SPARSE MULTIGRID MODAL ESTIMATION: INITIAL GRID SELECTION

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ABSTRACT

Modal retrieval problem can be addressed using sparse estimation techniques coupled with a multigrid approach. Selection of the initial grid, in the multigrid algorithm, is a critical problem that needs satisfactory solutions. In this paper we propose a strategy for selecting a coarse initial grid which guarantees convergence of the algorithm even starting with a single atom in the dictionary. The idea is to define the atoms so as their correlation with the signal modes is greater than a specified threshold. We show through simulations that the proposed solution provides accurate estimations even in presence of noise. Moreover, we investigate the convergence of the multigrid algorithm towards the true parameters.

Index Terms— modal estimation, adaptive sparse approximation, multigrid.

1. INTRODUCTION

Consider the modal retrieval problem in which the main aim is to estimate parameters of superimposed damped sinusoids. This topic has versatile applications in engineering, e.g., nuclear magnetic resonance (NMR) spectroscopy, wireless communications, sonar and radar. We will use the following complex modal signal model:

$$y(m) = \sum_{i=1}^{F} c_i a_i^m + e(m) \tag{1}$$

for $m = 0, \ldots, M-1$, where $a_i = e^{(-\alpha_i + j2\pi f_i)}$, with $\{\alpha_i\}_{i=1}^F$ the damping factors and $\{f_i\}_{i=1}^F$ the frequencies. $\{c_i\}_{i=1}^F$ are complex amplitudes and $e(m)$ is a white Gaussian noise with variance σ^2. The problem is to estimate the set of parameters $\{a_i, c_i\}_{i=1}^F$ from the observed sequence $y(m)$. This can be done by using three classes of possible approaches. The first class, parametric approaches, such as sub-space based methods [1, 2], whose use requires the choice of the number of components (model order) F before they can be applied to the data vector. The second class, nonparametric approaches, such as the periodograms, which do not require knowledge of F but they suffer from a limited spectral resolution. The third class is that of sparse approximation methods [3, 4, 5] which may be considered as semi-parametric. These later methods, as compared to the previous ones, can greatly enhance the estimation accuracy for noisy signals and their use does not necessarily require a priori knowledge of the model order F [6, 7]. This is why spectral analysis using sparse estimation methods has received considerable interest in the recent years.

Sparse approximation consists in finding a decomposition of a signal $\mathbf{y} \in \mathbb{C}^M$ as a linear combination of a limited number of elements from a dictionary $\mathbf{Q} \in \mathbb{C}^{M \times N}$, i.e., finding a coefficient vector $\mathbf{x} \in \mathbb{C}^N$ that satisfies $\mathbf{y} \approx \mathbf{Qx}$, where $\mathbf{Q}$ is overcomplete $(M < N)$. The sparsity condition on $\mathbf{x}$ ensures that the underdetermined problem does not have an infinite number of solutions. The dictionary $\mathbf{Q}$ can be chosen according to its ability to represent the signal with a limited number of coefficients or it can be imposed by the inverse problem at hand. In the present paper, we consider dictionaries whose atoms are function of some parameters. The different atoms of the dictionary are then formed by evaluating this function over a grid which has to be very fine to achieve a certain degree of resolution. This is the case for the modal estimation problem in which the atoms are formed by discretizing the frequency and damping factor axes. In this situation, the challenge is to get a good approximation without a prohibitive computational cost due to the huge size of the dictionary. To deal with this problem we proposed a multigrid approach [8, 7] which iteratively enhances the set of atoms in the dictionary. The goal of the multigrid algorithm is to improve resolution by avoiding computationally intractable operations. The estimation begins with a coarse dictionary which is then enhanced over several resolution levels according to the activated atoms at each level. However, different numerical simulations suggest that the construction of the initial dictionary (at the first level) should be done carefully in order to ensure convergence of the multigrid algorithm towards the signal modes $\{a_i\}_{i=1}^F$. In a recent work on grid selection problems, Stoica et al. [6] proposed to use a very fine grid to estimate the parameters of harmonic signals. As a result, the minimum number of atoms in the dictionary have to be greater than a threshold which is generally high. Alternatively, in this paper we propose to investigate the approach which consists of starting with a very coarse dictionary containing only a single atom and we derive the condition that ensures convergence to the true modes thanks to the multigrid approach.

