KALMAN VS H_∞ FILTER IN TERMS OF CONVERGENCE AND ACCURACY: APPLICATION TO CARRIER FREQUENCY OFFSET ESTIMATION

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ABSTRACT

H_∞ filtering is more and more used in the field of recursive estimation in signal processing. The purpose of this communication is to compare Kalman filtering and H_∞ filtering by considering their Ricatti-type equations. Our contribution is twofold: firstly, we show that the H_∞ filter can be seen as a Kalman filter with a model-noise covariance matrix that depends on the noise attenuation level and varies in time. Hence, this can explain the convergence properties of the H_∞ filter when estimating parameters. The convergence and accuracy properties of both Kalman and H_∞ filters are then illustrated by the estimation of a carrier frequency offset in a mobile communication system.

Index Terms— Kalman filter, extended Kalman filter, H_∞ filter, extended H_∞ filter, carrier frequency offset.

1. INTRODUCTION

In the field of recursive estimation, Kalman filtering (KF) and its variants have played a key role for more than 40 years. They have been used in a wide range of applications, from radar processing to GPS navigation, from biomedical application to speech enhancement. When dealing with a non-linear system, the extended Kalman filter (EKF) consists in analytically propagating the estimation through the system dynamics, by means of a first-order Taylor expansion, around the last state vector estimate, of the functions defining the state-space representation of the system. However, as the approximation may not be sufficient to describe the non-linearity, the EKF may sometimes diverge. To solve this problem, a second-order linearization can be considered and leads to the second-order EKF (SOEKF) [1]. Another solution is to use the iterative extended Kalman filter (IEKF). In that case, the measurement model is linearized around the updated state vector, instead of the predicted state vector. Then, the process is iterated until the state vector estimate does not change much.

However, the above approaches require the computation of the Jacobian and the Hessian matrices for the first-order and the second-order linearizations respectively. Therefore, the sigma-point Kalman filter (SPKF) [2], namely the unscented Kalman filter (UKF) and the central difference Kalman filter (CDKF), can be considered.[1] The UKF is based on the unscented transformation whereas the CDKF is based on the second-order Sterling polynomial interpolation formula. Note that brand new alternatives such as quadrature Kalman filter and cubature Kalman fiter could be also considered.

To relax the assumptions on the model, interacting multiple model of the 2^{nd} and 3^{rd} generations using EKF and its variants can be considered.

Nevertheless in the above approaches, the additive noise and the model noise must be white, uncorrelated and Gaussian. To relax the Gaussian assumption on the additive noise and the model noise, particle filter can be used. However, despite its popularity it is not well suited when estimating fixed parameters. In addition the computational cost may be high. As an alternative, H_∞ filter is designed to be robust against uncertainties. Its purpose is to minimize the peak error power in the frequency domain whereas the KF minimizes the average error power [3]. No Gaussian assumption on the additive noise and the model noise in the state-space representation of the system is required. The H_∞ approach was introduced in the field of control in 1981 [4]. For the last 12 years, several studies based on a Ricatti-type equation have been conducted by the signal processing community. Thus in [5], instead of using a KF, Shen *et al.* suggest using an H_∞ filter to enhance a speech signal disturbed by an additive noise and recorded by a single microphone. For this purpose, the signal is assumed to be modeled by an AR process. However, the AR parameters are unknown and hence need to be estimated. Shen *et al.* propose to estimate them directly from the noisy observations by using a second H_∞ filter. However, the resulting AR parameter estimates are biased. To avoid this bias problem in [5], Labarre *et al.* [6] suggest estimating both the AR model and its parameters. Although this leads to a non-linear estimation issue, they have developed a structure based on two mutually interactive H_∞ filters. The first one aims at estimating the AR model, while the second one updates the estimation of the AR parameters. In addition, in [7], the authors take advantage of the two mutually-interactive H_∞ filters based approach to jointly estimate the fading channel and its AR parameters. However, the authors do not obtain better performance than a KF based method.

Researchers have taken advantage of the similarity (i.e. the Ricatti-type equation) between KF and H_∞ filtering to address the H_∞ filter based non-linear estimation issue in a more general case. Three main

[1]In that case, the state distribution is approximated by a Gaussian distribution, which is characterized by the so-called sigma points. Then, the sigma points are propagated through the non-linear system. A weighted combination of the resulting values makes it possible to estimate the mean and the covariance matrix of the transformed random variable.

