AN ALTERNATING DESCENT ALGORITHM FOR THE OFF-GRID DOA ESTIMATION PROBLEM WITH SPARSITY CONSTRAINTS

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ABSTRACT

In this paper, we present an iterative alternating descent algorithm for the problem of off-grid direction-of-arrival (DOA) estimation under the spatial sparsity assumption. Using a secondary dictionary we approximate the off-grid DOAs exploiting the method of Taylor expansion. In that way, we overcome the limitation of the conventional sparsity-based DOA estimation approaches that the unknown directions belong to a predefined discrete angular grid. The proposed method (SOMP-LS) alternates between a sparse recovery problem solved using the Simultaneous Orthogonal Matching Pursuit algorithm and a least squares problem. Experiments demonstrate the performance gain of the proposed method over the conventional sparsity approach and other existing off-grid DOA estimation algorithms.

Index Terms— Array signal processing, direction-of-arrival estimation, sparse representations, Taylor expansion.

1. INTRODUCTION

Source localization has been an active research field due to its fundamental role in many signal processing areas ranging from radar and sonar to acoustic tracking. In array signal processing, where arrays of sensors are typically employed for the sampling of the spatial field, the problem of source localization is usually referred to as direction-of-arrival (DOA) estimation.

The classical array processing methods can be divided into two main categories, the parametric methods which are based on the maximum likelihood paradigm and the spectral based approaches often referred to as non-parametric approaches [1]. The former result in accurate estimates at the price of high computational complexity. On the other hand, non-parametric methods are computationally attractive. Among them the subspace-based method of multiple signal classification (MUSIC) stands as a powerful technique to the problem of spectral analysis and system identification. However, MUSIC results in decreased performance when the incoming sources are correlated or coherent [1].

Recent advances in the field of sparse representations has brought renewed interest to the problem of source localization. The concept of spatial sparsity for DOA estimation was first introduced in [2], where it was shown that the source localization problem can be cast as a sparse recovery problem in a redundant dictionary using the ℓ_1-SVD method. Under certain assumptions ℓ_1-SVD can achieve super-resolution even in the coherent sources scenario.

However, the sparse representation framework assumes that the sources arrive from directions that belong into a predefined discrete set of possible angles. Therefore, if the unknown DOAs are not in this angular grid the performance of these methods will degrade due to errors caused by mismatches. On the other hand, there is a trade off between the number of sensors and the spatial resolution. If the spatial resolution is too high, then the resulting dictionary will be highly redundant with highly correlated entries. Therefore, the model will not meet the compressed sensing requirement of incoherence for robust recovery [3].

In this paper, we investigate the problem of off-grid DOA estimation and propose a fast iterative alternating descent algorithm improving the performance of the DOA estimation based on sparsity constraints.

2. BACKGROUND

2.1. DOA estimation with sparsity constraints

Consider a uniform linear array (ULA) of M sensors with inter-sensor spacing d. For simplicity without loss of generality, we assume that K plane waves propagating from the far-field impinge on the array from the unknown angles $\theta_1, \theta_2, \ldots, \theta_K$ that we wish to estimate. Assuming no multipath propagation and that the signals are narrowband with central frequency f_c, each sensor captures a superimposition of the incoming signals with time delays (phase differences) of τ_p, which are functions of the signals' DOAs θ_i.

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The linear array response to the impinging plane wave can be expressed as:

$$\mathbf{a}(\theta_i) = \left[e^{j\omega_c\tau_1(\theta_i)}, e^{j\omega_c\tau_2(\theta_i)}, \ldots, e^{j\omega_c\tau_M(\theta_i)}\right]^T. \quad (1)$$

Substituting $\omega_c = 2\pi f_c$ and $\tau_p(\theta_i) = (p - \frac{M+1}{2})d\cos(\theta_i)/c$, where d is the sensor spacing which is chosen at half the wavelength $d = \lambda/2$, $\lambda = c/f_c$ is the wavelength and c is the speed of the propagation, we obtain:

$$\mathbf{a}(\theta_i) = \left[e^{-j\pi\frac{M-1}{2}\cos(\theta_i)}, \ldots, e^{j\pi\frac{M-1}{2}\cos(\theta_i)}\right]^T. \quad (2)$$