The paper is organized as follows. In section 2, we see how the modal retrieval problem may be addressed using sparse approximations. In section 3 we recall the principles of multigrid dictionary refinement. In section 4 we discuss different possible ways and conditions to construct the initial dictionary, then we present an efficient strategy to define the initial grid. In section 5 we present some simulation results to show the effectiveness of the proposed strategy and the convergence of the multigrd algorithm on randomly generated modal signals. Conclusions are drawn in section 6.

2. PROBLEM FORMULATION

The data model expressed in (1) can be written in a vector form as:

$$\mathbf{y} = \sum_{i=1}^{F} c_i \mathbf{a}(\alpha_i, f_i) + \mathbf{e} \quad (2)$$

where $\mathbf{a}(\alpha_i, f_i)$ is a vector of M elements:

$$\mathbf{a}(\alpha_i, f_i) = [1, e^{(-\alpha_i + j2\pi f_i)}, \ldots, e^{(-\alpha_i + j2\pi f_i)(M-1)}]^T. \quad (3)$$

Let α_{max} be an upper bound on $\{\alpha_i\}_{i=1}^F$ and let P the number of points of a uniform grid covering the damping factor interval $[0, \alpha_{max}]$. Similarly, let K be the number of points of a uniform grid covering the frequency interval $[0, 1[$. Finally, let

$$\mathbf{Q} = [\mathbf{q}(0,0), \ldots, \mathbf{q}(0, (K-1)\delta_f), \mathbf{q}(\delta_\alpha, 0), \ldots, \mathbf{q}(\delta_\alpha, (K-1)\delta_f), \ldots, \mathbf{q}((P-1)\delta_\alpha, (K-1)\delta_f)] \quad (4)$$

where $\mathbf{q}(\alpha, f) = \frac{\mathbf{a}(\alpha,f)}{||\mathbf{a}(\alpha,f)||_2}$, $\delta_\alpha = \alpha_{max}/P$ and $\delta_f = 1/K$. The matrix $\mathbf{Q}$, called dictionary, is constructed using $N = KP$ modes derived from the combination of the two uniform grids combining damping factors and frequencies. Each column $\{\mathbf{q}_n\}_{n=0}^{N-1}$ of $\mathbf{Q}$ is called atom. Thus, the dictionary $\mathbf{Q}$ is obtained from a 2-D grid. Therefore, we can approximate $\mathbf{y}$ in (2) by:

$$\mathbf{y} \approx \mathbf{Q}\mathbf{x} + \mathbf{e} \quad (5)$$

where $\mathbf{x} \in \mathbb{C}^N$ is a sparse vector, i.e. it contains a few non-zero elements and the rest of elements are equal to zero or negligible. Non-zero elements are nearly equal to $\{c_i\}$ and correspond to columns in $\mathbf{Q}$ that are equal (or close) to $\{\mathbf{a}(\alpha_i, f_i)\}$. Thus, the modal retrieval problem can be formulated as the sparse estimation of $\mathbf{x}$. It should be emphasized that by doing so, we implicitly assume that the dictionary includes the true signal modes. A first approach to ensure that (at least approximately) is to define $\mathbf{Q}$ on a very fine grid resulting in a high dimension dictionary. The main limitation of such an approach is to drastically increase the computational cost. We have proposed an alternative approach which consists in an adaptive refinement of the dictionary resulting in the so-called sparse multigrid approach [7]. Its principle is described in the next section.

Table 1. Sparse multigrid algorithm

- **Input.** A signal $\mathbf{y} \in \mathbb{C}^M$, a matrix $\mathbf{Q}_0 \in \mathbb{C}^{M \times N}$, a scalar λ and an integer L
- **Output.** A sparse coefficient vector $\mathbf{x}_{L-1} \in \mathbb{C}^N$.

For $l = 0$ up to $l = L-1$

$\mathbf{x}_l = \mathrm{SAM}\ (\mathbf{Q}_l, \mathbf{y}, \lambda)$

$\mathbf{Q}_{l+1} = \mathrm{ADAPT}\ (\mathbf{Q}_l, \mathbf{x}_l)$,

End For.