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approaches have thus emerged:
1) Initial works about the "extended H_∞ filter" (E-H_∞) were conducted by Burl [8]. Like the EKF, it consists of a first-order linearization around the last available estimation of the state vector. Various authors have used the E-H_∞ filter. Thus, Giremus *et al.* [9] have studied its relevance in the field of the global positioning system (GPS) navigation. However, the authors do not obtain noticeable improvements in terms of positioning error in comparison with KF.
2) Like SOEKF, the second-order extended H_∞ filter has been recently proposed in [10].
3) The "unscented H_∞ filter" [11] is implemented by embedding the unscented transformation into the "E-H_∞" architecture. According to the authors, the unscented H_∞ filtering can be carried out by using the statistical linear error propagation approach [12]. Like the UKF, the unscented H_∞ filter avoids the linearization step by using an unscented transformation. It is used to approximate the matrices that are involved in the definition of the H_∞ gain. In [13], the unscented H_∞ filter provides an initial alignment of an inertial navigation system. The authors show that H_∞ filtering approach is an "effective" method when the measurement noise is colored.
Note that to relax the assumptions on the model, interacting multiple model based on H_∞ filter has been proposed in [14].

In this paper, our purpose is to compare the H_∞ filter and the KF in the linear case when estimating model parameters, by comparing the Ricatti equations of both algorithms. To our knowledge, there is no work dealing with this kind of comparison. More particularly we show that H_∞ filtering can be seen as KF with a different model noise covariance. This hence explains the performance of the H_∞ filter and its convergence properties. Note that the result we obtain can be easily derived in the non-linear case when using the EKF and the E-H_∞.
The paper is organized as follows: In section 2, we recall the state space representation of the system required by KF and H_∞ filtering. Section 3 is dedicated to the comparative study between the Kalman filter and the H_∞ filter. Finally section 4 gives an illustration.
In the following I_N is the identity matrix of size N.

2. ABOUT THE STATE SPACE REPRESENTATION, THE KF AND THE H_∞ FILTER

KF is based on a state-space representation of the system and uses two equations to describe the system, when the state-space equations are linear:

$$\mathbf{x}(n) = \mathbf{\Phi}(n-1)\mathbf{x}(n-1) + \mathbf{\Gamma}\mathbf{w}(n) \tag{1}$$

$$\mathbf{y}(n) = \mathbf{\Psi}(n)\mathbf{x}(n) + \mathbf{b}(n) \tag{2}$$

where $\mathbf{x}(n)$ is the state vector of size U at time n and $\mathbf{y}(n)$ is the measurement vector of size K at time n. The model noise $\mathbf{w}$ and the observation noise $\mathbf{b}$ are uncorrelated white zero-mean Gaussian vectors. In this paper the covariance matrices of $\mathbf{w}$ and $\mathbf{b}$ are assumed to be $\boldsymbol{\mathcal{Q}} = \sigma_w^2\mathbf{I}_U$ and $\boldsymbol{\mathcal{R}} = \sigma_b^2\mathbf{I}_K$ respectively. In addition, $\mathbf{\Phi}(n-1)$ is the transition matrix of size $U \times U$ from time $n-1$ to n, $\mathbf{\Gamma}$ is the input gain matrix of size $U \times U$ and $\mathbf{\Psi}(n)$ is the measurement matrix $K \times U$ at time n.

In a H_∞ setting, given the state-space model in (1) and (2), let us introduce a third state-space equation to focus on a linear combination of the state-vector components:

$$\mathbf{z}(n) = \boldsymbol{\mathcal{L}}\mathbf{x}(n) \tag{3}$$

where $\boldsymbol{\mathcal{L}}$ is a linear transformation operator that can be either a matrix of size $U \times U$ or a row vector of size U. $\mathbf{z}(n)$ is hence a vector or a scalar.
Given (1)-(3), H_∞ filtering provides the estimation of the state vector, by minimizing the H_∞ norm of the transfer operator that maps the discrete-time noise disturbances to the estimation error, as follows:

$$J^\infty = \sup \frac{\sum_{n=0}^{N^{ob}-1} \mathbf{e}(n)^2}{\boldsymbol{\mathcal{V}}^{-1}\sum_{n=0}^{N^{ob}-1}\|\mathbf{b}(n)\|^2 + \boldsymbol{\mathcal{W}}^{-1}\sum_{n=0}^{N^{ob}-1}\|\mathbf{w}(n)\|^2} \tag{4}$$

where N^{ob} denotes the number of available observations, $\mathbf{e}(n) = \mathbf{z}(n) - \hat{\mathbf{z}}(n)$, and $\boldsymbol{\mathcal{V}}$ and $\boldsymbol{\mathcal{W}}$ are positive weighting matrices tuned by the practitioner to achieve performance requirements.

