located in the far-field w.r.t. the array. Each source is located by an angle-of-arrival denoted by $\theta_m$, $m = 1, 2$. The distance between $n$-th sensor w.r.t a reference sensor is denoted by $d_n$. Antenna array is assumed to be central-symmetric linear and the center of array is chosen as the reference sensor, *i.e.* $\sum_{n=1}^{N} d_n = 0$ and we denote $\sigma_a^2 = \frac{1}{N}\sum_{n=1}^{N} d_n^2$. In this scenario, the signal received at

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such array for the $l$-th snapshot is given by

$$\mathbf{y}(l)=\mathbf{a}(\omega_1)s_1(l)+\mathbf{a}(\omega_2)s_2(l)+\mathbf{n}(l), \tag{1}$$

where $l=1,\ldots,L$ with $L$ the number of snapshots. Let $y_n(l)$ define the $n$-th element of the vector $\mathbf{y}(l)$. We assume that $\mathbf{s}_1\neq\mathbf{s}_2$ and $||\mathbf{s}_1||^2=||\mathbf{s}_2||^2=L$. The steering vector has the following structures $\mathbf{a}(\omega_m)=[\exp(j\omega_m d_1)\ \ldots\ \exp(j\omega_m d_N)]^T$ where $\omega_m=\frac{2\pi}{\lambda}\sin\theta_m$ with $\lambda$ denoting the wavelength. The noise matrix for the $l$-th snapshot $\mathbf{n}(l)$ is assumed to be independent and identically distributed (i.i.d.) symmetric complex circular Gaussian with zero-mean and covariance matrix $\sigma^2\mathbf{I}_N$. Let us define $\mathbf{y}=[\mathbf{y}^T(1)\ldots\mathbf{y}^T(L)]^T$ and $\mathbf{n}=[\mathbf{n}^T(1)\ldots\mathbf{n}^T(L)]^T$, (1) can be rewritten as

$$\mathbf{y}=\mathbf{s}_1\otimes\mathbf{a}(\omega_1)+\mathbf{s}_2\otimes\mathbf{a}(\omega_2)+\mathbf{n}, \tag{2}$$

where $\otimes$ stands for the Kronecker product. In the following, we derive the ARL for this model in two different approaches, one based on a linearized binary hypothesis test and another based on the Cramér-Rao bound.

## 3. ARL BASED ON A LINEARIZED BINARY HYPOTHESIS TEST

If we denote $\delta=\omega_2-\omega_1$, the problem of resolving two closely spaced sources can be formulated as a binary hypothesis test as follows:

$$\begin{cases}\mathcal{H}_0: & \delta=0,\\ \mathcal{H}_1: & \delta\neq 0.\end{cases} \tag{3}$$

### 3.1. Linearized observation and new binary hypothesis test

Assume that $\delta$ is small, by using the first order Taylor expansion around the so-called center parameters $\omega_c=\frac{\omega_1+\omega_2}{2}$, i.e. $\mathbf{a}(\omega_1)\overset{1}{=}\mathbf{a}(\omega_c)-\frac{1}{2}\delta\dot{\mathbf{a}}(\omega_c)$ and $\mathbf{a}(\omega_2)\overset{1}{=}\mathbf{a}(\omega_c)+\frac{1}{2}\delta\dot{\mathbf{a}}(\omega_c)$ where symbol $\overset{1}{=}$ stands for first-order approximation and $\dot{\mathbf{a}}(\omega_c)=\frac{\partial\mathbf{a}(\omega_c)}{\partial\omega_c}$, one can obtain the linear approximation of (2) as follows [1]

$$\mathbf{y}\overset{1}{=}(\mathbf{a}(\omega_c)\otimes(\mathbf{s}_1+\mathbf{s}_2)+\frac{1}{2}\delta\dot{\mathbf{a}}(\omega_c)\otimes(\mathbf{s}_2-\mathbf{s}_1)+\mathbf{n}.$$

The linearized binary hypothesis test can be rewritten as

$$\begin{cases}\mathcal{H}_0: & \mathbf{y}\sim\mathcal{CN}(\mathbf{a}(\omega_c)\otimes(\mathbf{s}_1+\mathbf{s}_2),\sigma^2\mathbf{I}_{LN}),\\ \mathcal{H}_1: & \mathbf{y}\sim\mathcal{CN}(\mathbf{a}(\omega_c)\otimes(\mathbf{s}_1+\mathbf{s}_2)+\frac{1}{2}\delta\dot{\mathbf{a}}(\omega_c)\otimes(\mathbf{s}_2-\mathbf{s}_1),\sigma^2\mathbf{I}_{LN}).\end{cases} \tag{4}$$

[1]If the sources are equal, we need to consider a second-order Taylor expansion of the steering vectors. But this leads to untractable mathematical derivations.

