RECURSIVE FAN-TYPE FILTER DESIGN FROM 1D ANALOG TRANSFER FUNCTIONS

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ABSTRACT

In this paper a new design method for a class of 2D IIR fan-type filters based on spectral transformations is proposed. The design starts from an analog prototype filter with specified parameters and whose transfer function is factorized into biquads. Applying an appropriate frequency transformation to the 1D transfer function, the desired 2D filter is directly obtained in a factorized form as well. The approach is mainly analytical but also uses numerical approximations and is simple, efficient and versatile. These fan-type filters can also be regarded as components of directional filter banks. Using the determined frequency transformation, two-directional selective filters are obtained as well. Design examples are provided and simulation results are shown, for an application in medical image processing, namely the detection of blood vessels with a specified orientation from angiography images. For two-directional filters, an example is given of extracting lines with two different orientations from a test image.

***Index Terms*—**2D recursive filter design, frequency transformations, directional image filtering

1. INTRODUCTION

The field of two-dimensional filters has largely developed along the last three decades and various design methods were proposed by researchers [1]-[4]. Generally the currently-used design methods for 2D recursive filters rely on 1D filter prototypes, using spectral transformations from s to z plane via bilinear or Euler approximations, with the aim to obtain a 2D filter with a desired frequency response.

There are several classes of filters with orientation-selective frequency response, useful in tasks like edge detection, motion analysis, texture segmentation etc. In [5] a widely-used filter bank for the directional decomposition of images was proposed. These directional filter banks (DFB) provide an image decomposition in the frequency domain. Another type of DFB is presented in [6]. In [7] various 2D recursive filters are approached. Fan-shaped, also known as wedge-shaped filters find interesting applications. A design method for IIR and FIR fan filters is presented in an early paper [8]. An efficient design method for recursive fan filters is presented in [9]. An implementation of recursive fan filters using all-pass sections is given in [10]. In [11], an analytical least-squares technique for FIR filters, in particular fan-type, is proposed. In [12], IIR digital filters with complex coefficients are treated, while in [13] an analytical design method is proposed for directional filters.

In this work an analytical design method in the frequency domain for 2D fan-type filters is proposed, starting from a given 1D analog prototype filter, with a transfer function decomposed as a product of elementary functions. Since we envisage designing efficient 2D filters, of minimum order, recursive filters are used as prototypes, and the 2D fan-type filters will result recursive as well.

The paper is organized as follows: the analog 1D prototype filters used in design are introduced in section 2; the proposed design method is described in section 3, including design examples. In section 4 a fan-type filter with narrow aperture is used in directional filtering of a biomedical image, and also simulation results for two-directional filters are shown.

2. ANALOG PROTOTYPE FILTERS

A fundamental step in designing temporal and spatial filters is the approximation. We refer throughout this work only to recursive filters. Let us consider an analog prototype filter of order *N* with the general transfer function in variable *s*:

$$H_P(s) = \frac{P(s)}{Q(s)} = \sum_{i=0}^{M} p_i \cdot s^i \Bigg/ \sum_{j=0}^{N} q_j \cdot s^j \qquad (1)$$

Next we consider this general transfer function factorized into rational functions of first and second order. A second-order function (biquad) resulted from the factorization of the transfer function (1) will be of the form:

$$H_b(s) = k\left(s^2 + b_1 s + b_0\right) \Big/ \left(s^2 + a_1 s + a_0\right) \qquad (2)$$

where in general the second-order polynomials at the numerator and denominator have complex-conjugated roots. For usual approximations, e.g. Chebyshev or elliptic, usually in (2) $b_1 = 0$, so the numerator has imaginary zeros. For odd-order filters the denominator contains at least a first- order factor ($s + \alpha$). For an almost maximally-flat low-order filter, an elliptic filter is the best choice.

Let us consider an elliptic low-pass (LP) analog filter of order $N = 4$, cutoff frequency $\omega_c = 0.4\pi$, peak-to-peak ripple $R_p = 0.05$ db, stop-band attenuation $R_s = 36$ db.

