UNCERTAINTY PRINCIPLES, MINIMUM UNCERTAINTY SAMPLINGS AND TRANSLATIONS

Daniel Lantzberg, Florian Lieb, Hans-Georg Stark

Aschaffenburg Univ. of Applied Sciences
Faculty of Engineering, Würzburger Str. 45
63743 Aschaffenburg, Germany
Email: hans-georg.stark@h-ab.de

Ron Levie, Nir Sochen

School of Mathematical Sciences
Tel Aviv Univ., Tel Aviv 69978 Israel
Email: sochen@math.tau.ac.il

ABSTRACT

It has been shown recently, that the conventional variance based uncertainty measure associated with the wavelet transform can be arbitrarily small. Hence, no global minimizer exists.

In this paper we introduce a new discretization scheme in scale and time shifts, such that the total uncertainty of a corresponding function system has the lowest possible value. We also describe a generalized uncertainty principle inspired by the familiar uncertainty principle in time-frequency analysis. As an example we apply this concept to wavelet analysis, leading to a new affine uncertainty principle. We also introduce waveforms minimizing this principle.

Furthermore, we remark that the uncertainty measure associated with this new principle allows for decay estimates of the ambiguity function (reproducing kernel) associated with the wavelet transform.

Index Terms— Uncertainty principle, harmonic analysis.

1. INTRODUCTION

Throughout this paper we consider signals $f(t)$ as elements of the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ with $\langle \cdot, \cdot \rangle$ denoting the inner product.

On this space a unitary square integrable representation $\pi(g)$ $(g \in G)$ of a locally compact Lie group G is given such that the corresponding signal transform reads $f(t) \mapsto \left\langle \pi(g)\tilde{f}, f \right\rangle$ for a suitable $\tilde{f}(t)$. For a comprehensive review refer to [1].

Denoting with $g(\alpha_1, \alpha_2, \ldots)$ $(\alpha_k \in \mathbb{R})$ the dependency of the group elements on the real Lie-parameters we write for the corresponding one parameter subgroup representations $\pi(\alpha_k)$ $(k = 1, 2, \ldots)$. By Stone's theorem [2] these representation operators are linked to the corresponding self adjoint generators T_k via

$$\pi(\alpha_k) = e^{i\alpha_k T_k}. \tag{1}$$

This research has been (partially) supported by EU FET Open grant UNLocX (255931)

In this case the following inequality (the "general uncertainty principle") has a precise meaning [3] [4]:

$$v_f(T_1)v_f(T_2) \geq \frac{1}{4} \left|\langle f, [T_1, T_2]f\rangle\right|^2 \tag{2}$$

Here, given a self-adjoint operator T, we used the following notation for expectation value $e_f(T)$ and variance $v_f(T)$, respectively:

$$e_f(T) = \langle f, Tf\rangle \tag{3}$$

$$v_f(T) = \left\langle f, (T - e_f(T))^2 f \right\rangle. \tag{4}$$

The tuple $(e_f(T_1), v_f(T_1), e_f(T_2), v_f(T_2))$ can be visualized by the dashed "uncertainty rectangle" in the uncertainty plane as depicted in Figure 1.

Note that for simplicity throughout this paper we restrict ourselves to the parameters α_1, α_2 and the corresponding generators T_1, T_2, respectively. It is clear that (2) will be valid when picking any T_i and T_k from the set of generators.

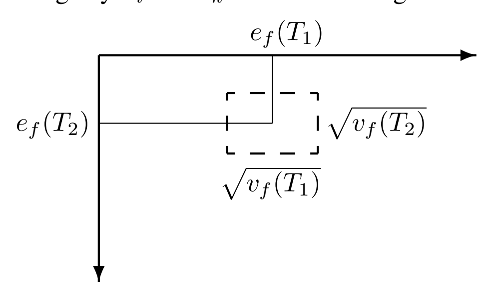

Fig. 1. Uncertainty rectangle

It has been shown recently [5] [6] that under certain circumstances

$$\inf_f v_f(T_1)v_f(T_2) = 0, \tag{5}$$

even if the lower bound $\frac{1}{4}\left|\langle f, [T_1, T_2]f\rangle\right|^2$ of inequality (2) is non-zero.