Following the concept of spatial sparsity, we discretize the angular space into $N >> K$ possible angles of arrival and construct an overcomplete dictionary of N atoms, corresponding to the impulse responses of the array:

$$\mathbf{A} = \left[\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_N)\right]. \quad (3)$$

The sensor measurements can then be modelled as:

$$\mathbf{y}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{n}(t). \quad (4)$$

Subsequently, according to the Lasso method [4] one can attempt to solve the optimization problem:

$$\min_{\mathbf{x}} \|\mathbf{y}(t) - \mathbf{A}\mathbf{x}(t)\|_2^2 + \lambda\|\mathbf{x}(t)\|_1 \quad (5)$$

where $\mathbf{x}(t)$ is a $N \times 1$ vector containing K non-zero entries.

In the case of multiple time snapshots, assuming that the plane waves impinge on the array from fixed locations, the problem can be formulated as a multi-measurement vector (MMV) sparse recovery problem.

2.2. Off-grid DOA estimation methods

As discussed in Section 1, one limitation of the above model is that it assumes that the unknown directions fall into the predefined angular grid. Recently, H. Zhu et al. [5] addressed the problem of off-grid DOA estimation and developed a method to solve the total least squares problem with sparsity constraints. In this work, the authors propose the sparse regularized total least squares (SRTLS) algorithm in order to solve the problem:

$$\begin{aligned}\hat{\mathbf{x}}(t) = \min_{\mathbf{x},\mathbf{n},\mathbf{E}} \|\left[\mathbf{E}, \mathbf{n}(t)\right]\|_F^2 + \lambda\|\mathbf{x}(t)\|_1 \\ \text{s. t.} \quad \mathbf{y}(t) + \mathbf{n}(t) = (\mathbf{A} + \mathbf{E})\mathbf{x}(t).\end{aligned} \quad (6)$$

SRTLS in an iterative fashion alternates between estimates of $\mathbf{x}$ and $\mathbf{E}$. It first solves the Lasso problem for fixed $\mathbf{E}$ using the interior point solver SeDuMi and then by setting the derivative of (6) with respect to $\mathbf{E}$ to zero it updates the matrix $\mathbf{E}$ using the formula:

$$\mathbf{E} = [\mathbf{y}(t) - \mathbf{A}\mathbf{x}(t)]\mathbf{x}^T(t)[\mathbf{I} + \mathbf{x}(t)\mathbf{x}(t)]^{-1}. \quad (7)$$

The algorithm terminates when the difference between two consecutive iterations becomes smaller than a threshold.

Furthermore, the work in [6] also addresses the problem of off-grid DOA estimation from a Bayesian perspective. The proposed sparse bayesian inference (SBI) algorithm is an iterative algorithm, applicable in both cases of single and multiple snapshots by modelling the off-grid mismatches as a Taylor approximation problem. Other related work includes the continuous Basis Pursuit algorithm [7].

In the following Sections, inspired by the SRTLS and SBI algorithms, we propose the SOMP-LS algorithm which uses the Simultaneous Orthogonal Matching Pursuit (SOMP) algorithm [8] at the first stage and then updates the dictionary with a least squares (LS) inversion.

3. THE PROPOSED APPROACH

3.1. Problem Formulation

Suppose now that the i-th plane wave impinges on the array from the angle $\tilde{\theta}_i$ that is not contained in the selected angular grid, namely $\tilde{\theta}_i \notin \{\theta_1, \ldots, \theta_N\}$. In such a case, as described in [6], the corresponding vector $\mathbf{a}(\tilde{\theta}_i)$ for the off-grid DOA can be approximated by the first order Taylor expansion:

$$\mathbf{a}(\tilde{\theta}_i) = \mathbf{a}(\theta_i) + \mathbf{b}(\theta_i)(\tilde{\theta}_i - \theta_i) \quad (8)$$

