A CLT FOR THE G-MUSIC DOA ESTIMATOR

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ABSTRACT

Subspace methods (e.g. MUSIC) are widely used in the context of DoA estimation using an array of M antennas. These methods perform well as long as the number of available samples N is much larger than M. However, their performance severely degrades when N is of the same order of magnitude than M. In this context, a DoA estimation method (called "G-MUSIC"), based on a new localization function estimate, was recently derived and shown to outperform the traditional methods for reasonable values of M, N. The consistency of both the localization function and DoA estimators was addressed in the asymptotic regime where M, N converge to infinity such that M/N converges to a positive constant. This paper addresses Central Limit Theorems for both the localization function and DoA estimators, in the previous asymptotic regime. Simulations confirm the validity of the results.

***Index Terms*—** DoA, MUSIC, CLT, Random Matrix Theory.

1. INTRODUCTION

The problem of estimating the Direction of Arrival (DoA) of K source signals from a set of N noisy observations collected by an array of M sensors has been widely studied in the past, and several algorithms have been proposed, among which the most popular are the so-called subspace methods, which are generally prefered over the Maximum Likelihood methods, because they offer a good trade-off between performances and computation costs. The traditional subspace estimation methods, e.g. the MUSIC method (Schmidt [1]), have been widely studied in the literature (see Stoica [2]), and the performances mainly characterized, in terms of consistency and asymptotic Gaussianity, in the asymptotic regime where the number of samples N converges to infinity while the number of antennas M remains constant. In practice, these traditional estimators are used in the context where $N >> M$, and perform well in this case. However, there exists several situations where using such an amount of samples is not conceivable, for example when the signals are stationnary only for a short period of time, or simply if the number of antennas is large. In the context where N is of the same order of magnitude than M, the performances of the traditional estimators severely degrade, mainly because they depend on the sample covariance matrix of the observations, which does not estimate properly the true covariance matrix of the observations in this context. Recently, based on random matrix theory results, a new subspace DoA estimation method ("G-MUSIC") was proposed, in the context of Gaussian temporally uncorrelated signals (see Mestre & Lagunas [3]), and later generalized to the case of deterministic unknown signals (see Vallet et al. [4]). In practice, this estimator outperforms the traditional ones, for realistic values of M, N. The G-MUSIC DoA estimator is based on a new estimator of the localization function, consistent in the asymptotic regime where $M, N \to \infty$ in such a way that $\frac{M}{N} \to c > 0$. The consistency of the DoA estimates, not addressed in [4], was later proved in Hachem et al. [5]. Recently, Mestre et al. [6] studied the asymptotic fluctuations of the G-MUSIC localization function estimator, and derived an expression of the Mean Square Error (MSE), in terms of a line integral which can be evaluated numerically. This work considered the estimator in [3], based on the constraint that the source signals are temporally uncorrelated. In this paper, we follow the same approach than [6] and analyze the asymptotic fluctuations of both the localization function and DoA estimators of [4], and also propose an expression of the MSE in terms of a line integral. Moreover, we provide an explicit approximation of this integral, in terms of eigenvalues and eigenvectors of the covariance matrix of the observations, which is accurate when the number of sources K is such that $K << N$. Some numerical examples are provided, which validate the results obtained.

The paper is organized as follows. In section 2, we introduce the model of signals used throughout the paper, and recall the problem of subspace DoA estimation. In section 3, we recall the main results concerning the G-MUSIC estimator. In section 4, we provide Central Limits Theorems (CLT) for both the G-MUSIC localization and DoA estimators, and give explicit approximations of the MSE. Finally, some numerical results are provided in section 5.

This work was partially supported by the French agency Direction Générale de l'Armement (DGA) and by the Catalan Government (grant 2009SGR1046).