Thus in these cases the area of the uncertainty rectangle depicted in Fig. 1 will be arbitrarily small; the affine group associated with the wavelet transform is an example.

This observation led to two concepts, which are described here. In section 2 we review the results from [6] and introduce mimimum uncertainty samplings allowing for generating function systems with lowest possible uncertainty under some given restrictions. In section 3 we introduce translation based, or, equivalently, adjoint translation uncertainty inequalities as an in our opinion proper localization concept. In section 3.2 we explore the link of this localization concept to concentration properties of reproducing kernels, which - generalizing corresponding ideas from time-frequency analysis [7] - we also call ambiguity functions.

We quote here only results, more details on mimum uncertainty samplings are described in [8], for adjoint translation uncertainty principles refer to [9].

2. VARIANCE-BASED UNCERTAINTY PRINCIPLES AND MINIMUM UNCERTAINTY SAMPLINGS

In this section we briefly recall the main results from [5] and [6] and sketch the idea of minimum uncertainty samplings.

Given the generators T_1 and T_2 introduced above, in [6] transform laws have been developed, describing how the tuple $(e_f(T_1), v_f(T_1), e_f(T_2), v_f(T_2))$ changes, if f is replaced by $\pi(g)f$. Graphically this may be illustrated by shifts and dilations of the uncertainty rectangle in Fig. 1.

In time-frequency analysis the transform group G is given by the Weyl-Heisenberg group and the generators $T_1 = i\frac{d}{dt}$ and $T_2 = t$ are associated with time- and frequency shifts, respectively. In this case $[T_1, T_2] = i\mathbf{1}$ and inequality (2) reads

$$v_f(T_1)v_f(T_2) \geq \frac{1}{4}, \tag{6}$$

with equality iff f is a Gaussian. Here $v_f(T_1)$ usually is evaluated in the Fourier domain, in this form the above inequality has been introduced to signal processing already in [10].

Starting from f we introduce the function system

$$\{e^{ik\Delta\omega t} f(t - m\Delta b)\}\ (k, m \in \mathbb{Z}).$$

This system is generated by a rectangular lattice in time- and frequency-shifts, respectively. The corresponding uncertainty rectangles are size invariant with respect to the group action and shifted in accordance with the lattice constants leading to a tesselation of the uncertainty plane as shown in Figure 2.

In wavelet analysis the transform group G is the affine group. The representation reads

$$\pi(\alpha, b)f(t) = e^{-\frac{\alpha}{2}} f\left(\frac{t-b}{e^{\alpha}}\right), \tag{7}$$

and the generators of pure translations and dilations, respectively, read $T_1 = i\frac{d}{dt}$ (as above) and $T_2 = i\left(\frac{1}{2} + t\frac{d}{dt}\right)$. They satisfy $[T_1, T_2] = iT_1$ leading to the inequality

$$v_f(T_1)v_f(T_2) \geq \frac{1}{4}\left|\langle f, T_1 f\rangle\right|^2 \tag{8}$$

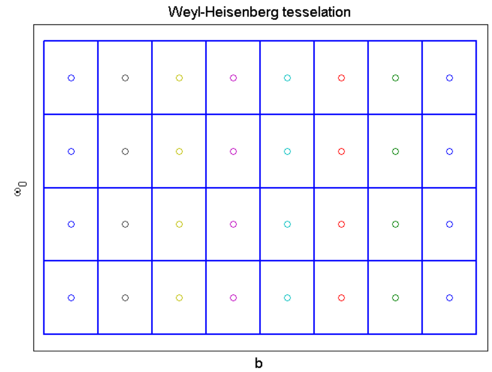

Fig. 2. Time-frequency-tesselation

In [5] [6] it was shown that for pure dilations of f we have $v_{\pi(\alpha,0)f}(T_1) = v_f(T_1)e^{-2\alpha}$ and $v_{\pi(\alpha,0)f}(T_2) = v_f(T_2)$, respectively. Thus obviously

$$\lim_{\alpha \longrightarrow \infty} v_{\pi(\alpha,0)f}(T_1)v_{\pi(\alpha,0)f}(T_2) = 0.$$