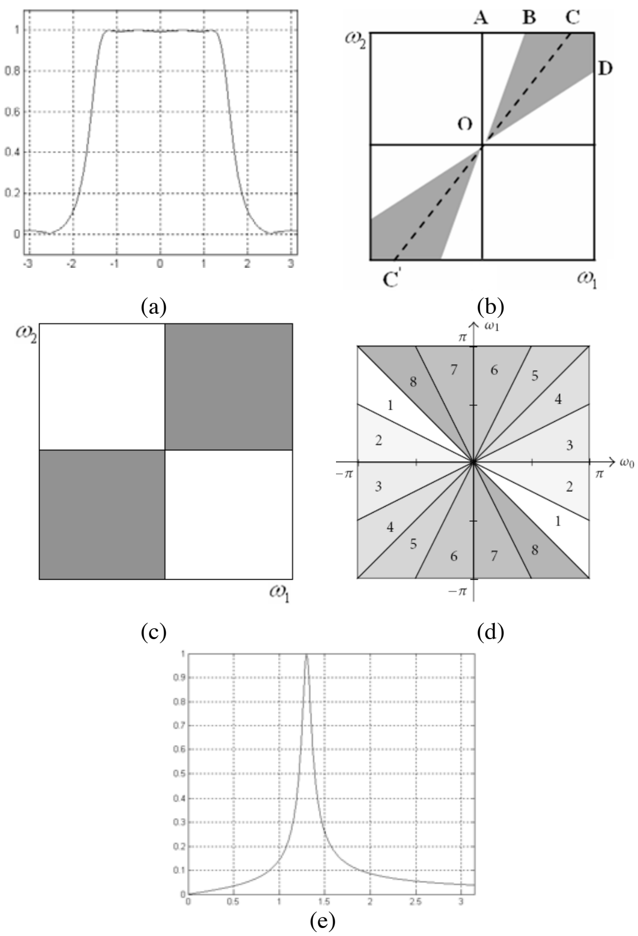


Fig.1. (a) Elliptic LP analog filter magnitude for order 4; (b) Ideal fan filter with given aperture, oriented at an angle φ; (c) Ideal two-quadrant fan filter (d) 8-band partitions of the frequency plane; (e) very selective (resonant) analog prototype filter

The transfer function of this LP filter can be factorized as a product of two biquad functions of the general form (2):

$$H_P(s) = k \cdot H_{b1}(s) \cdot H_{b2}(s) \tag{3}$$

where $k \cong 0.01$ and

$$H_{b1}(s) = \left(s^2 + 33.385\right) \Big/ \left(s^2 + 0.5894s + 2.2398\right) \tag{4}$$

$$H_{b2}(s) = \left(s^2 + 6.4226\right) \Big/ \left(s^2 + 1.9691s + 1.5266\right) \tag{5}$$

The frequency response magnitude of this low-pass filter on the range $(-\pi, \pi)$ is shown in Fig.1 (a).

Another useful analog prototype is the very selective (resonant or peaking) second-order filter of the form:

$$H_b(s) = \alpha s \Big/ \left(s^2 + \alpha s + \omega_0^2\right) \tag{6}$$

where ω_0 is the central frequency. The transfer function magnitude for such a filter with $\alpha = 0.1$ and $\omega_0 = 1.3$ is shown in Fig.1(e). This filter will be used here as a prototype for 2D two-directional filters.

3. DESIGN METHOD FOR FAN-TYPE FILTERS

In Fig.1(b) a general fan-type filter is shown, with an aperture angle $\prec BOD = \theta$, oriented along an axis CC' and its longitudinal axis forming an angle $\prec AOC = \varphi$ with frequency axis $O\omega_2$. A particular case is the two-quadrant fan filter, shown in Fig.1(c). Fig.1(d) shows a DFB with 8-band frequency partition [5], an angularly-oriented image decomposition which splits the frequency plane into fan-shaped sub-bands (channels).