### 3.2. Stein's lemma based analysis of ARL

From the Stein's lemma [8, 9], we have the asymptotic[2] relation between the $\mathcal{CD}$ relying the two probability density functions (pdf) for test (4) and a given probability of error for a binary hypothesis test as follows

$$\mathcal{CD}(p(y_n(l)|\mathcal{H}_0)||p(y_n(l)|\mathcal{H}_1))=-\lim_{NL\to\infty}\frac{1}{NL}\ln(P_e), \tag{5}$$

where $\mathcal{CD}(p(y_n(l)|\mathcal{H}_0)||p(y_n(l)|\mathcal{H}_1))$ is the Chernoff distance (for the sake of simplicity, we hereafter use $\mathcal{CD}_n(l)$ to denote the distance), $p(y_n(l)|\mathcal{H}_i)$ is the pdf of element $y_n(l)$ associated to hypothesis $\mathcal{H}_i$ and $P_e$ denotes a given probability of error. The Chernoff distance between two complex Gaussian distributions with parameterized means, *i.e.*

$$y_n(l)|\mathcal{H}_0\sim\mathcal{CN}((s_1(l)+s_2(l))\exp(j\omega_c d_n),\sigma^2),$$

and

$$y_n(l)|\mathcal{H}_1\sim\mathcal{CN}((s_1(l)+s_2(l))\exp(j\omega_c d_n)+\frac{1}{2}\delta\frac{\partial\exp(j\omega_c d_n)}{\partial\omega_c}(s_2(l)-s_1(l)),\sigma^2),$$

is given by [14]

$$\begin{aligned}\mathcal{CD}_n(l)&=\max_{0\le k\le 1}-\ln\int_\Omega[p(y_n(l)|\mathcal{H}_0)]^{1-k}[p(y_n(l)|\mathcal{H}_1)]^k dy_n(l)\\&=\max_{0\le k\le 1}\frac{k(1-k)}{\sigma^2}\left|\frac{1}{2}\delta\frac{\partial\exp(j\omega_c d_n)}{\partial\omega_c}(s_2(l)-s_1(l))\right|^2\\&=\frac{\delta^2}{16\sigma^2}d_n^2|s_2(l)-s_1(l)|^2.\end{aligned} \tag{6}$$

Note that we have used in the above derivation the fact that $\frac{\delta^2}{4\sigma^2}d_n^2|s_2(l)-s_1(l)|^2$ does not depend on $k$ and it is straightforward to see that $k(1-k)$ is maximized when $k=1/2$. Consequently, the $\mathcal{CD}$ between $p(\mathbf{y}|\mathcal{H}_0)$ and $p(\mathbf{y}|\mathcal{H}_1)$ is given by

$$\begin{aligned}\mathcal{CD}&=\sum_{n=1}^{N}\sum_{l=1}^{L}\mathcal{CD}_n(l)\\&=\sum_{n=1}^{N}\sum_{l=1}^{L}\frac{\delta^2}{16\sigma^2}d_n^2|s_2(l)-s_1(l)|^2\\&=\frac{\delta^2}{16\sigma^2}\sum_{n=1}^{N}d_n^2\sum_{l=1}^{L}|s_2(l)-s_1(l)|^2\\&=\frac{\delta^2}{16\sigma^2}N\sigma_a^2||\mathbf{s}_2-\mathbf{s}_1||^2.\end{aligned} \tag{7}$$

[2]Note the asymptotic context is not very severe since it is not necessary to consider a large number of sensors, $N$, and/or a large number of snapshots, $L$, but only a large product, $NL$, between these two quantities.