Therefore indeed in this case $\inf_f v_f(T_1)v_f(T_2) = 0$ even though the lower bound of uncertainty inequality (8) is nonzero.

Even though in the wavelet case there exists no minimizer of $v_f(T_1)v_f(T_2)$ we can introduce finite samplings $\{(\alpha_m, b_m)\ (m \in M)\}$ with $|M| < \infty$ such that, starting from a function f, the total uncertainty of the function system $\{\pi(\alpha_m, b_m)f\ (m \in M)\}$ is as small as possible under certain constraints [8]. The construction is described in the following.

First we introduce a sampling

$$\{(\alpha_k, b_k^n)\}_{k \in K} \tag{9}$$

with $|K| < \infty$ and $b_k^n = l_k + n \cdot \Delta b_k\ (n = 0, \ldots, N_k - 1)$. Thus on each scale level α_k there is an equally spaced b-sampling with a k-dependent sampling rate Δb_k, an offset l_k and N_k sampling points.

Then assume that $\{(\alpha_k, \Delta b_k, N_k)\}$ are kept fixed. A minimum uncertainty sampling is now characterized by a proper choice of the offsets l_k such that the total uncertainty

$$u_{tot} = \sum_{k,n} v_{\pi(\alpha_k, b_k^n)f}(T_1)v_{\pi(\alpha_k, b_k^n)f}(T_2) \tag{10}$$

is as small as possible. We now briefly sketch the computation procedure for these offsets, for more details refer to [8]. First, we introduce the scale factors $a = e^{\alpha}$ familiar in wavelet analysis [1] and illustrate the procedure in the b-a-plane rather than in the b-α-plane:

- Given f, first compute the "vertex ray" $b_{vertex} = a\gamma_f$. This ray is a straight line through the origin in the b-a-plane with an f-specific slope γ_f.

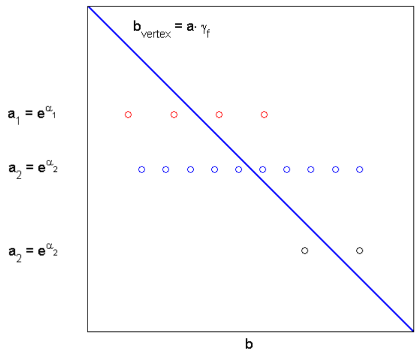

Fig. 3. Minimum uncertainty sampling

- Then choose l_k such, that for each k the points b_k^n are symmetrically balanced around the "vertex ray" as depicted in Figure 3. Indeed it can be shown [8] that then u_{tot} as defined in eq. (10) achieves the smallest possible value.

For more details, the computation procedure of γ_f, numerical experiments verifying the vertex computation and generalizations of this technique to other transform groups apart from the one dimensional affine group sketched above, refer to [8].