2. MODEL OF SIGNALS AND THE MUSIC METHOD

We assume that K narrow-band source signals are received by an array of M sensors, with $K < M$. At time n, the received signal writes

$$\mathbf{y}_n = \mathbf{A}\mathbf{s}_n + \mathbf{v}_n, \tag{1}$$

where $\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K)]$ is the matrix of the steering vectors $\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K)$ associated with the K sources with DoA $\theta_1, \ldots, \theta_K$. The vector $\mathbf{s}_n$ represents the transmitted signals, and is assumed deterministic non-observable, and $\mathbf{v}_n \sim \mathcal{N}_{\mathbb{C}^M}\left(\mathbf{0}, \sigma^2\mathbf{I}_M\right)$ represents the additive white Gaussian noise. [1] The steering vectors are assumed to follow the usual exponential model

$$\mathbf{a}(\theta) = \frac{1}{\sqrt{M}}\left[1, \mathrm{e}^{\mathrm{i}\theta}, \ldots, \mathrm{e}^{\mathrm{i}(M-1)\theta}\right]^T, \tag{2}$$

for all $\theta \in [-\pi, \pi]$, which corresponds to a uniform linear array of antennas. We assume that $N > M$ samples of the previous signal (1) are collected in the matrix $\mathbf{Y}_N = [\mathbf{y}_1, \ldots, \mathbf{y}_N]$, which can be written as

$$\mathbf{Y}_N = \mathbf{A}\mathbf{S}_N + \mathbf{V}_N, \tag{3}$$

with $\mathbf{S}_N = [\mathbf{s}_1, \ldots, \mathbf{s}_N]$ and $\mathbf{V}_N = [\mathbf{v}_1, \ldots, \mathbf{v}_N]$. We assume the matrix $\mathbf{S}_N$ to be full rank K, and thus $\mathbf{A}\mathbf{S}_N$ has rank K. We denote by $0 = \lambda_{1,N} = \ldots = \lambda_{M-K,N} < \lambda_{M-K+1,N} < \ldots < \lambda_{M,N}$ the eigenvalues of $\frac{\mathbf{A}\mathbf{S}_N\mathbf{S}_N^*\mathbf{A}^*}{N}$ (the non-zero eigenvalues are assumed to have multiplicity one without loss of generality). The associated eigenvectors are denoted $\mathbf{u}_{1,N}, \ldots, \mathbf{u}_{M,N}$. In the same way, we denote by $\hat{\lambda}_{1,N}, \ldots, \hat{\lambda}_{M,N}$ and $\hat{\mathbf{u}}_{1,N}, \ldots, \hat{\mathbf{u}}_{M,N}$ the eigenvalues and associated eigenvectors of the sample covariance matrix $\frac{\mathbf{Y}_N\mathbf{Y}_N^*}{N}$.

The MUSIC method is based on the observation that the DoA $\theta_1, \ldots, \theta_K$ are zeros the localization function

$$\eta_N(\theta) = \mathbf{a}(\theta)^*\mathbf{\Pi}_N\mathbf{a}(\theta), \tag{4}$$

where $\mathbf{\Pi}_N = \sum_{k=1}^{M-K}\mathbf{u}_{k,N}\mathbf{u}_{k,N}^*$ is the projector onto the noise subspace. The K DoA are estimated by taking the K deepest minima of an estimated localization function, which is given, in the context of the MUSIC method, by

$$\theta \mapsto \mathbf{a}(\theta)^*\hat{\mathbf{\Pi}}_N\mathbf{a}(\theta), \tag{5}$$

where $\hat{\mathbf{\Pi}}_N = \sum_{k=1}^{M-K}\hat{\mathbf{u}}_{k,N}\hat{\mathbf{u}}_{k,N}^*$ is the sample estimate of $\mathbf{\Pi}_N$. If $\limsup_N \|\frac{\mathbf{S}_N\mathbf{S}_N^*}{N}\| < \infty$, the law of large number implies that almost surely (a.s.),

$$\left\|N^{-1}\mathbf{Y}_N\mathbf{Y}_N^* - \left(N^{-1}\mathbf{A}\mathbf{S}_N\mathbf{S}_N^*\mathbf{A}^* + \sigma^2\mathbf{I}\right)\right\| \to 0 \tag{6}$$

as $N \to \infty$ while M is fixed, which ensures that (5) consistently estimates (4). However, in the asymptotic regime where $M, N \to \infty$ while $c_N = M/N \to c > 0$, the previous convergence is not valid, and (5) is not consistent anymore.