where $\theta_i \in \{\theta_1, \ldots, \theta_N\}$ is the nearest angle of the grid and $\mathbf{b}(\theta_i)$ is the first derivative of $\mathbf{a}(\theta_i)$ with respect to θ_i:

$$\mathbf{b}(\theta_i) = -j\pi\sin(\theta_i)\mathbf{p} \circ \mathbf{a}(\theta_i) \quad (9)$$

where $\mathbf{p} = \left[-\frac{M-1}{2}, -\frac{M-3}{2}, \ldots, \frac{M-1}{2}\right]^T$ and $\circ$ denotes the element wise Hadamard product. It follows that we can define the redundant $M \times N$ matrix $\mathbf{B}$ with atoms $\mathbf{b}(\theta_i)$ for all N angles of the grid. The off-grid DOA model can then be formulated:

$$\mathbf{y}(t) = [\mathbf{A} + \mathbf{B}\Delta_\theta]\mathbf{x}(t) + \mathbf{n}(t) \quad (10)$$

with $\Delta_\theta = \text{diag}(\boldsymbol{\delta})$, $\boldsymbol{\delta} = [\delta_1, \ldots, \delta_N]^T$ and $\delta_i = \tilde{\theta}_i - \theta_i$. In the above system of equations (10) both $\boldsymbol{\delta}$ and $\mathbf{x}(t)$ are unknowns and therefore after taking multiple time snapshots we need to solve:

$$\min_{\mathbf{X},\boldsymbol{\delta}} \|\boldsymbol{\delta}\|_2^2 + \|\mathbf{Y} - [\mathbf{A} + \mathbf{B}\Delta_\theta]\mathbf{X}\|_F^2 + \lambda\|\mathbf{X}\|_{1,2}. \quad (11)$$

However, the problem of equation (11) is non-convex and therefore it cannot be tackled using convex optimization. Thus, in what follows we propose an alternating descent algorithm that iteratively shifts between estimates of $\mathbf{X}$ and $\boldsymbol{\delta}$ until the update of the specific matrices is no longer significant.

3.2. Proposed Algorithm

A sub-optimal way to solve the non-convex optimization problem in equation (11) is to reduce the problem to convex by minimizing over one parameter at a time. To do this, we can first look for a solution to the requlared least squares by keeping the unknown vector $\boldsymbol{\delta}$ fixed and solve for $\mathbf{X}$. Therefore, at the k-th iteration of the algorithm we need to solve the MMV sparse recovery problem:

$$\min_{\mathbf{X}^k} \|\mathbf{X}^k\|_{1,2} \quad \text{s.t.} \quad \|\mathbf{Y} - [\mathbf{A} + \mathbf{B}\Delta_\theta^{k-1}]\mathbf{X}^k\|_F \leq \epsilon. \quad (12)$$

The sparse MMV problem (12) can be solved by mixed ℓ_1/ℓ_2 minimization or alternatively by greedy approaches such as SOMP, which assume joint sparsity over the multiple vectors.

Once $\mathbf{X}$ has been updated, we minimize over $\boldsymbol{\delta}$ keeping the current estimate of $\mathbf{X}$ fixed. In this case, the problem of equation (11) reduces to:

$$\min_{\boldsymbol{\delta}^k} \|\boldsymbol{\delta}^k\|_2^2 + \|\mathbf{Y} - [\mathbf{A} + \mathbf{B}\Delta_\theta^k]\mathbf{X}^k\|_F^2. \quad (13)$$

The problem (13) can be proven to have a closed form solution [5]. However, instead of taking the derivative and solving for $\boldsymbol{\delta}^k$, we note that for a single snapshot the problem of (13) is equivalent to the least squares problem:

$$\mathbf{y}(t) - \mathbf{A}\mathbf{x}^k(t) = \mathbf{B}\Delta_{\mathbf{x}(t)}^k \boldsymbol{\delta}^k \quad (14)$$

where $\Delta_{\mathbf{x}(t)}^k = \text{diag}\big([x_1^k(t), \ldots, x_N^k(t)]\big)$.