3. ADJOINT TRANSLATION UNCERTAINTY PRINCIPLES

3.1. General procedure and affine group

In this section we return to the general Lie-group representation introduced in Sect. 1 and in particular consider the one parameter subgroup representations together with the associated generators related by eq. (1). As mentioned in the introduction we restrict ourselves to $k = 1, 2$, i.e., we consider

$$\pi(\alpha_k) = e^{i\alpha_k T_k} \ (k = 1, 2).$$

The localization and uncertainty concept based on the tuple $(e_f(T_1), v_f(T_1), e_f(T_2), v_f(T_2))$, introduced in Sect. 1, applied to the Weyl-Heisenberg group corresponds to intuition for the following reason: Replacing $f(t)$ with $\pi(b)f(t) = f(t-b)$ (time shift) or $\pi(\omega_0)f(t) = e^{i\omega_0 t}f(t)$ (frequency shift), leads to a corresponding combined shift of the uncertainty rectangles as depicted in Fig. 2. Thus there is a bijective map of the "physical" time-frequency-plane [7] to the localization measure given by the above tuple.

This is no longer true for the affine group, for example [6]. In fact, it is possible there to find signals $f(t)$ such that the uncertainty rectangle remains at the same position, when $f(t)$ is dilated corresponding to $f(t) \mapsto \pi(\alpha)f(t) = e^{-\alpha/2}f(e^{-\alpha}t)$. Thus there is no simple bijection of the "physical" time-scale-plane of wavelet analysis [1] to the transform properties of the above localization tuple.

In general, for one parameter subgroup representations $\pi(\alpha_k) = e^{i\alpha_k T_k}$ of *any* transform group we look for a localization concept, which reflects the fact that all one parameter subgroup representations are representations of the one dimensional *translation* group, since

$$\pi(\alpha_k + \tilde{\alpha}_k) = \pi(\alpha_k)\pi(\tilde{\alpha}_k).$$

We sketch briefly the main results and refer to [9] for a more elaborate and general discussion.

First we note that, given $\pi(\alpha_k) = e^{i\alpha_k T_k}$, under mild restrictions on the spectrum of T_k there exist unitary transforms $\mathcal{D}_k : f(t) \mapsto \mathcal{D}_k f(\tau_k)$ such that the transformed subgroup representations act as translations:

Theorem 3.1. *Let the spectrum of T_k $(k = 1, 2)$ be continuous, non-degenerate and covering $\mathbb{R}$. Then unitary operators $\mathcal{D}_k : f(t) \mapsto \mathcal{D}_k f(\tau_k)$ exist such that the transformed generators act as derivatives. This is illustrated in the following diagram:*

$$\begin{tikzcd} f(t) \arrow[r, "T_k"] \arrow[d, "\mathcal{D}_k"] & T_k f(t) \arrow[d, "\mathcal{D}_k"] \\ \mathcal{D}_k f(\tau_k) \arrow[r, "\mathcal{D}_k T_k \mathcal{D}_k^*"'] & i\frac{d\mathcal{D}_k f}{d\tau_k}(\tau_k) \end{tikzcd}$$

Choose f such that $\mathcal{D}_k f(\tau_k)$ is smooth and compactly supported. Then the operators $\mathcal{D}_k \pi(\alpha_k)\mathcal{D}_k^* = \mathcal{D}_k e^{i\alpha_k T_k}\mathcal{D}_k^*$ act as translations:

$$\begin{tikzcd} f(t) \arrow[r, "\pi(\alpha_k)"] \arrow[d, "\mathcal{D}_k"] & \pi(\alpha_k) f(t) \arrow[d, "\mathcal{D}_k"] \\ \mathcal{D}_k f(\tau_k) \arrow[r, "\mathcal{D}_k \pi(\alpha_k) \mathcal{D}_k^*"'] & \mathcal{D}_k f(\tau_k - \alpha_k) \end{tikzcd}$$

Proof. Applying the spectral theorem [2], choose unitary $\mathcal{M}_k$ such that $\mathcal{M}_k T_k f(t) = m_k \mathcal{M}_k f(m_k)$. According to the above assumptions on the spectrum m_k covers $\mathbb{R}$. Multiplication is converted to differentiation by applying the Fourier transform $\mathcal{F}$. Therefore we may construct $\mathcal{D}_k$ as follows:

$$\mathcal{D}_k = \mathcal{F}\mathcal{M}_k \tag{11}$$

Then obviously $\mathcal{D}_k T_k f(\tau_k) = i\frac{d\mathcal{D}_k f}{d\tau_k}(\tau_k)$.