[1] $\mathcal{N}_{\mathbb{R}^k}(\boldsymbol{\alpha}, \mathbf{\Gamma})$ represents the k-variate normal distribution with mean $\boldsymbol{\alpha} \in \mathbb{R}^k$ and covariance $\mathbf{\Gamma}$, and $\mathbf{x} + \mathrm{i}\mathbf{y}$ follows the $\mathcal{N}_{\mathbb{C}^k}(\boldsymbol{\alpha} + \mathrm{i}\boldsymbol{\beta}, \mathbf{\Gamma})$ distribution if $\mathbf{x}, \mathbf{y}$ are independent and $\mathbf{x} \sim \mathcal{N}_{\mathbb{R}^k}(\boldsymbol{\alpha}, \frac{\mathbf{\Gamma}}{2})$, $\mathbf{y} \sim \mathcal{N}_{\mathbb{R}^k}(\boldsymbol{\beta}, \frac{\mathbf{\Gamma}}{2})$.

3. THE G-MUSIC METHOD

From now on, we consider the following regime: we assume that $M = M(N)$, $K = K(N)$ are functions of N, such that $c_N = M/N \to_N c \in (0, 1)$, and $K < M$. We also assume that $\limsup_N \|\frac{\mathbf{S}_N\mathbf{S}_N^*}{N}\| < \infty$. In general settings, we will consider the consistent estimation of

$$\eta_N = \mathbf{d}_{1,N}^*\mathbf{\Pi}_N\mathbf{d}_{2,N}, \tag{7}$$

where $\mathbf{d}_{1,N}, \mathbf{d}_{2,N} \in \mathbb{C}^M$ are deterministic vectors such that

$$\limsup_{N\to\infty} \max\left\{\|\mathbf{d}_{1,N}\|, \|\mathbf{d}_{2,N}\|\right\} < \infty. \tag{8}$$

Before describing the G-MUSIC estimator, we first review some well-known results concerning the asymptotic behaviour of the sample eigenvalues. Let $\hat{\mu}_N = \frac{1}{M}\sum_{k=1}^{M}\delta_{\hat{\lambda}_{k,N}}$ be the empirical eigenvalue distribution of $\frac{\mathbf{Y}_N\mathbf{Y}_N^*}{N}$, where δ_x is the Dirac measure at x. The Stieltjes transform of $\hat{\mu}_N$ is given by

$$\hat{m}_N(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\hat{\mu}_N(\lambda)}{\lambda - z} = \frac{1}{M}\mathrm{Tr}\,\mathbf{Q}_N(z),$$

with $\mathbf{Q}_N(z) = \left(\frac{\mathbf{Y}_N\mathbf{Y}_N^*}{N} - z\mathbf{I}\right)^{-1}$. From Dozier & Silverstein [7] [8], there exists a deterministic probability measure μ_N, with support $\mathrm{supp}(\mu_N) \subset \mathbb{R}^+ = [0, \infty)$, such that a.s.,

$$\hat{\mu}_N - \mu_N \xrightarrow[N\to\infty]{w} 0,$$

where "w" denotes the weak convergence. Equivalently, for $z \in \mathbb{C}\backslash\mathbb{R}^+$, $\hat{m}_N(z) - m_N(z) \to_N 0$ a.s., where $m_N(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_N(\lambda)}{\lambda - z}$ is the Stieltjes transform of μ_N, which satisfies the equation $m_N(z) = \frac{1}{M}\mathrm{Tr}\,\mathbf{T}_N(z)$ with

$$\mathbf{T}_N(z) = \left(1 + \sigma^2 c_N m_N(z)\right)\left(\frac{\mathbf{A}\mathbf{S}_N\mathbf{S}_N^*\mathbf{A}^*}{N} - w_N(z)\mathbf{I}\right)^{-1}$$

and

$$w_N(z) = z\left(1 + \sigma^2 c_N m_N(z)\right)^2 - \sigma^2(1 - c_N)\left(1 + \sigma^2 c_N m_N(z)\right).$$