From (5) and (7), we obtain

$$\frac{\delta^2}{16\sigma^2} N\sigma_a^2 ||\mathbf{s}_2 - \mathbf{s}_1||^2 = -\lim_{NL\to\infty} \ln(P_e).$$

Finally, the ARL based on the $\mathcal{CD}$ is given by

$$\delta_C = \sqrt{\frac{-16\sigma^2 \ln(P_e)}{N\sigma_a^2 ||\mathbf{s}_2 - \mathbf{s}_1||^2}} = \sqrt{\frac{-8\sigma^2 \ln(P_e)}{N\sigma_a^2 (L - \mathcal{R}\{\mathbf{s}_1^H \mathbf{s}_2\})}}. \quad (8)$$

## 4. ARL BASED ON THE LINEARIZED CRAMÉR-RAO BOUND

In this section, we apply the Smith's criterion [4, 15] and the Lee's criterion [3] which are based on the estimation accuracy concept to determine the ARL for the considered model. Generally speaking, Lee and Smith's criteria are not easy to derive since the off-diagonal terms of the CRB for general arrays are a function of the ARL. So, to obtain the ARL in the sense of Lee and Smith, we have to analytically solve a polynomial of high order [16, 17] after linearization. But as we show in the appendix for central-symmetric arrays, the CRB is invariant to the ARL and thus the derivation of these criteria is considerably simplified. Using the linearized derivation of the CRB (see the appendix for details) for the considered model, the ARL based on the Lee's criterion is given by

$$\delta_L = 2\text{max}\left(\sqrt{[\mathbf{CRB}]_{1,1}}, \sqrt{[\mathbf{CRB}]_{2,2}}\right) = \sqrt{\frac{2\sigma^2}{N\sigma_a^2\left(L - \frac{\mathcal{R}^2\{\mathbf{s}_1^H\mathbf{s}_2\}}{L}\right)}}, \quad (9)$$

and the ARL based on the Smith's criterion [15] is given by

$$\delta_S = \sqrt{\gamma([\mathbf{CRB}]_{1,1} + [\mathbf{CRB}]_{2,2} - 2[\mathbf{CRB}]_{1,2})} = \sqrt{\frac{\gamma\sigma^2}{N\sigma_a^2 (L - \mathcal{R}\{\mathbf{s}_1^H \mathbf{s}_2\})}}, \quad (10)$$

where $\gamma$ is a translation factor [5] which can be estimated numerically by solving the equation $Q^{-1}_{\mathcal{X}_1^2}(P_{fa}) = Q^{-1}_{\mathcal{X}_1^2(\gamma)}(P_d)$, where $Q^{-1}_{\mathcal{X}_1^2}$ is the inverse of the right tail of the chi-square distribution, denoted by $\mathcal{X}_1^2$, and $P_{fa}$ and $P_d$ are the probability of false alarm and detection, respectively.

## 5. ANALYTIC COMPARISONS

### 5.1. Ratios between the ARL

It can be seen that the ARL based on three criteria are proportional and the factors can be derived from (8), (9) and (10) as

$$\frac{\delta_C}{\delta_S} = 2\sqrt{\frac{-2\ln(P_e)}{\gamma}}, \quad \frac{\delta_C}{\delta_L} = 2\sqrt{-\ln(P_e)\beta}, \quad \frac{\delta_L}{\delta_S} = \sqrt{\frac{2}{\gamma\beta}}. \quad (11)$$

where $\beta = 1 + \frac{\mathcal{R}\{\mathbf{s}_1^H\mathbf{s}_2\}}{L}$. While the first factor depends only on the probability of error and $\gamma$ (which depends on the probability of false alarm and of detection), the other factors that related to the Lee's criterion depend also on the number of snapshots and the sources.

### 5.2. Unified expressions of the ARL

If the two sources are orthogonal, *i.e.* $\mathbf{s}_1^H\mathbf{s}_2 = 0$ or/and if the number of snapshots is large enough *i.e.*, $L \gg \mathcal{R}\{\mathbf{s}_1^H\mathbf{s}_2\}$, we have $\beta \approx 1$ and the ARL given here can be unified according to

$$\delta = \sqrt{\frac{\xi\sigma^2}{NL\sigma_a^2}},$$

where

$$\xi = -8\ln(P_e) \quad \text{for the Chernoff's ARL} \quad (12)$$
$$\xi = \gamma(P_{fa}, P_d) \quad \text{for the Smith's ARL} \quad (13)$$
$$\xi = 2 \quad \text{for the Lee's ARL} \quad (14)$$

and the ratios between these ARL are given by

$$\frac{\delta_C}{\delta_S} = 2\sqrt{\frac{-2\ln(P_e)}{\gamma}}, \quad \frac{\delta_C}{\delta_L} = 2\sqrt{-\ln(P_e)}, \quad \frac{\delta_L}{\delta_S} = \sqrt{\frac{2}{\gamma}}. \quad (15)$$

One can see the ARL obtained above do not depend on the sources but on the configuration of the array, the power of noise, the probability of false alarm and the probability of detection.