To avoid the difficulties in minimizing (4), the following suboptimal H_∞ problem is usually considered:

$$J^\infty < \Xi^2 \tag{5}$$

where Ξ^2 is the prescribed noise attenuation level.

At that stage, $\mathbf{P}^\infty(n+1|n)$ satisfies the following Riccati equation for the H_∞ filter:

$$\begin{aligned}\mathbf{P}^\infty(n+1|n) &= \mathbf{\Phi}(n)\mathbf{P}^\infty(n|n)\mathbf{\Phi}^H(n) + \mathbf{\Gamma}\boldsymbol{\mathcal{W}}\mathbf{\Gamma}^H \\ &= \mathbf{\Phi}(n)\mathbf{P}^\infty(n|n-1)\{\mathbf{I}_U \\ &\quad - \begin{bmatrix} \mathbf{\Psi}^H(n) & \boldsymbol{\mathcal{L}}^H \end{bmatrix} \mathbb{M}^{-1} \begin{bmatrix} \mathbf{\Psi}(n) \\ \boldsymbol{\mathcal{L}} \end{bmatrix} \\ &\quad \times \mathbf{P}^\infty(n|n-1)\}\mathbf{\Phi}^H(n) + \mathbf{\Gamma}\boldsymbol{\mathcal{W}}\mathbf{\Gamma}^H\end{aligned} \tag{6}$$

where $\mathbb{M}$ is defined in (7) at the top of the next page.

When using the H_∞ filter, the state vector can be estimated recursively as follows:

$$\hat{\mathbf{x}}(n|n) = \hat{\mathbf{x}}(n|n-1) + \mathbf{K}^\infty(n)\tilde{\mathbf{y}}(n) \tag{8}$$

where $\hat{\mathbf{x}}(n|n-1)$ and $\hat{\mathbf{x}}(n|n)$ are respectively the *a priori* and *a posteriori* estimations of the state vector at time n and $\mathbf{K}^\infty(n)$ is the H_∞ filter gain defined as:

$$\mathbf{K}^\infty(n) = \{\mathbf{P}^{\mathbf{xy}\infty}(n)\}\{\mathbf{P}^{\mathbf{yy}\infty}(n)\}^{-1} \tag{9}$$

where $\mathbf{P}^{\mathbf{yy}\infty}(n)$ satisfies:

$$\mathbf{P}^{\mathbf{yy}\infty}(n) = \mathbf{\Psi}(n)\mathbf{P}^\infty(n|n-1)\mathbf{\Psi}^H(n) + \boldsymbol{\mathcal{V}} \tag{10}$$

and $\mathbf{P}^{\mathbf{xy}\infty}(n)$ is:

$$\mathbf{P}^{\mathbf{xy}\infty}(n) = \mathbf{P}^\infty(n|n-1)\mathbf{\Psi}^H(n) \tag{11}$$

Equ. (6) is true provided that:

$$\mathbf{P}^\infty(n+1|n)^{-1} + \mathbf{\Psi}^H(n)\mathbf{\Psi}(n) - \Xi^{-2}\boldsymbol{\mathcal{L}}^H\boldsymbol{\mathcal{L}} > 0 \tag{12}$$

3. KF VS H_∞ FILTERING

In the following, $\mathbf{K}(n)$ denotes the Kalman gain. In addition, $\mathbf{P}(n|n) = (I_U - \mathbf{K}(n)\mathbf{\Psi}(n))\mathbf{P}(n|n-1)$ and $\mathbf{P}(n|n-1)$ denote the *a posteriori* and *a priori* error covariance matrices when using a KF.
Remark: for the sake of space, we do not recall the equation of Kalman filtering as they are well known.