The 1-D analog filter discussed in section 2 will be used as prototype. The general fan-type filter can be derived from a LP prototype using the frequency mapping:

$$\omega \to f_\varphi(\omega_1, \omega_2) = \frac{a \cdot (\omega_1 \cdot \cos\varphi - \omega_2 \cdot \sin\varphi)}{(\omega_1 \cdot \sin\varphi + \omega_2 \cdot \cos\varphi)} \tag{7}$$

In (7), a is the *aperture coefficient*, given by $a = 1/\mathrm{tg}(\theta/2)$, where θ is the aperture angle of the fan-type filter. This frequency mapping can be written in the complex frequency variables $s_1 = j\omega_1$ and $s_2 = j\omega_2$:

$$s \to f_\varphi(s_1, s_2) = \frac{ja \cdot (s_1 \cdot \cos\varphi - s_2 \cdot \sin\varphi)}{(s_1 \cdot \sin\varphi + s_2 \cdot \cos\varphi)} \tag{8}$$

The most common method to obtain a discrete filter from an analog prototype is the bilinear transform. If the sample interval takes the value $T = 1$, the double bilinear transform for s_1 and s_2 in the complex plane (s_1, s_2) has the form:

$$s_1 = 2(z_1 - 1)/(z_1 + 1) \qquad s_2 = 2(z_2 - 1)/(z_2 + 1) \tag{9}$$

Even if this method is straightforward, the designed 2D filter, corresponding to the transfer function in z_1, z_2, will present noticeable linearity distortions towards the limits of the frequency plane as compared to the ideal frequency response. This is mainly due to the frequency warping effect introduced by the bilinear transform, expressed by the continuous-time to discrete-time frequency mapping:

$$\omega = (2/T) \cdot \mathrm{arctg}\left(\omega_a T/2\right) \tag{10}$$

where ω is a frequency of the discrete-time filter and ω_a is the corresponding frequency of the continuous-time filter. In order to correct this error, a pre-warping will be applied. Taking $T = 1$ in relation (10) we substitute the mappings:

$$\omega_1 \to 2 \cdot \mathrm{arctg}\left(\omega_1/2\right) \quad \omega_2 \to 2 \cdot \mathrm{arctg}\left(\omega_2/2\right) \tag{11}$$

In handling the nonlinear functions (11), rational approximations would be more suitable. An efficient one is Chebyshev-Padé, which gives an uniform approximation over an entire specified range. We get the following accurate approximation on the range $[-\pi, \pi]$:

$$\mathrm{arctg}\left(\omega/2\right) \cong 0.4751 \cdot \omega \Big/ \left(1 + 0.05 \cdot \omega^2\right) \tag{12}$$

Substituting the nonlinear mappings (11) with expression (12) into (8) we get the 1D to 2D mapping which includes pre-warping along both axes of the frequency plane:

$$s \to F_\varphi(s_1, s_2) = \frac{ja \cdot \left(s_1(1-0.05s_2^2)\cos\varphi - s_2(1-0.05s_1^2)\sin\varphi\right)}{\left(s_1(1-0.05s_2^2)\sin\varphi + s_2(1-0.05s_1^2)\cos\varphi\right)} \tag{13}$$

We now apply the bilinear transform (9) along the two axes and obtain the mapping $s \to F_\varphi(z_1, z_2)$ in matrix form:

$$s \to F_\varphi(z_1,z_2) = k \cdot M_\varphi(z_1,z_2) / N_\varphi(z_1,z_2)$$
$$= k \cdot \frac{\begin{bmatrix} z_1^{-1} & 1 & z_1 \end{bmatrix} \times \mathbf{M}_\varphi \times \begin{bmatrix} z_2^{-1} & 1 & z_2 \end{bmatrix}^T}{\begin{bmatrix} z_1^{-1} & 1 & z_1 \end{bmatrix} \times \mathbf{N}_\varphi \times \begin{bmatrix} z_2^{-1} & 1 & z_2 \end{bmatrix}^T} \tag{14}$$

where the 3×3 matrices $\mathbf{M}_\varphi$ and $\mathbf{N}_\varphi$ are given by:

$$\mathbf{M}_\varphi = \cos\varphi \cdot \begin{bmatrix} -1 & -3 & -1 \\ 0 & 0 & 0 \\ 1 & 3 & 1 \end{bmatrix} - \sin\varphi \cdot \begin{bmatrix} -1 & 0 & 1 \\ -3 & 0 & 3 \\ -1 & 0 & 1 \end{bmatrix} \tag{15}$$

$$\mathbf{N}_\varphi = \sin\varphi \cdot \begin{bmatrix} -1 & -3 & -1 \\ 0 & 0 & 0 \\ 1 & 3 & 1 \end{bmatrix} + \cos\varphi \cdot \begin{bmatrix} -1 & 0 & 1 \\ -3 & 0 & 3 \\ -1 & 0 & 1 \end{bmatrix} \tag{16}$$