Considering now T time snapshots, it is straightforward to vectorize the resulting T least squares problems:

$$\begin{bmatrix} \mathbf{y}(1) - \mathbf{A}\mathbf{x}^k(1) \\ \vdots \\ \mathbf{y}(T) - \mathbf{A}\mathbf{x}^k(T) \end{bmatrix} = \begin{bmatrix} \mathbf{B}\Delta_{\mathbf{x}(1)}^k \\ \vdots \\ \mathbf{B}\Delta_{\mathbf{x}(T)}^k \end{bmatrix} \boldsymbol{\delta}^k. \quad (15)$$

Therefore, at the k-th iteration the update to $\boldsymbol{\delta}$ will be:

$$\boldsymbol{\delta}^k = \mathbf{B}_{\Delta_{\mathbf{X}}^k}^\dagger \mathbf{R}^k \quad (16)$$

where $\mathbf{B}_{\Delta_{\mathbf{X}}^k} = [\mathbf{B}\Delta_{\mathbf{x}(1)}^k, \ldots, \mathbf{B}\Delta_{\mathbf{x}(T)}^k]^T$ and $\mathbf{R}^k = \mathbf{Y} - \mathbf{A}\mathbf{X}^k$.

The proposed algorithm assumes that the sparsity level, namely the number K of the impinging on the ULA sources is known a priori. After obtaining T time snapshots an optional pre-processing step follows to reduce the dimensionality of the MMV problem by applying singular value decomposition (SVD) to the $M \times T$ measurement matrix $\mathbf{Y}$. Therefore, by thresholding its singular values and obtaining the largest K among them, we form the $M \times K$ measurement matrix $\mathbf{Y}_{SV}$ reducing the dimensionality of the problem, in a similar manner to the ℓ_1-SVD algorithm.

The proposed alternating descent algorithm (Algorithm 1) is initialized with $\boldsymbol{\delta}^0 = \mathbf{0}_{N\times 1}$ and the K-term approximation to the problem (12) is obtained by running the SOMP algorithm for K iterations. Next, $\boldsymbol{\delta}$ is updated through equation (16). The algorithm iterates between these two steps and terminates when the difference between two consecutive updates of $\boldsymbol{\delta}$ falls below some chosen threshold. The final values of $\boldsymbol{\delta}$ provide an approximate estimate of the difference between the nearest $\theta_i \in \{\theta_1, \ldots, \theta_N\}$ and the true DOAs $\tilde{\theta}_i \notin \{\theta_1, \ldots, \theta_N\}$ for $i = 1, \ldots, K$.

Algorithm 1 SOMP-LS Alternating Descent algorithm

Input: $\mathbf{A}, \mathbf{B}, \mathbf{Y}, K, mIts$
Initialize: $k \leftarrow 0$, $\boldsymbol{\delta}^0 \leftarrow \mathbf{0}$,
while $k \leq mIts$ **do**
    $k \leftarrow k + 1$
    $\mathbf{X}^k \leftarrow \text{SOMP}([\mathbf{A} + \mathbf{B}\Delta_\theta^k], \mathbf{Y}, K)$
    $\mathbf{R}^k = \mathbf{Y} - \mathbf{A}\mathbf{X}^k$, $\mathbf{B}_{\Delta_{\mathbf{X}}^k} = [\mathbf{B}\Delta_{\mathbf{x}(1)}^k, \ldots, \mathbf{B}\Delta_{\mathbf{x}(T)}^k]^T$
    $\boldsymbol{\delta}^k = \mathbf{B}_{\Delta_{\mathbf{X}}^k}^\dagger \mathbf{R}^k$
    if $\|\boldsymbol{\delta}^k - \boldsymbol{\delta}^{k-1}\| \leq \epsilon$ **then** exit; **end if**
end while
Output: $\mathbf{X}, \boldsymbol{\delta}$

When compared to SRTLS, our proposed method replaces the Lasso solver at the regularization step with the greedy algorithm that allows for faster convergence especially in the case that multiple snapshots are considered. Therefore, the K-term approximation of SOMP provides faster convergence due to its algorithmic simplicity.