From [4], the support of μ_N is given by

$$\mathrm{supp}(\mu_N) = \bigcup_{q=1}^{Q}\left[x_{q,N}^-, x_{q,N}^+\right],$$

$$\vartheta_N(k,l) = -\frac{N\delta(k,l)}{2(M-K)} + \frac{\sigma^2}{\pi}\int_{x_{1,N}^-}^{x_{1,N}^+} \frac{\mathrm{Im}\left(w_N(x)\right)|w_N'(x)|^2\left(|w_N(x)|^2\tilde{v}_N(x) + \lambda_{k,N}\lambda_{l,N}v_N(x) + (\lambda_{k,N}+\lambda_{l,N})(1-u_N(x))\right)}{|\lambda_{k,N}-w_N(x)|^2|\lambda_{l,N}-w_N(x)|^2}\mathrm{d}x, \quad (13)$$

with $1 \leq Q \leq K+1$ and where $x_{q,N}^-, x_{q,N}^+$ are the $2Q$ positive local extrema of the function

$$\phi_N(w) = w(1-\sigma^2 c_N f_N(w))^2 + \sigma^2(1-c_N)(1-\sigma^2 c_N f_N(w)),$$

where $f_N(w) = \frac{1}{M}\sum_{k=1}^{M}\frac{1}{\lambda_{k,N}-w}$, and which satisfy $x_{q,N}^- < x_{q,N}^+ < x_{q+1,N}^-$. From [8], $m_N(z)$ can be extended continuously to the real axis when $z \in \mathbb{C}^+ \to x \in \mathbb{R}$, and we denote the limit $m_N(x)$. We define $w_N(x)$ in the same way. To introduce the G-MUSIC method, we need to formulate an additional assumption related to the separation between the signal and noise eigenvalues.

A-1 *There exists $t_1^-, t_1^+, t_2^-, t_2^+ > 0$ such that*

$$t_1^- < \liminf_{N\to\infty} x_{1,N}^- \leq \limsup_{N\to\infty} x_{1,N}^+ < t_1^+$$
$$t_2^- < \liminf_{N\to\infty} x_{2,N}^- \leq \limsup_{N\to\infty} x_{Q,N}^+ < t_2^+,$$

and for N large enough, $w_N(t_2^-) < \lambda_{M-K+1,N}$.

Assumption **A-1** is related with the Signal to Noise Ratio (SNR) when K is independent of N and $\lambda_{M-K+k,N} \to_N \gamma_k$ for $k = 1,\ldots,K$, if we define the SNR to be the ratio $\frac{\gamma_K}{\sigma^2}$. Indeed, it is shown in this case (see Loubaton & Vallet [9]) that Assumption **A-1** is equivalent to the condition $\frac{\gamma_K}{\sigma^2} > \sqrt{c}$. Assumption **A-1** has also important consequences on the behaviour of the sample eigenvalues, and implies that a.s., the sample eigenvalues split in two groups, namely

$$t_1^- < \liminf_{N\to\infty} \hat{\lambda}_{1,N} \leq \limsup_{N\to\infty} \hat{\lambda}_{M-K,N} < t_1^+, \quad (9)$$

and $\liminf_N \hat{\lambda}_{M-K+1,N} > t_2^-$. From [4], under assumption **A-1**, (7) can be expressed as

$$\eta_N = \frac{1}{2\pi\mathrm{i}}\oint_{\partial\mathcal{R}} \mathbf{d}_{1,N}^*\mathbf{T}_N(z)\mathbf{d}_{2,N}\frac{w_N'(z)}{1+\sigma^2 c_N m_N(z)}\mathrm{d}z,$$

where $\partial\mathcal{R}$ is the clockwise oriented boundary of the rectangle

$$\mathcal{R} = \left\{x + \mathrm{i}y : x \in \left[t_1^- - \epsilon, t_1^+ + \epsilon\right], y \in [-\delta,\delta]\right\},$$

for some $\epsilon > 0$ s.t. $0 < t_1^- - \epsilon, t_1^+ + \epsilon < t_2^-$ and $\delta > 0$. Define $\hat{w}_N(z) = z(1+\sigma^2 c_N \hat{m}_N(z))^2 - \sigma^2(1+\sigma^2 c_N \hat{m}_N(z))$. In [4], it is shown that

$$\sup_{z\in\partial\mathcal{R}}\left|\mathbf{d}_{1,N}^*\mathbf{Q}_N(z)\mathbf{d}_{2,N}\frac{\hat{w}_N'(z)}{1+\sigma^2 c_N \hat{m}_N(z)} - \mathbf{d}_{1,N}^*\mathbf{T}_N(z)\mathbf{d}_{2,N}\frac{w_N'(z)}{1+\sigma^2 c_N m_N(z)}\right| \xrightarrow[N\to\infty]{a.s.} 0.$$