Substituting the mapping (14) into the expression of the biquad transfer function $H_b(s)$, we get the 2D transfer function $H_{W1}(z_1, z_2)$ in the matrix form:

$$H_{W1}(z_1,z_2) = \left(\mathbf{z_1} \times \mathbf{B_1} \times \mathbf{z_2}^T\right) \big/ \left(\mathbf{z_1} \times \mathbf{A_1} \times \mathbf{z_2}^T\right) \tag{17}$$

where $\mathbf{z_1}$ and $\mathbf{z_2}$ are the vectors:

$$\mathbf{z_1} = \begin{bmatrix} 1 & z_1 & z_1^2 & z_1^3 & z_1^4 \end{bmatrix} \quad \mathbf{z_2} = \begin{bmatrix} 1 & z_2 & z_2^2 & z_2^3 & z_2^4 \end{bmatrix} \tag{18}$$

and the 5×5 matrices $\mathbf{B_1}$, $\mathbf{A_1}$ are given by the expressions:

$$\mathbf{B_1} = b_0 \cdot \mathbf{N}_\varphi * \mathbf{N}_\varphi - a^2 \cdot \mathbf{M}_\varphi * \mathbf{M}_\varphi \tag{19}$$

$$\mathbf{A_1} = a_0 \cdot \mathbf{N}_\varphi * \mathbf{N}_\varphi - a^2 \cdot \mathbf{M}_\varphi * \mathbf{M}_\varphi + j \cdot a \cdot a_1 \cdot \mathbf{M}_\varphi * \mathbf{N}_\varphi \tag{20}$$

The 2D transfer function for the each biquad is complex. This however should not be a serious shortcoming, since IIR filters with complex coefficients are also used [11].

The characteristics of a fan-type filter designed with this method and based on the prototype filter of order 4 given by (3)-(5) is shown in Fig.2, for the indicated parameters.

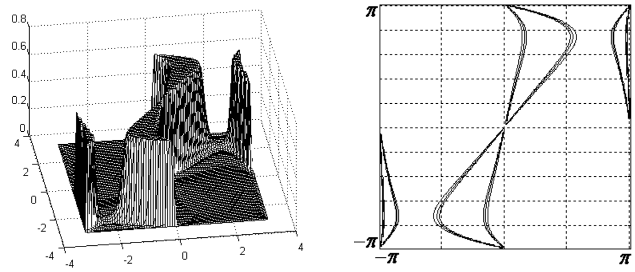
As can be noticed, the filter characteristic corrected by pre-warping has a good linearity, however it still twists towards the margins of the frequency plane. These marginal linearity distortions can be corrected using an additional low-pass (LP) filter. For instance, we can also choose an 1D elliptic digital filter of order 3, with pass-band ripple 0.1dB, stop-band attenuation 40 dB and cutoff frequency 0.6. It has the coefficients given by the vectors:

$$\mathbf{B}_1 = [0.3513 \quad 1.01 \quad 1.01 \quad 0.3513] \tag{21}$$

$$\mathbf{A}_1 = [1 \quad 0.9644 \quad 0.6701 \quad 0.088] \tag{22}$$

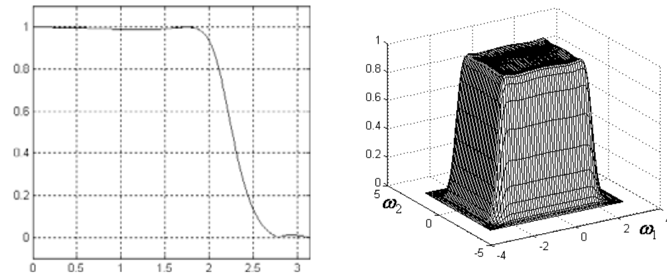
The 2D low-pass filter is separable and results by applying successively the 1D filter along the two frequency axes; its matrices (also called *templates*) result as: $\mathbf{B}_C = \mathbf{B}_1^T \otimes \mathbf{B}_1$ and $\mathbf{A}_C = \mathbf{A}_1^T \otimes \mathbf{A}_1$.