At the second step of dictionary update we exploit the interpolation dictionary $\mathbf{B}$ and estimate the vector $\boldsymbol{\delta}$ of size $N \times 1$ instead of the $M \times N$ matrix $\mathbf{E}$. As shown in the following section, experiments favour the updating rule of equation (16) instead of the SRTLS update of (7).

4. EXPERIMENTAL RESULTS

In this Section, we present experimental results for the evaluation of the proposed off-grid DOA alternating descent algorithm. The algorithm is compared against the SBI algorithm and the ℓ_1-SVD algorithm, which as already discussed assumes that K sources arrive from angles that exactly match K DOAs of the selected angular grid. For a fair comparison with the SRTLS approach, we also derived the SOMP-TLS algorithm, which under the same update rule with the proposed method for $\mathbf{X}$ replaces equation (16) with:

$$\boldsymbol{\delta}^k = \text{diag}\{\mathbf{B}^\dagger[\mathbf{Y} - \mathbf{A}\mathbf{X}^k](\mathbf{X}^k)^T[\mathbf{I} + \mathbf{X}^k\mathbf{X}^k]^{-1}\}. \quad (17)$$

In the following experiments, we considered a ULA of $M = 8$ sensors and the angular space $[0°, 180°]$ was uniformly discretized with resolution of $2°$, resulting in a grid of $N = 91$ potential angles of arrival $\{0°, 2°, \ldots, 180°\}$. Therefore, the redundant dictionaries $\mathbf{A}$ and $\mathbf{B}$ were of size 8×91.

4.1. Sources with off-grid DOAs

In the first experiment we considered two zero mean narrowband far-field sources with equal power levels arriving on the

Table 1. Elapsed times (sec) of tested algorithms.

	$T = 50$		$T = 200$	
Alg/SNR	10dB	-10dB	10dB	-10dB
SOMP-LS	0.0056	0.0076	0.0048	0.0061
SOMP-TLS	0.0072	0.0109	0.0080	0.0080
SBI-SVD	0.1596	0.6141	0.1467	0.4361
ℓ_1-SVD	0.4406	0.3152	0.4432	0.3404

Table 2. Elapsed times (sec) of tested algorithms.

	$T = 50$		$T = 200$	
Alg/SNR	10dB	-10dB	10dB	-10dB
SOMP-LS	0.0034	0.0078	0.0041	0.0057
SOMP-TLS	0.0058	0.0105	0.0038	0.0073
SBI-SVD	0.1564	0.6008	0.1425	0.4127
ℓ_1-SVD	0.4020	0.3354	0.4117	0.3283

ULA from directions 60.3° and 88.6° and therefore the sparsity level was set at $K = 2$. Initially, the number of time snapshots was $T = 200$ but the experiment was also repeated for fewer time samples by setting $T = 50$. For all tested algorithms, we assumed that the sparsity level K is known a priori and the dimensionality of the measurements was reduced using the SVD method and thresholding the largest K singular values corresponding to the signal subspace. The additive noise at the sensors was white gaussian and the noise level varied from -10 dB to 20 dB with a step size of 5 dB.

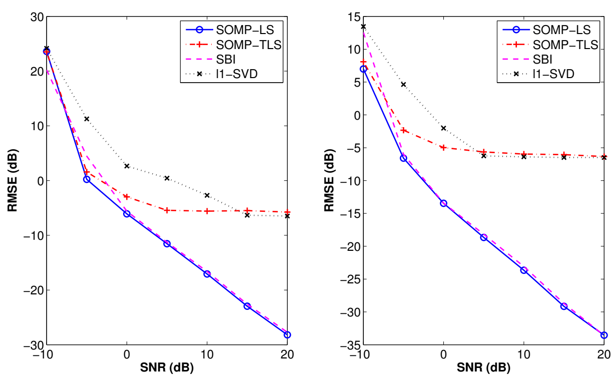

Fig. 1. Average RMSE of DOA estimation of $K = 2$ sources with off-grid directions (dB) vs SNR (dB) for $T = 50$ (left) and $T = 200$ (right) time snapshots.