This of course implies that $\hat{\eta}_N - \eta_N \to_N 0$ a.s., where

$$\hat{\eta}_N = \frac{1}{2\pi\mathrm{i}}\oint_{\partial\mathcal{R}} \mathbf{d}_{1,N}^*\mathbf{Q}_N(z)\mathbf{d}_{2,N}\frac{\hat{w}_N'(z)}{1+\sigma^2 c_N \hat{m}_N(z)}\mathrm{d}z \quad (10)$$

is therefore a consistent estimator of (7). We notice that the integrand in (10) is meromorphic with poles at $\hat{\lambda}_{1,N},\ldots,\hat{\lambda}_{M,N}$ as well the zeros of the function $z \mapsto 1 + \sigma^2 c_N \hat{m}_N(z)$, denoted $\hat{\omega}_{1,N},\ldots,\hat{\omega}_{M,N}$. It is shown in [4] that under assumption **A-1**, these zeros follow a property similar to (9), i.e a.s.

$$t_1^- < \liminf_{N\to\infty} \hat{\omega}_{1,N} \leq \limsup_{N\to\infty} \hat{\omega}_{M-K,N} < t_1^+, \quad (11)$$

and $\liminf_N \hat{\omega}_{M-K+1,N} > t_2^-$. Thus, all the poles can be located with respect to the integration contour $\mathcal{R}$, which ensures that the integral can be solved using residue theorem, to obtain an explicit formula in terms of $\hat{\mathbf{u}}_{k,N}, \hat{\lambda}_{k,N}$ and $\hat{\omega}_{k,N}$ (see [4]). By setting $\mathbf{d}_{1,N} = \mathbf{d}_{2,N} = \mathbf{a}(\theta)$, we of course obtain a consistent estimator $\hat{\eta}_N(\theta)$ of the localization function $\eta_N(\theta)$ defined in (4). The G-MUSIC DoA estimator are thus defined as the K deepest local minima of $\theta \mapsto \hat{\eta}_N(\theta)$, and we will denote these estimators $\hat{\theta}_{1,N},\ldots,\hat{\theta}_{K,N}$. In [5], it was further shown that these estimators are consistent with the property

$$N\left(\hat{\theta}_{k,N} - \theta_k\right) \xrightarrow[N\to\infty]{a.s.} 0. \quad (12)$$

4. 2ND ORDER ANALYSIS

In this section, we provide a CLT for both the localization function (10) and the DoA estimators $\hat{\theta}_{1,N},\ldots,\hat{\theta}_{K,N}$.

Before stating the main result, we need to introduce some new quantities. For $x \in \mathbb{R}$, we define

$$u_N(x) = \frac{\sigma^2}{N}\sum_{k=1}^{M}\frac{\lambda_{k,N}}{|\lambda_{k,N} - w_N(x)|^2}$$

as well as

$$v_N(x) = \frac{\sigma^2}{N}\sum_{k=1}^{M}\frac{1}{|\lambda_{k,N} - w_N(x)|^2}$$

$$\vartheta_N(k,l) = \frac{\sigma^4 c_N \left(\lambda_{k,N}\lambda_{l,N} + (\lambda_{k,N}+\lambda_{l,N})\sigma^2 + \sigma^4\right)\left(\lambda_{k,N}\lambda_{l,N} + \sigma^4 c_N\right)}{2\left(\lambda_{k,N}^2 - \sigma^4 c_N\right)\left(\lambda_{l,N}^2 - \sigma^4 c_N\right)\left(\lambda_{k,N}\lambda_{l,N} - \sigma^4 c_N\right)}(1-\delta(k,l)) + \epsilon_N(k,l), \quad (16)$$

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and $\tilde{v}_N(w) = v_N(x) + \sigma^2(1-c_N)|w_N(x)|^{-2}$. Note that $u_N(x)$, $v_N(x)$ and $\tilde{v}_N(x)$ are well-defined since $w_N(x) \notin \{\lambda_{1,N},\ldots,\lambda_{M,N}\}$ for all $x \in \mathbb{R}$ (see [4]). For $1 \leq k,l \leq M$, we define $\vartheta_N(k,l)$ by the formula (13) given at the top of the page, where $\delta(k,l) = 1$ if $k,l \in \{1,\ldots,M-K\}$ and 0 otherwise. Finally, we define