3. MULTIGRID DICTIONARY REFINEMENT

To achieve a high-resolution modal estimation, a possible way is to define a high-resolution dictionary often resulting in a prohibitive computational burden. Rather, it is also possible to adaptively refine a coarse one through a multigrid scheme. This results in the algorithm sketched in table 1 where the key step is the adaptation of the dictionary according to the previous one and the estimated vector $\mathbf{x}$. The algorithm amounts to insert (resp. remove) atoms in (resp. from) $\mathbf{Q}$ and to rerun the sparse approximation algorithm. We propose two refinement procedures. The first one consists in inserting new atoms in the $\mathbf{Q}$ matrix in the neighborhood of active ones. In other words, we first restore the signal $\mathbf{x}_{(l)}$ related to the dictionary $\mathbf{Q}_{(l)}$ by applying a sparse approximation method (SAM) at level l. Then we refine the dictionary by inserting atoms in between pairs of $\mathbf{Q}_{(l)}$, in the neighborhood of each activated atom and we apply again the SAM at level $l+1$ to restore $\mathbf{x}_{(l+1)}$ with respect to the refined dictionary $\mathbf{Q}_{(l+1)}$. Thus we refine iteratively the dictionary until the maximum level $l = L-1$ is reached. This procedure is illustrated in figure 1(a) where the dictionary atoms depend on two parameters f and α. The disadvantage of this procedure is that the size of the dictionary is increasing as new atoms are constantly added between two resolution levels. Hence, the computational cost will be increasing. To cope with this limitation, we propose a second procedure consisting in adding new atoms as in the first procedure and deleting distant non-active ones (fig. 1(b)).

The multigrid dictionary refinement is proposed in the context of modal analysis. However, it is worth noticing that this idea can be straightforwardly extended to any dictionary obtained by sampling a continuous function over a grid. We discuss the selection of the initial grid in the following section.

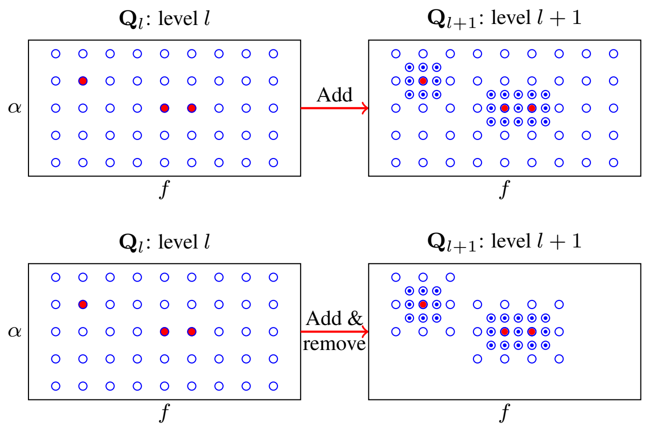


Fig. 1. Two multigrid schemes.

4. SELECTION OF THE INITIAL GRID

The initial dictionary in the multigrid scheme is of great importance since a bad definition of it may cause completely wrong estimation. Therefore, it should respond to at least two conditions. First, in order to ensure convergence to the true modes, it is necessary that each signal mode has a non negligible correlation with at least one atom $\mathbf{q}_n$ from $\mathbf{Q}$. Second, to reduce numerical complexity, the dictionary should contain only a few atoms defined from a coarse sampling of the variables (α, f). These conditions may be fulfilled by defining broadband atoms in the initial dictionary, viz. whatever the number of atoms were included in the dictionary, the spectrum of the atoms have to cover all the spectral range $[0, 1[$ in order to ensure a certain correlation with the signal modes. To clarify this idea, let us consider a rather elementary dictionary containing only one atom $\mathbf{q}(\alpha_0, f_0) \in \mathbb{C}^M$ formed from the signal:

$$q(m) = \beta_0 e^{(-\alpha_0 + j2\pi f_0)m}$$

where $\beta_0 = 1/||\mathbf{a}(\alpha_0, f_0)||_2$ is a normalization constant. The spectrum of this damped atom is:

$$\begin{aligned} Q(f) &= \sum_{m=0}^{M-1} q(m) e^{-j2\pi f m} \\ &= \beta_0 \frac{1 - e^{(-\alpha_0 + j2\pi(f_0 - f))M}}{1 - e^{(-\alpha_0 + j2\pi(f_0 - f))}} \end{aligned}$$