$$
\begin{aligned}
\mathbb{M} &= \begin{bmatrix} \boldsymbol{\mathcal{V}} & \mathbf{0} \\ \mathbf{0} & -\Xi^2 \mathbf{I}_U \end{bmatrix} + \begin{bmatrix} \mathbf{\Psi}(n) \\ \boldsymbol{\mathcal{L}} \end{bmatrix} \mathbf{P}^{\infty}(n|n-1) \begin{bmatrix} \mathbf{\Psi}^H(n) & \boldsymbol{\mathcal{L}}^H \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{\Psi}(n)\mathbf{P}^{\infty}(n|n-1)\mathbf{\Psi}^H(n) + \boldsymbol{\mathcal{V}} & \mathbf{\Psi}(n)\mathbf{P}^{\infty}(n|n-1)\boldsymbol{\mathcal{L}}^H \\ \boldsymbol{\mathcal{L}}\mathbf{P}^{\infty}(n|n-1)\mathbf{\Psi}^H(n) & \boldsymbol{\mathcal{L}}\mathbf{P}^{\infty}(n|n-1)\boldsymbol{\mathcal{L}}^H - \Xi^2 \mathbf{I}_U \end{bmatrix}
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\mathbb{M}^{-1} &= \begin{bmatrix} \mathbf{I}_K & -\mathbf{K}^H(n) \\ \underline{\mathbf{0}} & \mathbf{I}_U \end{bmatrix} \begin{bmatrix} \{\mathbf{P}^{\mathbf{yy}}(n)\}^{-1} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \mathbf{\Upsilon}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I}_K & \underline{\mathbf{0}} \\ -\mathbf{K}(n) & \mathbf{I}_U \end{bmatrix} \\
&= \mathbb{W}(n)\mathbb{X}(n)\mathbb{Y}(n)
\end{aligned} \tag{19}
$$

Let us first assume that:

$$\mathbf{P}^{\infty}(n|n-1) = \mathbf{P}(n|n-1) \tag{13}$$

and

$$\boldsymbol{\mathcal{L}} = \mathbf{I}_U \tag{14}$$

In addition, as it is often done when dealing with H_∞ filter in signal processing, let us set $\boldsymbol{\mathcal{W}}$ and $\boldsymbol{\mathcal{V}}$ to $\boldsymbol{\mathcal{Q}}$ and $\boldsymbol{\mathcal{R}}$ respectively. This implies that $\mathbf{P}^{\mathbf{yy}\infty}(n)$ introduced in (10) is equal to the innovation covariance matrix $\mathbf{P}^{\mathbf{yy}}(n)$ when using KF:

$$\mathbf{P}^{\mathbf{yy}\infty}(n) = \mathbf{P}^{\mathbf{yy}}(n) = \mathbf{\Psi}(n)\mathbf{P}(n|n-1)\mathbf{\Psi}^H(n) + \boldsymbol{\mathcal{R}} \tag{15}$$

Then, the H_∞ and Kalman gains are equal:

$$\mathbf{K}^{\infty}(n) = \mathbf{K}(n) = \mathbf{P}(n|n-1)\mathbf{\Psi}^H(n)(\mathbf{P}^{\mathbf{yy}}(n))^{-1} \tag{16}$$

Then, let us compare $\mathbf{P}^{\infty}(n+1|n)$ and $\mathbf{P}(n+1|n)$. For this purpose, let us use (6) where the matrix $\mathbb{M}$ must be inverted. Among the approaches to invert $\mathbb{M}$ such as the matrix inversion lemma, we suggest using the one based on the Schur complement [15] for the sake of simplicity [2]. Thus, let us define $\mathbf{\Upsilon}(n)$ the Schur complement of $\mathbf{P}^{\mathbf{yy}\infty}(n)$ in $\mathbb{M}$ as follows:

$$
\begin{aligned}
\mathbf{\Upsilon}(n) \;=\;& \{\mathbf{P}^{\infty}(n|n-1) - \Xi^2 \mathbf{I}_U\} \\
& - \; \mathbf{P}^{\infty}(n|n-1)\mathbf{\Psi}^H(n)\{\mathbf{P}^{\mathbf{yy}\infty}(n)\}^{-1} \\
& \times \mathbf{\Psi}(n)\mathbf{P}^{\infty}(n|n-1)
\end{aligned} \tag{17}
$$

Given (13) and (15) and the relation between $\mathbf{P}(n|n-1)$ and $\mathbf{P}(n|n)$, one obtains:

$$
\begin{aligned}
\mathbf{\Upsilon}(n) &= \mathbf{P}(n|n-1) \\
& \quad - \; \{\mathbf{P}(n|n-1)\mathbf{\Psi}^H(n)\{\mathbf{P}^{\mathbf{yy}}(n)\}^{-1}\}\mathbf{\Psi}(n) \\
& \quad \times \mathbf{P}(n|n-1) - \Xi^2 \mathbf{I}_U \\
&= \mathbf{P}(n|n-1) - \mathbf{K}(n)\mathbf{\Psi}(n)\mathbf{P}(n|n-1) - \Xi^2 \mathbf{I}_U \\
&= \mathbf{P}(n|n) - \Xi^2 \mathbf{I}_U
\end{aligned} \tag{18}
$$