The second part of the theorem follows from expanding $e^{i\alpha_k \mathcal{D}_k T_k \mathcal{D}_k^*}$. $\square$

A simple and well-known example is modulation (frequency shift), which is converted into translation by the Fourier transform:

$$\begin{array}{ccc} f(t) & \xrightarrow{\quad \pi(\omega_0) \quad} & e^{i\omega_0 t} f(t) \\ \Big\downarrow \mathcal{F} & & \Big\downarrow \mathcal{F} \\ \hat{f}(\omega) & \xrightarrow[\mathcal{F}\pi(\omega_0)\mathcal{F}^*]{} & \hat{f}(\omega - \omega_0) \end{array}$$

The transforms associated with $\mathcal{D}_k$ $(k = 1, 2)$ we call *adjoint translation representations* associated with T_k. As localization / uncertainty measures we now propose the tuple

$$(e_f(\breve{T}_1), v_f(\breve{T}_1), e_f(\breve{T}_2), v_f(\breve{T}_2)),$$

where the *adjoint localization operators* $\breve{T}_k$ $(k = 1, 2)$ act as *multiplication* operators on the corresponding transforms:

$$\mathcal{D}_k \breve{T}_k f(\tau_k) = \tau_k \mathcal{D}_k f(\tau_k). \tag{12}$$

Again, $v_f(\breve{T}_1), v_f(\breve{T}_2)$ will satisfy the uncertainty inequality

$$v_f(\breve{T}_1) v_f(\breve{T}_2) \geq \frac{1}{4} \left| \left\langle f, [\breve{T}_1, \breve{T}_2] f \right\rangle \right|^2 \tag{13}$$

and the general approach for constructing corresponding minimizers and / or equalizers reads:

- Given T_k compute $\mathcal{D}_k T_k \mathcal{D}_k^* = i \frac{\partial}{\partial \tau_k}$.
- Compute $\breve{T}_k = \mathcal{D}_k^* \tau_k \mathcal{D}_k$.
- Compute equalizers and/or minimizers of (13). Here an *equalizer* f satisfies $v_f(\breve{T}_1) v_f(\breve{T}_2) = \frac{1}{4} \left| \left\langle f, [\breve{T}_1, \breve{T}_2] f \right\rangle \right|^2$, whereas for a *minimizer* f the variance product $v_f(\breve{T}_1) v_f(\breve{T}_2)$ is as small as possible.

The procedure outlined above can be applied to wavelet transforms, i.e., the affine group with generators $T_1 = i \frac{d}{dt}$ and $T_2 = i \left(\frac{1}{2} + t \frac{d}{dt} \right)$, respectively, satisfying $[T_1, T_2] = i T_1$. Again, we only sketch results, for details and proofs refer to [9].

Starting from $f(t)$ we consider the scale time sampling $(\alpha_k, b_k^m) = (k \cdot \Delta\alpha, m \cdot \Delta b \cdot e^{k \cdot \Delta\alpha})$ and introduce the corresponding function system

$$\{\pi(\alpha_k, b_k^m) f\} \ (k, m \in \mathbb{Z})$$

Visualizing the tuple $(e_f(\breve{T}_1), v_f(\breve{T}_1), e_f(\breve{T}_2), v_f(\breve{T}_2))$ again by uncertainty rectangles, the resulting tesselation then is shown in Fig. 4. In fact we observe now a strong resemblance to the corresponding situation in time-frequency analysis, as illustrated in Fig. 2. The only difference is that the step width of time shifts is scaled on each α_k-level.