Fig. 1 illustrates the average RMSE of the DOA estimation problem for all tested algorithms against the noise level for the two considered cases of $T = 50$ and $T = 200$ snapshots. The results have been averaged over 100 trials. As expected, ℓ_1-SVD showcases the worst performance with the largest error in most of the cases as the directions of the sources do not fall into the predefined angular grid. Among the off-grid DOA estimation algorithms, the proposed method achieves the best performance. It outperforms SOMP-TLS, showing that the update rule of equation (16) results in better approximations than the one in (17). SOMP-LS also performs slightly better than the SBI algorithm in both cases.

The average convergence time of each algorithm for two noise levels (10dB and -10dB) is shown in Table 1. In both cases examined ($T = 50$ and $T = 200$), the proposed approach was the fastest and when compared to SBI, SOMP-LS was at least 25 times faster. In most of the cases, SOMP-LS also achieved faster convergence than SOMP-TLS.

4.2. Sources with on-grid DOAs

For the second experiment, we kept the same setting as in the first experiment, but this time we assumed that the $K = 2$ sources arrive on the ULA from directions θ_i such that $\theta_i \in \{0°, 2°, \ldots, 180°\}$ for $i = 1, 2$. Subsequently, in the specific experiment we attempted to examine the error introduced by the off-grid DOA estimation, when the DOAs of the sources is a subset of the discrete angular grid.

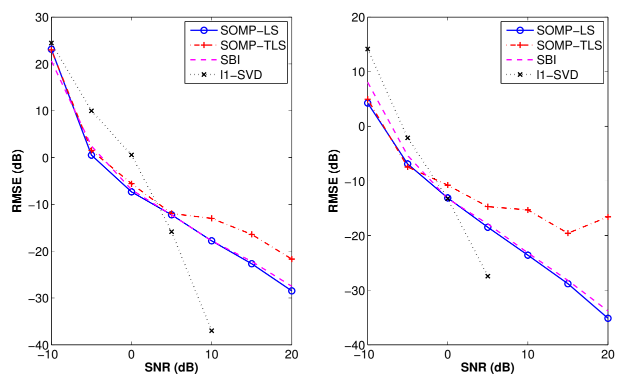

Fig. 2. Average RMSE of DOA estimation of $K = 2$ sources with on-grid directions (dB) vs SNR (dB) for $T = 50$ (left) and $T = 200$ (right) time snapshots.

The simulation average results over 100 iterations are summarized in Fig. 2. As can be seen, the off-grid DOA estimation algorithms perform better than ℓ_1-SVD for high noise levels below 0 dB, but when the SNR is above 0 dB ℓ_1-SVD achieves the best estimation of the unknown parameters. However, even in this scenario the performance of the off-grid algorithms are acceptable as the average RMSE is less than -10 dB. Among the off-grid DOA estimation methods SOMP-LS and SBI showcase better performance than SOMP-TLS.

Table 2 summarizes the average convergence times for all algorithms. Once again, the proposed method achieves the fastest overall convergence.

4.3. Off-grid DOA estimation with correlated sources

Finally, we carried out an experiment to evaluate the algorithms in the scenario when the impinging sources are highly correlated. To do this, we kept the same setting as in the first experiment, considering $K = 2$ correlated sources with directions $60.3°$ and $88.6°$.

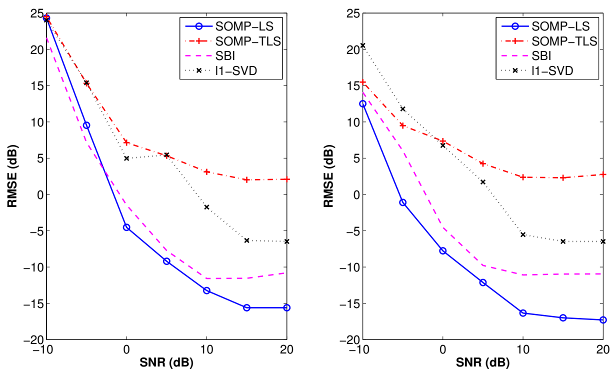

Fig. 3. Average RMSE of DOA estimation of $K = 2$ correlated sources with off-grid directions (dB) vs SNR (dB) for $T = 50$ (left) and $T = 200$ (right) time snapshots.