## 6. NUMERICAL ILLUSTRATIONS

In this section, some numerical results are presented to analyze and compare the behavior of the ARL determined by the different approaches proposed above. The scenario is the following: the sensor array is central-symmetric uniform linear array and is composed of $N = 10$ sensors, with inter-element spacing (in unit of wavelengths) is $0.5$, the number of snapshots $L = 100$, the probability of false alarm and the probability of detection are $P_{fa} = 0.01$ and $P_d = 0.99$, respectively, we obtain the probability of error $P_e = \frac{1}{2}P_{fa} + \frac{1}{2}(1 - P_d) = 0.01$. First, figure 1 plots the ARL $\delta$ versus the SNR(dB)

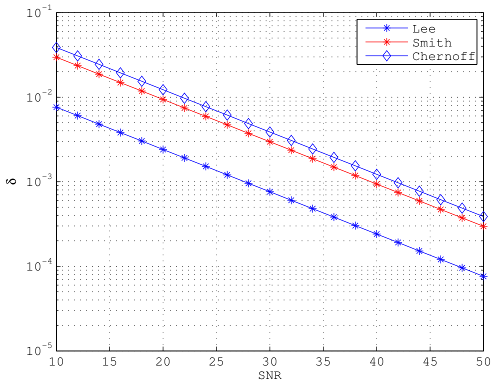


**Fig. 1**. ARL versus the SNR(dB)

One can see on figure 1 that the ARL based on the Chernoff distance and the Smith's criterion are very closed. Considering a fixed value of the probability of detection $P_d = 0.99$, we plot on figure 2 the ARL based on the Chernoff distance and the Smith's criterion versus the probability of false alarm. One can see that at a certain value of $P_{fa}$, the ARL based on these two criteria can be identical.

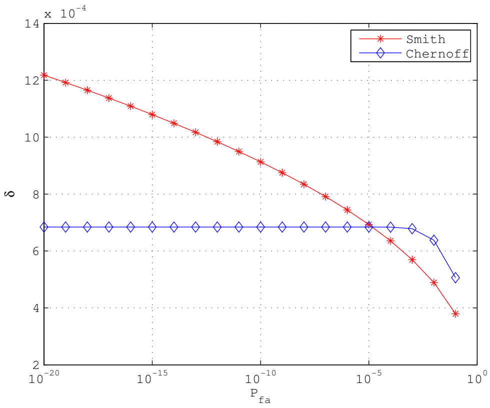


**Fig. 2**. ARL versus $P_{fa}$ at SNR $= 45dB$

## 7. CONCLUSION

In this paper, we have derived a closed-form expression of the ARL, denoted by $\delta$, for two closely spaced targets in the context of array processing for two standard approaches, namely the Lee and the Smith's criteria and for a new method based on the Stein's lemma. The latter is based on the link between the Chernoff distance and a given probability of error associated to the binary hypothesis test: $\mathcal{H}_0 : \delta = 0$ versus $\mathcal{H}_1 : \delta \neq 0$. The analysis has provided new interesting insights on the ARL in this context. It has been seen that the ARL based on the Chernoff distance and the Smith's criterion have a similar behavior and they are proportional by a factor which depends on the probabilities of false alarm and of detection and not of the signal parameters. We also show that for orthogonal sources and/or a large number of snapshots, it is possible to give a unified expression of the ARL for the three considered approaches.

## 8. APPENDIX: DERIVATION OF THE LINEARIZED CRB

From (2), we have $\mathbf{y} \sim \mathcal{CN}(\mathbf{a}(\omega_1)\otimes\mathbf{s}_1 + \mathbf{a}(\omega_2)\otimes\mathbf{s}_2, \sigma^2\mathbf{I}_{LN})$. The vector of unknown parameters is $\boldsymbol{\omega} = [\omega_1\ \omega_2]^T$. It is well known that the CRB matrix is the the inverse of the Fisher information matrix (FIM), defined by $\mathbf{F}(\boldsymbol{\omega})$. Hence, to obtain the CRB, first, we derive the set of elements of the FIM using the Slepian-Bang formula (see, *e.g.*, [18]) as follows