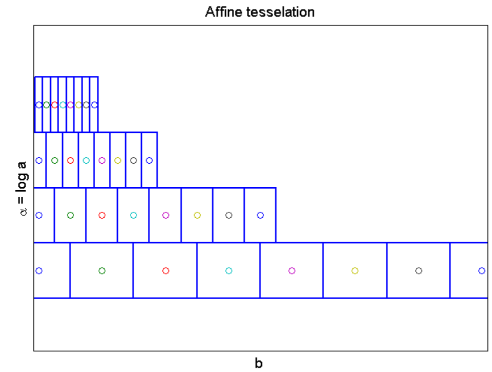

Fig. 4. Scale-time-tesselation

The corresponding uncertainty inequality for *analytic* signals $f(t)$ (i.e., signals $f(t)$ such that $supp(\hat{f}) \subseteq \mathbb{R}^+$) now reads [9]

$$v_f(\breve{T}_1) v_f(\breve{T}_2) \geq \frac{1}{4} \left| C_f \right|^2, \tag{14}$$

with

$$C_f = \int_0^\infty \frac{\left| \hat{f}(\omega) \right|^2}{\omega} d\omega.$$

For completeness we mention that - with different notation - this inequality also appears in [11], without reference to the adjoint translation idea presented here. Note that the admissibility condition of wavelet theory [1] reads $C_f < \infty$.

In [9], equalizers of (14), i.e., functions $f(t)$ satisfying

$$v_f(\breve{T}_1) v_f(\breve{T}_2) = \frac{1}{4} \left| C_f \right|^2$$

are computed. They are given in the Fourier domain as

$$\hat{f}(\omega) = C e^{\omega(i\alpha + \mu(\beta + 1 - \log \omega))}, \tag{15}$$

where $\alpha, \beta \in \mathbb{R}$, $\mu > 0$ and C is a normalization constant. Note that $C \to 0$ for $\mu \to \infty$. Therefore the above equality is valid only for $\mu \to \infty$ since then $\hat{f}(0) = C \to 0$, which implies $C_f < \infty$.

Moreover, in [9] a function sequence $\{f_n\}_{n \in \mathbb{N}}$ is constructed such that the following holds:

$$\lim_{n \to \infty} (v_{f_n}(\breve{T}_1) v_{f_n}(\breve{T}_2)) = \lim_{n \to \infty} (v_{f_n}(\breve{T}_1) + v_{f_n}(\breve{T}_2)) = 0.$$

Thus, asymptotically we obtain a minimizer of the variance product such that area and perimeter of the uncertainty rectangle tend to zero. The construction works in the Fourier domain as follows:

Introduce the bump function

$$\hat{f}(\omega) = \begin{cases} 0.5\omega^2 & 0 \leq \omega < 1 \\ 0.5(-3+6\omega-2\omega^2) & 1 \leq \omega < 2 \\ 0.5(3-\omega)^2 & 2 \leq \omega \leq 3 \\ 0 & else \end{cases}$$

Then choose $\kappa(n)$ ($n \in \mathbb{N}$) such that $n = o(\kappa(n))$, an example is $\kappa(n) = n^2$. Finally define f_n by

$$\hat{f}_n(\omega) = \frac{1}{\sqrt{n}}\hat{f}\left(\frac{\omega - \kappa(n)}{n}\right)$$

and normalize each f_n such that $\|f_n\| = 1$.

3.2. Adjoint Translation Uncertainty Principle and Ambiguity Function

In time-frequency analysis the concept of ambiguity functions has emerged as an alternative notion of locality [7]. Given a signal $f(t)$, the ambiguity function matches $f(t)$ with copies shifted in time and frequency:

$$\mathcal{A}(\omega_0, b) = \langle f, \pi(\omega_0, b)f \rangle ,$$

where $\pi(\omega_0, b)f(t) = e^{i\omega_0 t}f(t-b)$. For certain applications one is interested in signals $f(t)$ giving rise to very peaky ambiguity functions, i.e., the decay properties of $|\mathcal{A}(\omega_0, b)|$ are of considerable interest.