While ℓ_1-SVD performance remains almost intact, the performance of the off-grid DOA estimation methods deteriorate with SOMP-TLS algorithm resulting in poor performance, as shown in Fig. 3. However, SOMP-LS is the least affected algorithm by the correlated sources, outperforming all other methods in the specific scenario. This performance gain also seems to increase as the number of the available time snapshots becomes larger ($T = 200$).

5. CONCLUSIONS

We have presented an alternating descent algorithm for the problem of off-grid DOA estimation under the sparse representations framework. In the typical DOA estimation approach based on the sparsity assumption, the angular space is discretized forming a grid of potential DOAs for the incoming sources. According to this model, the obtained sensor measurements are decomposed in a redundant dictionary, which contains the impulse responses between the ULA and all directions of the discrete set. However, the unknown directions are continuous and therefore mismatches might occur.

In the proposed framework, we formulated the problem of off-grid DOA estimation using the Taylor expansion to approximate the true DOAs reducing the mismatch errors of the standard approach. The proposed method, is an alternating descent algorithm that at the first stage attempts to identify the nearest directions of the sources to the ones included in the initial grid using the SOMP algorithm. It then updates the dictionary and the corresponding angular grid. The process is repeated until the overall error is no longer reduced.

Experimental results has proven that SOMP-LS[1] can overcome the resolution limitations of the standard sparsity model. When compared to other off-grid DOA methods such as the SBI algorithm, it also achieved slightly better performance, while in the coherent sources scenario the proposed algorithm outperformed all other tested methods. The simplicity of the greedy algorithm at the first stage combined with a single least squares inversion at the second stage of the algorithm resulted in the fastest convergence among the compared algorithms.

In future work, we will attempt to investigate the algorithm's performance for close spaced sources and consider alternative sparse coding schemes.

6. REFERENCES

[1] H. Krim and M. Viberg, "Two decades of array signal processing research," *IEEE Signal processing magazine*, vol. 13, no. 4, pp. 67–94, 1996.

[2] D. Malioutov, M. Cetin, and A.S. Willsky, "A sparse signal reconstruction perspective for source localization with sensor arrays," *IEEE Transactions on Signal Processing*, vol. 53, no. 8, pp. 3010–3022, 2005.

[3] E. J. Candes and M. B. Wakin, "An introduction to compressive sampling," *IEEE Signal Processing Magazine*, vol. 25, no. 2, pp. 21–30, 2008.

[4] R. Tibshirani, "Regression shrinkage and selection via the lasso," *Journal of the Royal Statistical Society. Series B (Methodological)*, pp. 267–288, 1996.

[5] H. Zhu, G. Leus, and G. B. Giannakis, "Sparse regularized total least squares for sensing applications," in *Proc. IEEE International Workshop on Signal Processing Advances in Wireless Communications*, 2010, pp. 1–5.

[6] L. Yang, Z. Xie and C. Zhang, "Off-grid direction of arrival estimation using sparse bayesian inference," *Arxiv preprint, available online at http://arxiv.org/abs/1108.5838*, 2011.

[7] C. Ekanadham, D. Tranchina, and E. P. Simoncelli, "Sparse decomposition of transformation-invariant signals with continuous basis pursuit," in *Proc. IEEE International Conference on Acoustics, Speech and Signal Processing*, 2011, pp. 4060–4063.

[8] J.A. Tropp, A.C. Gilbert, and M.J. Strauss, "Simultaneous sparse approximation via greedy pursuit," in *Proc. IEEE International Conference on Acoustics, Speech, and Signal Processing*, 2005, vol. 5, pp. v/721–v/724.

[1] For MATLAB code see: https://code.soundsoftware.ac.uk/hg/doa-ad/