The correction filter has the transfer function:



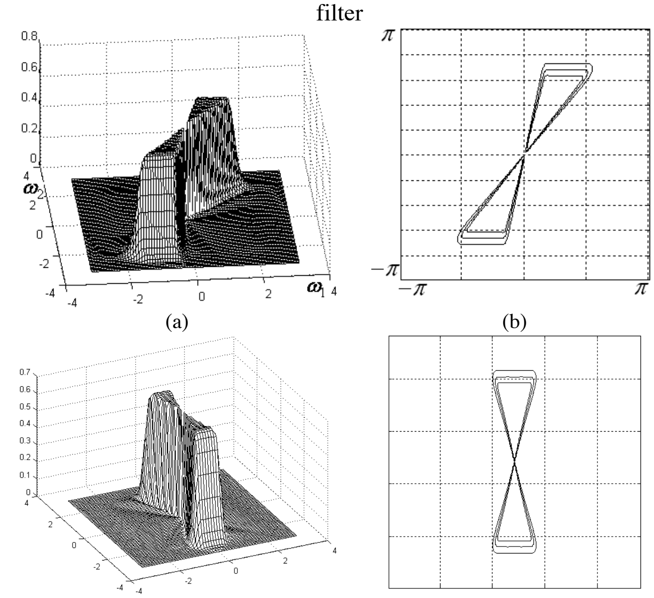
(a) (b)

Fig.2. Frequency response magnitude and contour plot for a fan-shaped filter with aperture $\theta = 0.1\pi$ and inclination angle $\varphi = \pi/7$



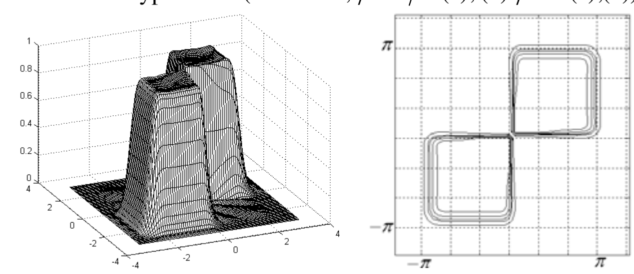
(a) (b)

Fig.3. (a) Low-pass 1D filter; (b) resulted low-pass 2D correction filter



(a) (b) (c) (d)

Fig.4. Frequency response magnitude and contour plot of two corrected fan-type filters ($\theta = 0.1\pi$, $\varphi = \pi/7$ (a), (b) $\varphi = 0$ (c),(d))



(a) (b)

Fig.5. Frequency response magnitude (a) and contour plot for the 2-quadrant fan filter

$$H_C(z_1,z_2) = \left(\mathbf{z_1} \times \mathbf{B}_C \times \mathbf{z_2}^T\right) \Big/ \left(\mathbf{z_1} \times \mathbf{A}_C \times \mathbf{z_2}^T\right) \quad (23)$$

where $\mathbf{z_1} = \begin{bmatrix} 1 & z_1 & z_1^2 & z_1^3 \end{bmatrix}$ and $\mathbf{z_2} = \begin{bmatrix} 1 & z_2 & z_2^2 & z_2^3 \end{bmatrix}$.

The 1D and resulted 2D square-shaped correction filter characteristics are shown in Fig.3 and are almost maximally-flat, as required. Two corrected fan-type filters with the given parameters have the magnitudes and the contour plots shown in Fig.4. The initial distortions have been eliminated.

With the same correction filter, we obtain the two-quadrant fan filter, shown in Fig.5, by setting the aperture angle $\theta = \pi/2$ and inclination $\varphi = \pi/4$.