It can be shown that the maximum of $|Q(f)|$ is

$$\max_f |Q(f)| = Q(f_0) = \beta_0 \frac{1 - e^{-\alpha_0 M}}{1 - e^{-\alpha_0}}$$

and the minimum is

$$\min_f |Q(f)| = Q(f_0 \pm \tfrac{1}{2}) = \beta_0 \frac{1 - (-e^{-\alpha_0})^M}{1 + e^{-\alpha_0}}$$

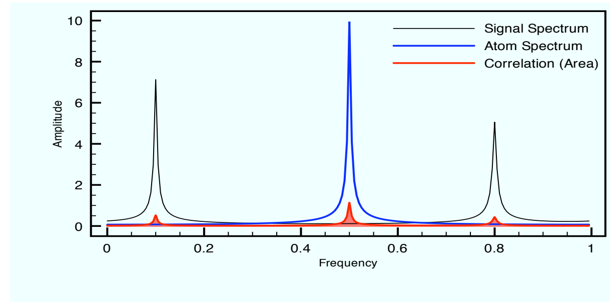
Now, we want $Q(f)$ to cover all the spectral range in the way that it has some correlation with any possible peaks existing in the signal, as illustrated in figure 2. Hence, given a signal $y(m)$ with spectrum $Y(f)$, we want to maximize the correlation $|\int Q^*(f)Y(f)df|$ irrespective of modes position in $y(m)$. This can be done by keeping the minimum $\min_f |Q(f)|$ greater than zero. More precisely, the minimum correlation ρ may be controlled using the inequality:

$$\frac{\min_f |Q(f)|}{\max_f |Q(f)|} \geq \rho \tag{6}$$

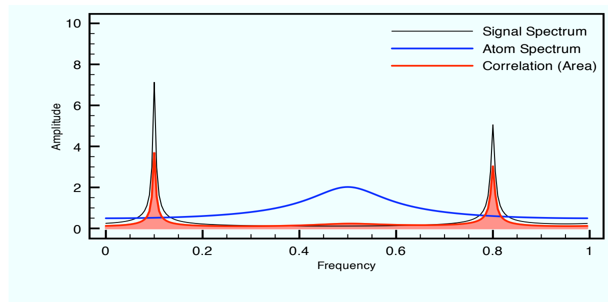
where $0 < \rho < 1$. It can be verified that the solution of this inequality is given by:

$$\alpha_0 \geq \ln(1 + \rho) - \ln(1 - \rho) \tag{7}$$

This result will be used in the next section to show the convergence of the multigrid algorithm even starting with a single atom. Note that, by doing so, the limits of the frequency ($f_{\min} = 0, f_{\max} = 1$) and the damping factor ($\alpha_{\min} = 0, \alpha_{\max} = \alpha_0$) are utilized to calculate the position of the new atoms in the second level of the multigrid scheme.



(a) A single narrowband atom



(b) A single broadband atom

Fig. 2. Schematic representation of the correlation between signal modes and a single atom.

5. SIMULATION RESULTS

In this section, we present results of computer simulations to illustrate the usefulness of the proposed strategy for the construction of the initial dictionary. First, we present obtained results on a simulated noisy modal signal to prove convergence to true modes. Then we analyze the convergence by estimating parameters of random generated modal signals. Finally, we analyze convergence to true modes with different noise levels.

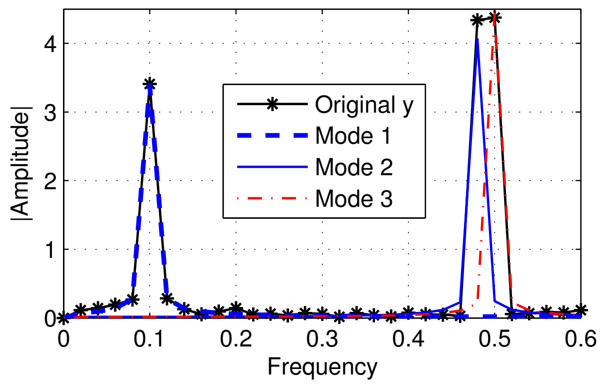


Fig. 3. FFT of y and estimated modes, $\text{SNR}_1 = 20$ dB.