$$\Gamma_N(k,l) = \mathrm{Re}\left(\eta_{k,N}^{(1,2)}\eta_{l,N}^{(1,2)}\right) + \frac{\eta_{k,N}^{(1,1)}\eta_{l,N}^{(2,2)} + \eta_{l,N}^{(1,1)}\eta_{k,N}^{(2,2)}}{2}$$

where $\eta_{k,N}^{(i,j)} = \mathbf{d}_{i,N}^*\mathbf{u}_{k,N}\mathbf{u}_{k,N}^*\mathbf{d}_{j,N}$, and

$$\Gamma_N = \sum_{k=1}^{M}\sum_{l=1}^{M}\vartheta_N(k,l)\Gamma_N(k,l). \quad (14)$$

The main result is the following.

Theorem 1. *Assume* **A-1** *holds. Then* $\vartheta_N(k,l) \geq 0$ *for all* k,l *and*[2] $\mathrm{Re}\left(\hat{\eta}_N - \eta_N\right) = \mathcal{O}_{\mathbb{P}}\left(\sqrt{\frac{\Gamma_N}{N}}\right) + o_{\mathbb{P}}\left(\frac{1}{\sqrt{N}}\right)$. *If*

$$\liminf_{N\to\infty}\Gamma_N > 0, \quad (15)$$

we have

$$\frac{\sqrt{N}\mathrm{Re}\left(\hat{\eta}_N - \eta_N\right)}{\sqrt{\Gamma_N}} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}_{\mathbb{R}}(0,1).$$

Moreover, when K *is independent of* N, $\vartheta_N(k,l)$ *can be approximated by* (16) *given at the top of the page, where* $\max_{k,l}|\epsilon_N(k,l)| \to_N 0$.

The proof of theorem 1 is omitted due to space constraints. Since $|\mathrm{Re}(z_1 z_2)| \leq \frac{1}{2}\left(|z_1|^2 + |z_2|^2\right)$ for $z_1, z_2 \in \mathbb{C}$, we have of course $\Gamma_N(k,l) \geq 0$. We remark that the main purpose of (15) is to ensure that the fluctuations of $\hat{\eta}_N - \eta_N$ are of the order $\mathcal{O}\left(N^{-1/2}\right)$. Indeed, there exist several situations where the fluctuations can be faster than $\mathcal{O}\left(N^{-1/2}\right)$, e.g. when K is independent of N and $\mathbf{d}_{1,N} = \mathbf{d}_{2,N} = \mathbf{u}_{1,N}$, then (16) implies $\Gamma_N = \mathcal{O}\left(N^{-1}\right)$ and $\mathrm{Re}\left(\hat{\eta}_N - \eta_N\right) = o_{\mathbb{P}}\left(\frac{1}{\sqrt{N}}\right)$. Moreover, when K is independent of N, assumption **A-1** also implies that

$$\liminf_{N\to\infty}\min_{k,l\geq M-K+1}\vartheta_N(k,l) > 0,$$

and (15) is therefore ensured by the sufficient condition

$$\liminf_{N\to\infty}\mathrm{Re}\left(\left(\mathbf{d}_{1,N}^*\mathbf{\Pi}_N^{\perp}\mathbf{d}_{2,N}\right)^2\right) + \mathbf{d}_{1,N}^*\mathbf{\Pi}_N^{\perp}\mathbf{d}_{1,N}\mathbf{d}_{2,N}^*\mathbf{\Pi}_N^{\perp}\mathbf{d}_{2,N} > 0, \quad (17)$$

where $\mathbf{\Pi}_N^{\perp} = \mathbf{I} - \mathbf{\Pi}_N = \mathbf{A}\left(\mathbf{A}^*\mathbf{A}\right)^{-1}\mathbf{A}^*$ is the projector onto the signal subspace. We are now in position to obtain from theorem 1 a CLT for the localization function estimate (10) and we assume for the remainder that K is independent of N. If $\mathbf{d}_{1,N} = \mathbf{d}_{2,N} = \mathbf{a}(\theta)$ with $\theta \in [-\pi,\pi]\backslash\{\theta_1,\ldots,\theta_K\}$, it is easy to see that $\mathbf{a}(\theta)^*\mathbf{\Pi}_N^{\perp}\mathbf{a}(\theta) \to_N 0$, and thus $\Gamma_N \to_N 0$. On the other hand, if $\mathbf{d}_{1,N} = \mathbf{d}_{2,N} = \mathbf{a}(\theta_n)$ and $\eta_{k,N}(\theta_n) = \mathbf{a}(\theta_n)^*\mathbf{u}_{k,N}\mathbf{u}_{k,N}^*\mathbf{a}(\theta_n)$, for $n = 1,\ldots,K$, it is straightforward to see that (17) is satisfied in this case. This leads the following corollary.