$$
\begin{aligned}
\mathbf{F}(\omega_1, \omega_1) &= \frac{2}{\sigma^2}\mathcal{R}\Big\{(\dot{\mathbf{a}}(\omega_1)^H \otimes \mathbf{s}_1^H)(\dot{\mathbf{a}}(\omega_1) \otimes \mathbf{s}_1)\Big\} \\
&= \frac{2\sigma_a^2 NL}{\sigma^2}, \\
\mathbf{F}(\omega_2, \omega_2) &= \frac{2}{\sigma^2}\mathcal{R}\Big\{(\dot{\mathbf{a}}(\omega_2)^H \otimes \mathbf{s}_2^H)(\dot{\mathbf{a}}(\omega_2) \otimes \mathbf{s}_2)\Big\} \\
&= \frac{2\sigma_a^2 NL}{\sigma^2}, \\
\mathbf{F}(\omega_1, \omega_2) &= \frac{2}{\sigma^2}\mathcal{R}\Big\{(\dot{\mathbf{a}}(\omega_1)^H \otimes \mathbf{s}_1^H)(\dot{\mathbf{a}}(\omega_2) \otimes \mathbf{s}_2)\Big\} \\
&= \frac{2}{\sigma^2}\mathcal{R}\Big\{\sum_{n=1}^{N} d_n^2 \exp(\frac{j}{2}d_n\delta)\mathbf{s}_1^H\mathbf{s}_2\Big\} \\
&\stackrel{1}{=} \frac{2}{\sigma^2}\mathcal{R}\Big\{\sum_{n=1}^{N} d_n^2 (1 + \frac{j}{2}d_n\delta)\mathbf{s}_1^H\mathbf{s}_2\Big\} \\
&= \frac{2N\sigma_a^2}{\sigma^2}\mathcal{R}\{\mathbf{s}_1^H\mathbf{s}_2\} + \delta\mathcal{R}\{j\mathbf{s}_1^H\mathbf{s}_2\}\sum_{n=1}^{N} d_n^3, \\
\mathbf{F}(\omega_2, \omega_1) &= \frac{2}{\sigma^2}\mathcal{R}\Big\{(\dot{\mathbf{a}}(\omega_2)^H \otimes \mathbf{s}_2^H)\mathbf{R}^{-1}(\dot{\mathbf{a}}(\omega_1) \otimes \mathbf{s}_1)\Big\} \\
&= \frac{2}{\sigma^2}\mathcal{R}\Big\{\sum_{n=1}^{N} d_n^2 \exp(\frac{j}{2}d_n\delta)\mathbf{s}_2^H\mathbf{s}_1\Big\} \\
&\stackrel{1}{=} \frac{2}{\sigma^2}\mathcal{R}\Big\{\sum_{n=1}^{N} d_n^2 (1 + \frac{j}{2}d_n\delta)\mathbf{s}_2^H\mathbf{s}_1\Big\} \\
&= \frac{2N\sigma_a^2}{\sigma^2}\mathcal{R}\{\mathbf{s}_2^H\mathbf{s}_1\} + \delta\mathcal{R}\{j\mathbf{s}_2^H\mathbf{s}_1\}\sum_{n=1}^{N} d_n^3.
\end{aligned}
$$

The off terms of the CRB are linearized thanks to a first-order Taylor expansion. As the antenna array is central-symmetric linear and the center of array is chosen as the reference sensor, we have $\sum_{n=1}^{N} d_n^3 = 0$. Consequently,

$$\mathbf{F}(\omega_1, \omega_2) \stackrel{1}{=} \frac{2N}{\sigma^2}\sigma_a^2 \mathcal{R}\{\mathbf{s}_1^H \mathbf{s}_2\}.$$

Inverting the FIM, one can obtain the elements of the CRB as follows

$$\begin{aligned}\mathbf{CRB} &= \mathbf{F}^{-1}(\boldsymbol{\omega}) \\ &\stackrel{1}{=} \frac{1}{\mathbf{F}(\omega_1, \omega_1)^2 - \mathbf{F}(\omega_1, \omega_2)^2} \\ &\qquad \cdot \begin{bmatrix} \mathbf{F}(\omega_1, \omega_1) & -\mathbf{F}(\omega_1, \omega_2) \\ -\mathbf{F}(\omega_1, \omega_2) & \mathbf{F}(\omega_1, \omega_1) \end{bmatrix}.\end{aligned}$$

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