[2] $\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$, then $\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{I}_K & -\mathbf{A}^{-1}\mathbf{B} \\ \underline{\mathbf{0}} & \mathbf{I}_U \end{bmatrix} \begin{bmatrix} \mathbf{A}^{-1} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \mathbf{S}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I}_K & \underline{\mathbf{0}} \\ -\mathbf{C}\mathbf{A}^{-1} & \mathbf{I}_U \end{bmatrix}$ where $\mathbf{A}$ is a matrix of size $K \times K$, $\mathbf{B}$ is a matrix of size $K \times U$, $\mathbf{C}$ is a matrix of size $U \times K$, $\mathbf{D}$ is a matrix of size $U \times U$, $\mathbf{S} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ is the Schur complement of $\mathbf{A}$ in $\mathbf{M}$ and $\underline{\mathbf{0}}$ is a zero matrix.

Then, $\mathbb{M}^{-1}$ can be expressed as the product of three matrices, the coefficients of which are defined from the coefficients of M and the Schur complement $\mathbf{\Upsilon}(n)$. See (19) at the top.

At that stage, given (19), we can rewrite the Riccati recursion (6) as:

$$
\begin{aligned}
\mathbf{P}^{\infty}(n+1|n) &= \mathbf{\Phi}(n)\mathbf{P}(n|n-1)\mathbf{\Phi}^H(n) + \mathbf{\Gamma}\boldsymbol{\mathcal{Q}}\mathbf{\Gamma}^H \\
& \quad -\mathbf{\Phi}(n)\mathbf{P}(n|n-1) \begin{bmatrix} \mathbf{\Psi}^H(n) & \mathbf{I}_U \end{bmatrix} \mathbb{W}(n)\mathbb{X}(n)\mathbb{Y}(n) \\
& \quad \times \begin{bmatrix} \mathbf{\Psi}(n) \\ \mathbf{I}_U \end{bmatrix} \mathbf{P}(n|n-1)\mathbf{\Phi}^H(n) \\
&= \mathbf{\Phi}(n)\mathbf{P}(n|n-1)\mathbf{\Phi}^H(n) + \mathbf{\Gamma}\boldsymbol{\mathcal{Q}}\mathbf{\Gamma}^H \\
& \quad -\mathbf{\Phi}(n) \begin{bmatrix} \mathbf{P}(n|n-1)\mathbf{\Psi}^H(n) & \mathbf{P}(n|n) \end{bmatrix} \\
& \quad \times \begin{bmatrix} \{\mathbf{P}^{\mathbf{yy}}(n)\}^{-1} & 0 \\ 0 & \mathbf{\Upsilon}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{\Psi}(n)\mathbf{P}(n|n-1) \\ \mathbf{P}(n|n) \end{bmatrix} \mathbf{\Phi}^H(n)
\end{aligned} \tag{20}
$$

Using (18), this leads to:

$$
\begin{aligned}
\mathbf{P}^{\infty}(n+1|n) &= \mathbf{\Phi}(n)\mathbf{P}(n|n-1)\mathbf{\Phi}^H(n) + \mathbf{\Gamma}\boldsymbol{\mathcal{Q}}\mathbf{\Gamma}^H \\
& \quad -\mathbf{\Phi}(n) \left[\mathbf{P}(n|n-1)\mathbf{\Psi}^H(n)\{\mathbf{P}^{\mathbf{yy}}(n)\}^{-1} \;\; \mathbf{P}(n|n)\mathbf{\Upsilon}^{-1}\right] \\
& \quad \times \begin{bmatrix} \mathbf{\Psi}(n)\mathbf{P}(n|n-1) \\ \mathbf{P}(n|n) \end{bmatrix} \mathbf{\Phi}^H(n) \\
&= \mathbf{\Phi}(n)\mathbf{P}(n|n-1)\mathbf{\Phi}^H(n) \\
& \quad -\mathbf{\Phi}(n)\mathbf{K}(n)\mathbf{\Psi}(n)\mathbf{P}(n|n-1)\mathbf{\Phi}^H(n) \\
& \quad -\mathbf{\Phi}(n)\mathbf{P}(n|n)\mathbf{\Upsilon}^{-1}\mathbf{P}(n|n)\mathbf{\Phi}^H(n) + \mathbf{\Gamma}\boldsymbol{\mathcal{Q}}\mathbf{\Gamma}^H \\
&= \mathbf{\Phi}(n)\mathbf{P}(n|n)\mathbf{\Phi}^H(n) - \mathbf{\Phi}(n)\mathbf{P}(n|n)\{\mathbf{P}(n|n) - \Xi^2 \mathbf{I}_U\}^{-1} \\
& \quad \times \mathbf{P}(n|n)\mathbf{\Phi}^H(n) + \mathbf{\Gamma}\boldsymbol{\mathcal{Q}}\mathbf{\Gamma}^H
\end{aligned} \tag{21}
$$