In general terms, given an arbitrary transform group G the ambiguity function may be written as $\mathcal{K}(g) = \langle f, \pi(g)f\rangle$; we use the symbol $\mathcal{K}$ instead of $\mathcal{A}$, because the ambiguity function is related to the reproducing kernel of this representation [1]. The concept of adjoint translation uncertainty principles introduced above in Sect. 3 allow for decay estimates of $|\mathcal{K}(g)|$ in terms of $v_f(\breve{T}_k)$ $(k = 1, 2)$ [9].

As an example we again consider the affine group / wavelet transform with $T_1 = i\frac{d}{dt}$ and $T_2 = i\left(\frac{1}{2} + t\frac{d}{dt}\right)$. Then an example of such an estimate reads

$$|\langle f, \pi(\alpha, 0)f\rangle| \leq \min\left(1, 4\frac{\sqrt{v_f(\breve{T}_2)}}{\alpha} + 4\frac{v_f(\breve{T}_2)}{\alpha^2}\right).$$

Again, the reader is referred to [9] for more general statements and proofs.

4. CONCLUSION

Starting from ”conventional” variance based uncertainty inequalities of the form (2) we introduced the concept of minimum uncertainty samplings in order to produce function systems with lowest possible uncertainty obtained by a suitable discretization of the underlying transform group.

Adjoint translation uncertainty principles introduced in Sect. 3 seem to be a appropriate tool to overcome certain shortcomings of ”traditional” variance based uncertainty principles [12] [13] [6]. The resulting equalizing or minimizing waveforms (we have shown first results for wavelet transforms) shall be fed into applications in the course of the FET Open project UNLocX.

5. REFERENCES

[1] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, 1992.

[2] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Vol. 1: Functional Analysis*, Academic Press, 1980.

[3] K. Kraus, “A further remark on uncertainty relations,” *Zeitschrift fuer Physik*, vol. 201, pp. 134 – 141, 1967.

[4] J.G. Christensen, “The uncertainty principle for operators determined by lie groups,” *Journal of Fourier Analysis and Applications (JFAA)*, vol. 10, no. 5, pp. 541 – 544, 2004.

[5] P. Maass C. Sagiv N. Sochen and H.-G. Stark, “Do uncertainty minimizers attain minimal uncertainty?,” *Journal of Fourier Analysis and Applications (JFAA)*, vol. 16, no. 3, pp. 448 – 469, 2010.

[6] H.-G. Stark and N. Sochen, “Square integrable group representations and the uncertainty principle,” *Journal of Fourier Analysis and Applications (JFAA)*, vol. 17, pp. 916 – 931, 2011.

[7] K. Groechenig, *Foundations of Time-Frequency Analysis*, Birkhaeuser, 2001.

[8] H.-G. Stark D. Lantzberg and F. Lieb, “Variance based uncertainty principles and minimum uncertainty samplings,” Preprint Aschaffenburg UAS, February 2012.

[9] D. Lantzberg R. Levie F. Lieb N. Sochen and H.-G. Stark, “Adjoint translations and the uncertainty principle,” in preparation.

[10] D. Gabor, “Theory of communication,” *J. IEEE*, vol. 93, pp. 429 – 459, 1946.

[11] P. Flandrin, “Inequalities in mellin-fourier signal analysis,” in *Wavelet Transforms and Time-Frequency Signal Analysis*, L. Debnath, Ed., pp. 289 – 319. Birkhäuser, 2001.

[12] S. Dahlke and P. Maass, “The affine uncertainty principle in one and two dimensions,” *Comput. Math. Appl.*, vol. 30, no. 3-6, pp. 293 – 305, 1995.

[13] C. Sagiv N.A. Sochen and Y.Y. Zeevi, “The uncertainty principle: Group theoretic approach, possible minimizers and scale space properties,” *J. Math. Imaging Vision*, vol. 26, no. 1 – 2, pp. 149 – 166, 2006.