Corollary 1. *Under assumption* **A-1** *and if* K *is independent of* N,

$$\frac{\sqrt{N}\hat{\eta}_N(\theta_k)}{\sqrt{2\sum_{k,l}\vartheta_N(k,l)\eta_{k,N}(\theta_n)\eta_{l,N}(\theta_n)}} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}_{\mathbb{R}}(0,1),$$

and for $\theta \in [-\pi,\pi]\backslash\{\theta_1,\ldots,\theta_K\}$,

$$\hat{\eta}_N(\theta) - \eta_N(\theta) = o_{\mathbb{P}}\left(\frac{1}{\sqrt{N}}\right).$$

We finally provide the CLT for the DoA estimators. For that purpose, we use the classical Δ-method. Using (12) and a Taylor expansion around θ_n ($n = 1,\ldots,K$), we obtain a.s. that

$$\hat{\eta}_N'\left(\hat{\theta}_{n,N}\right) = \hat{\eta}_N'\left(\theta_n\right) + \left(\hat{\theta}_{n,N} - \theta_n\right)\hat{\eta}_N^{(2)}\left(\theta_n\right) + \frac{1}{2}\left(\hat{\theta}_{n,N} - \theta_n\right)^2\hat{\eta}_N^{(3)}\left(\tilde{\theta}_{n,N}\right),$$

for N large enough, where $\tilde{\theta}_{n,N}$ is between $\hat{\theta}_{n,N}$ and θ_n. Since by definition, we have $\hat{\eta}_N'\left(\hat{\theta}_{n,N}\right) = 0$, we thus obtain

$$\hat{\theta}_{n,N} - \theta_n = -\frac{\hat{\eta}_N'\left(\theta_n\right)}{\hat{\eta}_N^{(2)}\left(\theta_n\right) + \frac{\hat{\theta}_{n,N}-\theta_n}{2}\hat{\eta}_N^{(3)}\left(\tilde{\theta}_{n,N}\right)}. \quad (18)$$

We first characterize the asymptotic behaviour of the denominator of (18). Since $\sup_\theta\left\|\mathbf{a}^{(k)}(\theta)\right\| \sim M^k$, we deduce from (9) and (11) that $N^{-3}\hat{\eta}_N^{(3)}\left(\tilde{\theta}_{n,N}\right) = \mathcal{O}(1)$ a.s. Therefore, (12) implies

$$\left(\hat{\theta}_{n,N} - \theta_n\right)\frac{\hat{\eta}_N^{(3)}\left(\tilde{\theta}_{n,N}\right)}{N^2} \xrightarrow[N\to\infty]{a.s.} 0.$$

From section 3, we can write

$$\frac{1}{N^2}\hat{\eta}_N^{(2)}\left(\theta_n\right) = 2\frac{\mathbf{a}'(\theta_n)^*}{N}\mathbf{\Pi}_N\frac{\mathbf{a}'(\theta_n)}{N} + o_{\mathbb{P}}(1),$$

[2] $\mathcal{O}_{\mathbb{P}}(1)$ represents boundedness in probability (tightness).

and thus we obtain

$$N^{3/2}\left(\hat{\theta}_{n,N}-\theta_n\right)=-\frac{\frac{1}{\sqrt{N}}\hat{\eta}_N'(\theta_n)}{2\frac{\mathbf{a}'(\theta_n)^*}{N}\mathbf{\Pi}_N\frac{\mathbf{a}'(\theta_n)}{N}+o_{\mathbb{P}}(1)}. \quad (19)$$

Finally, a straightforward application of theorem 1 to the numerator of (19), by setting $\mathbf{d}_{1,N}=N^{-1}\mathbf{a}'(\theta_n)$ and $\mathbf{d}_{2,N}=\mathbf{a}(\theta_n)$ leads the following result.