Hence, the solution of the Riccati equation when using the H_∞ filter satisfies:

$$
\begin{aligned}
\mathbf{P}^{\infty}(n+1|n) &= \mathbf{\Phi}(n)\mathbf{P}(n|n)\mathbf{\Phi}^H(n) \\
& \quad + \boldsymbol{\mathcal{Q}}^{\Xi}(n) + \mathbf{\Gamma}\boldsymbol{\mathcal{Q}}\mathbf{\Gamma}^H \\
&= \mathbf{\Phi}(n)\mathbf{P}(n|n)\mathbf{\Phi}^H(n) + \boldsymbol{\mathcal{Q}}^{\Xi w}
\end{aligned} \tag{22}
$$

where

$$\boldsymbol{\mathcal{Q}}^{\boldsymbol{\Xi}}(n) = -\mathbf{\Phi}(n)\mathbf{P}(n|n)\{\mathbf{P}(n|n) - \Xi^2 \mathbf{I}_U\}^{-1}\mathbf{P}(n|n)\mathbf{\Phi}^H(n) \tag{23}$$

Given (22), one has:

$$\mathbf{P}^{\infty}(n+1|n) = \mathbf{P}(n+1|n) + \boldsymbol{\mathcal{Q}}^{\Xi}(n) \tag{24}$$

Therefore, the H_∞ filter can be seen as a KF with a model-noise covariance matrix equal to $\boldsymbol{\mathcal{Q}}^{\Xi w} = \boldsymbol{\mathcal{Q}}^{\Xi}(n) + \mathbf{\Gamma}\boldsymbol{\mathcal{Q}}\mathbf{\Gamma}^H$

$= -\mathbf{\Phi}(n)\mathbf{P}(n|n)\{\mathbf{P}(n|n) - \Xi^2\mathbf{I}_U\}^{-1}\mathbf{P}(n|n)\mathbf{\Phi}^H(n) + \mathbf{\Gamma}\mathbf{\mathcal{Q}}\mathbf{\Gamma}^H$.
For parameter tracking, the larger the coefficients of the state-noise covariance matrix are, the easier it is to track the parameter variations, especially when the parameters are subject to abrupt variations. Nevertheless, the larger they are, the larger the variance of the estimated parameters over time is.
To analyze how the H_∞ filter behaves in comparison with KF, let us introduce the eigenvalue decomposition of $\mathbf{P}(n|n)$:

$$\mathbf{P}(n|n) = \mathbf{G}\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_U \end{bmatrix}\mathbf{G}^{-1}$$

where $\{\lambda_u\}_{u\in\{1,2,\ldots,U\}}$ are the eigenvalues of $\mathbf{P}(n|n)$. Hence, one has:

$$\mathbf{P}(n|n) - \Xi^2\mathbf{I}_U = \mathbf{G}\begin{bmatrix} \lambda_1 - \Xi^2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_U - \Xi^2 \end{bmatrix}\mathbf{G}^{-1}$$