The other type of 2D filter using the same fan filter frequency mapping is a 2D filter which is orientation-selective along two directions in the frequency plane. It is based on a selective peaking (resonant) IIR filter prototype as given in section 2. Applying the same frequency transformation $s \to F_\varphi(z_1,z_2)$ determined before and given by (14) to the prototype filter (6) we get the 2D transfer function $H_{W2}(z_1,z_2)$ in the matrix form:

$$H_{W2}(z_1,z_2) = \left(\mathbf{z_1} \times \mathbf{B}_2 \times \mathbf{z_2}^T\right) \Big/ \left(\mathbf{z_1} \times \mathbf{A}_2 \times \mathbf{z_2}^T\right) \quad (24)$$

which is identical to (17), but the 5×5 matrices $\mathbf{B_2}$ and $\mathbf{A_2}$ have now the form:

$$\mathbf{B_2} = a \cdot \alpha \cdot \mathbf{M}_\varphi * \mathbf{N}_\varphi \quad (25)$$

$$\mathbf{A_2} = a \cdot \alpha \cdot \mathbf{M}_\varphi * \mathbf{N}_\varphi + j\left(a^2 \cdot \mathbf{M}_\varphi * \mathbf{M}_\varphi - \omega_0^2 \cdot \mathbf{N}_\varphi * \mathbf{N}_\varphi\right) \quad (26)$$

As can be noticed, the matrix $\mathbf{A_2}$ corresponding to the denominator has complex elements. As is the case of the maximally-flat fan filter designed before, the resulted two-directional filters feature marginal distortions which can be removed using the correction filter described by (21)-(23).

In Fig.6 the frequency response magnitudes and contour plots of two corrected two-directional filters are shown, for different aperture and orientation angles. The first filter is a particular case, being oriented along the two frequency axes, therefore it can be used to detect simultaneously horizontal and vertical lines from an image, as will be shown in the next section through simulation results.

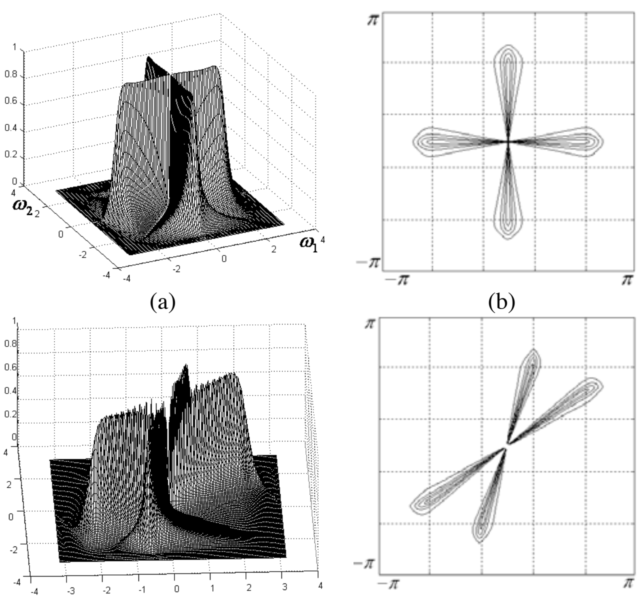


Fig.6. Frequency response magnitude and contour plot of two corrected two-directional filters with parameters: $\theta = \pi/2$, $\varphi = \pi/4$ (a),(b) and $\theta = \pi/6$, $\varphi = \pi/5$ (c),(d)

As a general remark, using an analog prototype instead of a digital one, as is currently done, simplifies the design in this case, because the frequency mapping results simpler and leads to a 2-D filter of lower complexity.

A full stability analysis for the 2D filters designed using the proposed method will be done in further work. In principle, since the design starts from stable prototypes and the 2D filters result through transformations that conserve stability, like bilinear transform, the 2D filters should also result stable. They could however become unstable if the numerical approximations used, for instance (12), introduce large errors. In such a case we may have to increase the precision of approximation by taking more higher order terms, which will increase in turn the filter complexity.

4. APPLICATIONS AND SIMULATION RESULTS

Let us apply the designed fan-shaped filters, regarded as components of a DFB, in filtering a typical retinal vascular image. Clinicians usually search in angiograms relevant features like number and position of vessels (arteries, capillaries). An angular-oriented filter bank may be used in analyzing angiography images by detecting vessels with a given orientation. Let us consider the retinal fluorescein angiogram from Fig.7(a), featuring some pathological elements which indicate a diabetic retinopathy. This image is filtered using 5 fan-type filters with narrow aperture and different orientation angles, designed using the described method. Fig.7 (b)-(f) show the directionally filtered images. The vessels whose frequency spectrum overlaps more or less with the filter characteristic remain visible, while the others are blurred, an effect of the directional LP filtering.