Corollary 2. *Assume* **A-1** *and* K *independent of* N*. Then*

$$N^{3/2}\sqrt{\frac{\left(\mathbf{d}_{1,N}^*\mathbf{\Pi}_N\mathbf{d}_{1,N}\right)^2}{\Gamma_N}}\left(\hat{\theta}_{n,N}-\theta_n\right)\xrightarrow[N\to\infty]{\mathcal{D}}\mathcal{N}_{\mathbb{R}}(0,1),$$

where Γ_N *is defined by* (14) *by taking* $\mathbf{d}_{1,N}=N^{-1}\mathbf{a}'(\theta_n)$ *and* $\mathbf{d}_{2,N}=\mathbf{a}(\theta_n)$.

5. SIMULATIONS

In this section, we illustrate numerically the result of corollary 2. We consider $M=20$, $N=40$, and 2 sources with DoA $\theta_1=0.5$ and $\theta_2=1$. The steering vectors follow the model (2). The rows of matrix $\mathbf{S}_N$ are realizations of mutually independent Gaussian AR(1) processes with correlation coefficient 0.9. The SNR is defined to be $-10\log(\sigma^2)$. In all the following experiments, we use the formula (13) given in theorem 1 to compute the variance Γ_N.

In figure 1, we have plotted the histogram of $N^{3/2}(\hat{\theta}_{1,N}-\theta_1)$ (5000 trials) as well as the density of a Gaussian distribution with zero mean and variance $(\mathbf{d}_{1,N}^*\mathbf{\Pi}_N\mathbf{d}_{1,N})^{-2}\Gamma_N$, for a SNR of 10 dB.

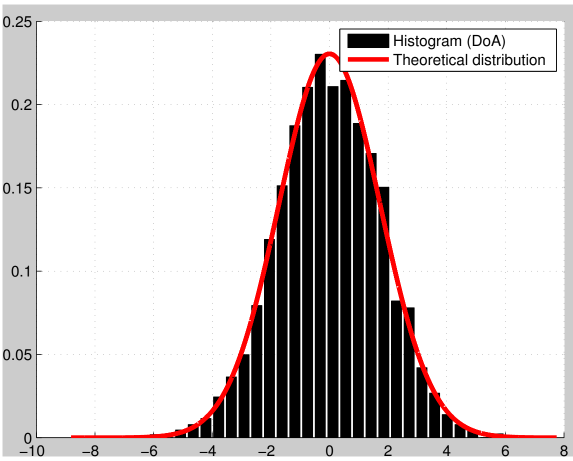

Fig. 1. Histogram and theoret. distrib. of $N^{3/2}(\hat{\theta}_{1,N}-\theta_1)$

In figure 2, we have plotted the theoretical and empirical MSE of the first angle estimate, i.e $\mathbb{E}[(\hat{\theta}_{1,N}-\theta_1)^2]$, versus the SNR. The Cramer-Rao Bound (CRB), derived in [2], is also represented. Assumption **A-1** is fulfilled from SNR=4dB.

Figures 1 and 2 show that the results obtained in corollary 2, as well as the expression of the variance Γ_N, are accurate for realistic values of M, N.

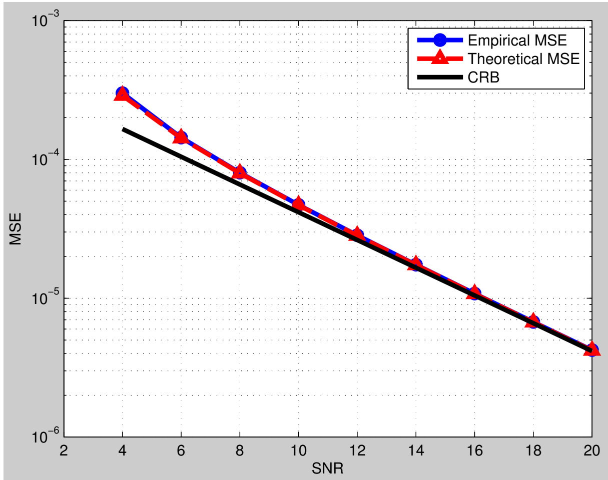

Fig. 2. Theoretical and empirical MSE of $\hat{\theta}_{1,N}$

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