Therefore:

$$\mathbf{\mathcal{Q}}^\Xi(n) = \bar{\mathbf{\Phi}}(n)\begin{bmatrix} -\frac{{\lambda_1}^2}{\lambda_1 - \Xi^2} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\frac{{\lambda_U}^2}{\lambda_U - \Xi^2} \end{bmatrix}(\bar{\mathbf{\Phi}}(n))^H \quad (25)$$

where $\bar{\mathbf{\Phi}}(n) = \mathbf{\Phi}(n)\mathbf{G}$ and $\{-\frac{{\lambda_u}^2}{\lambda_u - \Xi^2}\}_{u\in\{1,2,\ldots,U\}}$ are the eigenvalues of $\mathbf{\mathcal{Q}}^\Xi(n)$. It should be noted that when Ξ tends to $+\infty$, $\mathbf{\mathcal{Q}}^\Xi(n)$ tends to be a zero matrix and $\mathbf{P}^\infty(n+1|n)$ tends to $\mathbf{P}(n+1|n)$.
If $\Xi^2 > \lambda_u$, for $u \in \{1, 2, \ldots, U\}$, $\mathbf{\mathcal{Q}}^\Xi(n)$ is a positive-definite matrix. The solution to the Ricatti equation for the H_∞ filter can be seen as an upper bound of the Kalman *a priori* error covariance matrix. The same kind of consideration can be done when comparing the EKF and the E-H_∞.

4. ILLUSTRATION

In this section, we carry out a comparative study between variants of the KF and the H_∞ filter to estimate the carrier frequency offset (CFO) in mobile communication. Indeed, despite the great success of the code division multiple access (CDMA) as a multiple access technique, the current trend is to use the orthogonal frequency division multiple access (OFDMA). In that case, the input data stream is split into a number of streams that are transmitted in parallel over a large number of orthogonal subcarriers. The frequency-selective fading over the entire bandwidth of the transmitted signal has hence the advantage of being converted in frequency-flat fading over each subcarrier. These schemes are particularly well adapted to mobile wireless communication to provide high-data rate services.
However, there are two unwanted phenomena. 1/ the channel remains unknown and hence has to be estimated. 2/ the relative transmitter-receiver motion and a difference between the local-oscillator (LO) frequencies at the transmitter and the receiver lead to a CFO. These CFOs that affect the received signal no longer guarantee orthogonality between subcarriers. To avoid the resulting intercarrier interference (ICI), a CFO estimation and correction step must be introduced at the receiver by using a training sequence (i.e. a sequence known both at the transmitter and the receiver).
In what follows, an orthogonal frequency division multiple access (OFDMA) IEEE 802.16 WirelessMAN™ uplink system composed of $U = 4$ users sharing $K = 128$ subcarriers is considered. A transmission over a Rayleigh slow-fading frequency-selective channel composed of $L_u = 3\ \forall u$ multipaths is supposed and a cyclic prefix $N_g = K/8 \geq \underset{u}{max}(L_u)$ is added to the OFDMA symbol. BPSK is used to modulate the information bits. The carrier frequency is at $f_c = 2.6$GHz and the bandwidth is set to $W = 20$MHz. The users' normalized carrier frequency offset (CFO) errors are set to $\epsilon_u = 0.02\ \forall u$. Then, let us introduce SNR$= 10\log(\frac{\sigma_u^2}{\sigma_B^2})$, where σ_u^2 is the mean power of the received signal from the uth user.
Let us define the vector $\boldsymbol{\epsilon}$ of size U and the vector $\mathbf{h}$ of size $L = \sum_{u=1}^{U} L_u$ by storing the CFOs and the channel state informations (CSIs) of the U users in the system as follows:

$$\boldsymbol{\epsilon} = [\epsilon_1, \epsilon_2, \ldots, \epsilon_u, \ldots, \epsilon_U] \quad (26)$$
$$\mathbf{h} = [\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_u, \ldots, \mathbf{h}_U] \quad (27)$$

where $\mathbf{h}_u = [h_u(0), h_u(1), \ldots, h_u(l), \ldots, h_u(L_u - 1)]$.
Let us define the vector $\mathbf{x}(n)$ of size $\mathcal{U} = U + 2L$, which is the state vector of the system:

$$\mathbf{x}(n) = \begin{bmatrix} \boldsymbol{\epsilon}(n) & Re\{\mathbf{h}(n)\} & Im\{\mathbf{h}(n)\} \end{bmatrix}^T \quad (28)$$

Now, let us introduce the state-space representation of the system:
State equation:

$$\mathbf{x}(n) = \mathbf{x}(n-1) + \mathbf{w}(n) \quad \forall \quad n \in [-N_g, K-1] \quad (29)$$