An example of image filtering with a two-directional filter is also shown. This type of filter can be used in simultaneously detecting perpendicular lines from an image. The test image in Fig.8(a) contains straight lines with different orientations and lengths, and a few curves. We use the filter shown in Fig.6(a),(b). Depending on filter selectivity, only the lines with the spectrum oriented more or less along the filter pass-bands will remain in the filtered image. In the output image shown in Fig.8(b), the lines roughly oriented horizontally and vertically are preserved, while the others are filtered out or appear very blurred. The joints of detected lines appear as darker pixels and can be detected, if after filtering a proper threshold is applied.

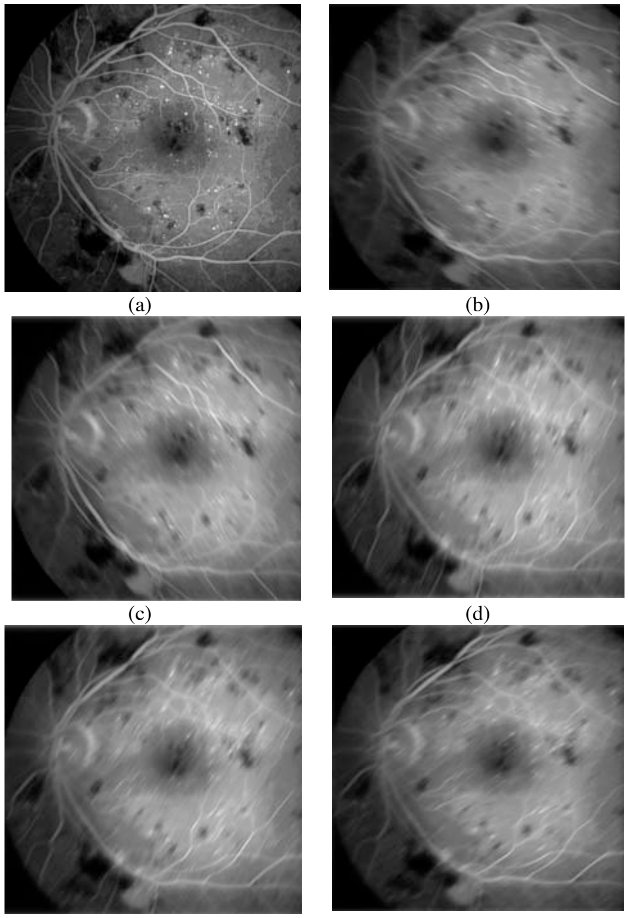


Fig.7 (a) Retinal fluorescein angiogram; (b)-(f) images resulted as output of the filter bank channels

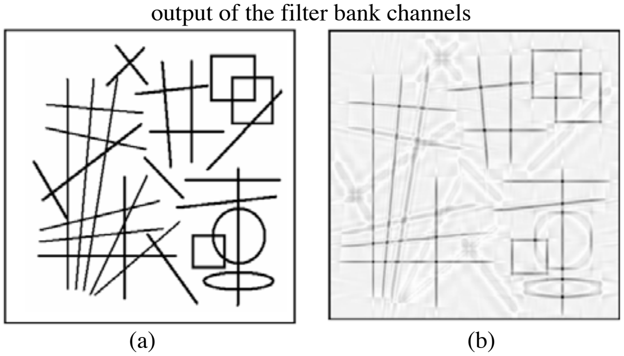


Fig.8. (a) Test image; (b) filtered image using a two-directional filter with $\theta = \pi / 2$, $\varphi = \pi / 4$

5. CONCLUSIONS

An efficient analytical design method for recursive fan-type filters was approached. Two types of filters were proposed: fan filters and two-directional selective filters, both specified by an aperture and orientation angle. The design is based on analog low-pass and band-pass prototypes with specified parameters. A particular frequency mapping is derived, which is applied to the 1D prototype in order to obtain the 2D filter. The method includes the bilinear transform applied on the two axes of the frequency plane and uses a frequency pre-warping to compensate for filter distortions. This design technique is rather simple, efficient and flexible, since by starting from different specifications, the new 2D filter matrices result directly, by applying the determined frequency mapping, and there is no need to resume the whole design procedure. Results of directional filtering using narrow fan-type filters were provided for a retina angiography image.

6. REFERENCES

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