5.1. Convergence

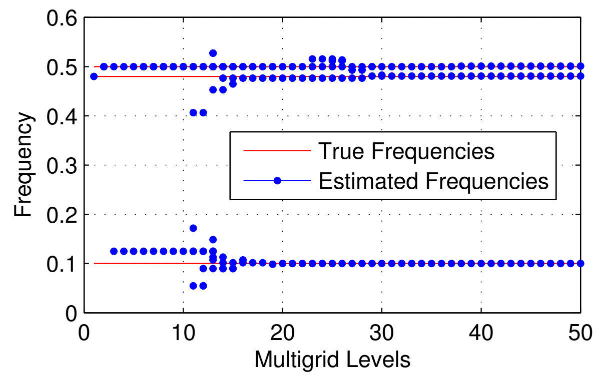
In the first example, the modal signal $\mathbf{y}$ is composed of $M = 50$ samples and made up of three modes (superimposed damped sinusoids) having the same amplitudes $c_i = 1$. The variance of the additive Gaussian noise is fixed such that the SNR of mode 1 is $\text{SNR}_1 = 20$ dB. Table 2 shows the parameters of the three simulated modes and their estimates using the multigrid algorithm.

Table 2. First example: Simulated and estimated modes

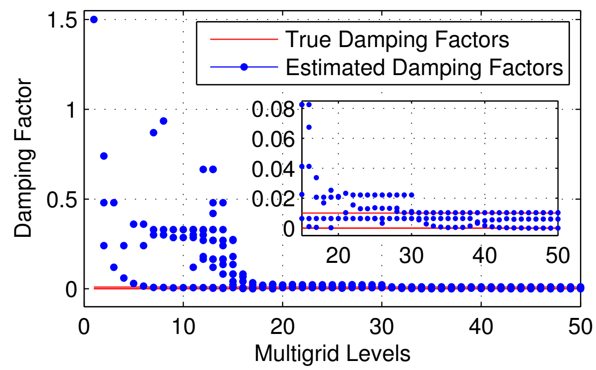
	True		Estimates	
Modes	f	α	f	α
1	0.100	0.010	0.100	0.010
2	0.480	0.000	0.480	0.006
3	0.500	0.000	0.501	0.000

The initial grid is defined using the result of the previous section. We choose $\rho = 0.635$, which gives $\alpha_0 = 1.5$, and we put $f_0 = 0.48$ (recall that f_0 can take any value in $[0, 1[$). For the sparse approximation we use the single best replacement (SBR) method [9] because its good performances for modal retrieval [8]. We apply the multigrid algorithm over 50 levels; the algorithm converges before reaching this maximum level but the objective here is to see its behavior with extra levels after convergence. These settings will be used for all remaining simulations. In figure 3 we present the spectrum of the simulated noisy modal signal and the spectrum of each estimated component. Figures 4(a)–(b) present the estimated frequencies and damping factors at each level. It can be seen that the algorithm converges towards the true frequencies. Concerning the damping factors, two of them are very well estimated (α_1 and α_3); the third is slightly biased ($\hat{\alpha}_2 = 0.006$ versus $\alpha_2 = 0$). Finally, the algorithm converges at level $L = 40$.

In order to show the importance of the definition of the initial dictionary, let us consider the case $\rho = 0.25$ corresponding to $\alpha_0 = 0.51$. For the same settings as before, the results are shown in figure (5). We see clearly that the multigrid algorithm does not converge and fails to detect the first mode.



(a) Estimated frequencies



(b) Estimated damping factors. The insert shows details for levels 15 to 50.

Fig. 4. Estimated parameters at each multigrid level: Convergence to true parameters.

5.2. Resolution

Here we simulate 1000 noise-free signals of 30 samples, each one contains two harmonic sinusoids whose frequencies are generated randomly. The distance between frequencies in the same signal is forced to be greater than or equal to Fourier resolution. Figure 6 presents the estimation error for the two modes in each trial. The standard deviation of the estimation error for the two modes has an order of magnitude of 10^{-4}. The mean squared error (MSE) is equal to 8.603×10^{-8} and 9.758×10^{-8} for mode 1 and mode 2, respectively. We can conclude that whatever the position of the signal frequencies, the proposed strategy converges to the true modes.