Measurement equation:

$$R(n) = \sum_{u=1}^{U} e^{j2\pi\frac{\epsilon_u n}{K}} \sum_{l=0}^{L_u-1} h_u(l)X_u(n-l) + \mathbf{b}(n) \quad (30)$$

where $\mathbf{w}(n)$ is an AWGN matrix with zero-mean and covariance matrix $\sigma_w^2\mathbf{I}_\mathcal{U}$, $\mathbf{b}(n)$ is an AWGN with zero-mean and variance σ_b^2, $X_u(n)$ is the nth sample of the transmitted OFDMA symbol and $-N_g \leq n \leq K-1$.
When using H_∞ filtering: $\Xi = 10^2$, $\boldsymbol{\mathcal{V}} = \frac{\sigma_b^2}{2}\mathbf{I}_2$ and $\boldsymbol{\mathcal{W}} = \mathbf{I}_\mathcal{U}\sigma_w^2$. One assumes that there is a state noise $\mathbf{w}(n)$ with a very small variance, e.g. $\sigma_w^2 = 10^{-3}$. For the CFO, the initialization parameters of the algorithm is $\hat{\epsilon}_u(0) = 0\ \forall u$.
Figure 1 shows that the IEKF converges faster than the other Kalman approaches. This is due to the fact that the measurement model is linearized around the updated state vector, instead of the predicted state vector.
Remark: when the noise characteristics are available, the EKF provides quite similar results in comparison with the "E-H_∞" and there is no real difference between the SPKF and the "unscented H_∞ filter". For that reason, in figure 1 we only show the results of the Kalman filter based approaches.
In real cases, the variance of the additive is not necessarily exactly known. Therefore, we suggest studying the robustness of the Kalman and H_∞ filter based methods against uncertainties. Figure 2 shows the robustness of the H_∞ filtering taken the variance of the additive noise is not known. For an error of 5dB over the variance

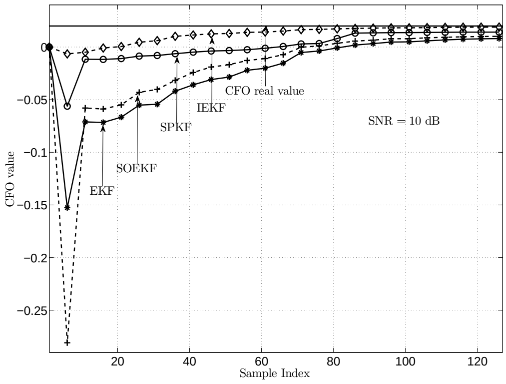

Fig. 1. Recursive CFO estimation for a joint CFO/channel estimation using optimal filtering, when the noise statistics are available

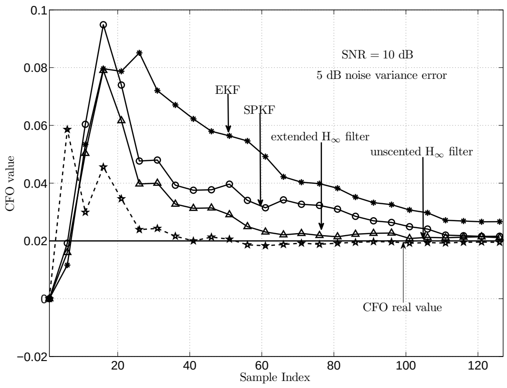

Fig. 2. Comparison between the Kalman filtering approaches and the H_∞ filtering approaches in terms of convergence speed, when the noise statistics are not available

of the additive noise, H_∞ filtering based approaches converge faster than the Kalman ones. However, the computational complexity of the H_∞ filtering based approaches is higher than the SPKF. It should be noted that when using the extended H_∞ and the unscented H_∞ approaches, the choice of the noise attenuation level Ξ plays a key role. If it is set to a high value, there is no real difference between the Kalman algorithm and the H_∞ based approach, whereas no H_∞ solution may exist if Ξ is set to a small value.

5. CONCLUSIONS AND PERSPECTIVES

Like Kalman filter, H_∞ filter requires matrices to be *a priori* defined. Thus, in the linear case, the transition and the measurement matrices must be *a priori* known. In addition, a prescribed noise attenuation level must be set by the practitionner. It must be suitably chosen to guarantee the existence of the filter.

The theoretical comparative study presented in the paper will be the basis of a sensitivity analysis of both H_∞ and Kalman filters. H_∞ filtering is a compromise between convergence, accuracy and computational cost. The key issue is the selection of the noise attenuation level. Concerning the estimation of both the channel and the CFO in an OFDMA uplink communication system, IEKF is the KF based approaches that converges the faster.

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