5.3. Detection rate

In this simulation, the signal $\mathbf{y}$ contains two modes at $a_1 = e^{j2\pi 0.23}$ and $a_2 = e^{-0.1+j2\pi 0.46}$ with $(c_1, c_2) = (1, 1)$. The number of samples is $M = 30$. The parameters of the signal are estimated by the multigrid algorithm at different noise powers. We emphasize here that the mode a_2 as compared

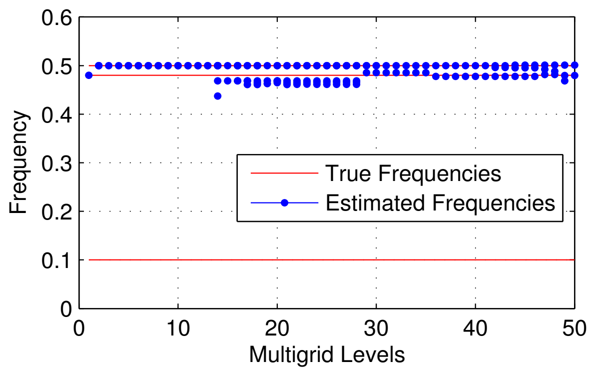


Fig. 5. Estimated frequencies at each level: the multigrid algorithm fails with $\rho = 0.25$.

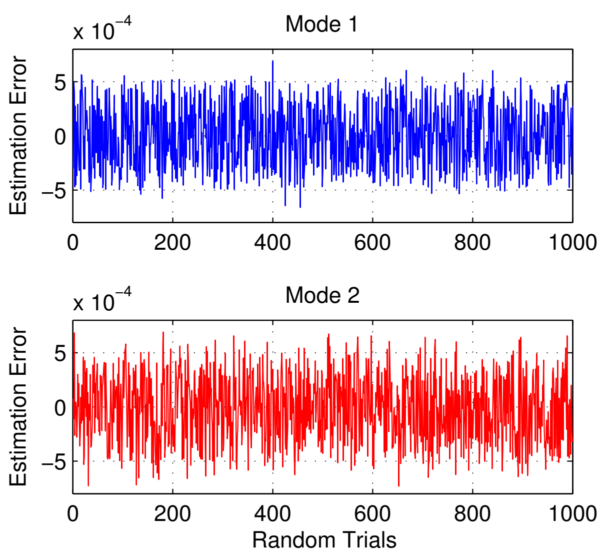


Fig. 6. Estimation error of 1000 randomly generated modal signals. Mode 1 (top), mode 2 (bottom).

to a_1 vanishes rapidly because of its damping factor. We do 100 Monte Carlo trials for each noise level. Then we compute the rate of successful estimations (convergence rate) at each level. The obtained results are presented in figure 7. We can see that, thanks to the proposed strategy, the algorithm converges to true modes with a rate upper than 80% for an SNR_1 more than 9 dB; and the rate of convergence is almost equal to 100% with an SNR_1 upper than 18 dB.

6. CONCLUSION

We proposed an efficient strategy to select the initial grid in the sparse multigrid modal estimation approach. The proposed solutions consist in starting estimation with a dictionary only containing one atom. Then, it was shown, through simulations, that this strategy provides accurate estimations, guarantees convergence to true damped sinusoids parameters and ensures to have a modal dictionary still be coarse. As

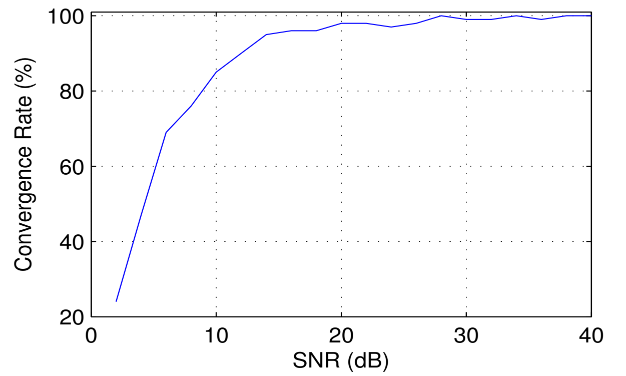


Fig. 7. Monte Carlo trials: convergence rate to $(\alpha_1, f_1) = (0.00, 0.23)$ and $(\alpha_2, f_2) = (0.1, 0.46)$.

future work, we are planing to theoretically study the convergence of the multigrid approach.

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