# SPARSE BLIND DECONVOLUTION BASED ON SCALE INVARIANT SMOOTHED $\ell_0$-NORM

*Kenji Nose-Filho*[1], *Christian Jutten*[2], *João M. T. Romano*[1]

[1] School of Electrical and Computer Engineering, University of Campinas (UNICAMP), Brazil
[2] GIPSA-lab, University Joseph Fourier (UJF), France
kenjinose@gmail.com, christian.jutten@gipsa-lab.grenoble-inp.fr, romano@decom.fee.unicamp.br

## ABSTRACT

In this work, we explore the problem of blind deconvolution in the context of sparse signals. We show that the $\ell_0$-norm works as a contrast function, if the length of the impulse response of the system is smaller than the shortest distance between two spikes of the input signal. Demonstrating this sufficient condition is our basic theoretical result. However, one of the problems of dealing with the $\ell_0$-norm in optimization problems is the requirement of exhaustive or combinatorial search methods, since it is a non continuous function. In order to propose an alternative for that, Mohimani et al. (2009) proposed a smoothed and continuous version of the $\ell_0$-norm. Here, we propose a modification of this criterion in order to make it scale-invariant and, finally, we derive a gradient-based algorithm for the modified criterion. Results with synthetic data suggests that the imposed conditions are sufficient but not strictly necessary.

***Index Terms*—** Blind Deconvolution, Smoothed $\ell_0$-norm, Sparse Signals

## 1. INTRODUCTION

Deconvolution plays a fundamental role in signal processing. Its applications vary from seismic reflection [1–4], ultrasonic inspection [5], telecommunications [6] and optics. In this work, we deal with blind deconvolution scenario, in which the impulse response of the convolution system, as well as the input signal, is unknown. This is an ill-posed problem and some considerations are usually made for its regularization [1, 2, 7, 8].

In this paper, we explore the assumption that the input signal $\mathbf{s}(n)$ is sparse, i.e., composed of a few spikes of unknown amplitude and position, separated by zero-terms [2–4].

For this, we propose a scale invariant version of the smoothed $\ell_0$-norm [9]. The $\ell_0$-norm is a common metric used to quantify sparse signals, since it corresponds to the number of non-zero elements of a given vector. Formally speaking, it is actually not a norm because it is not positive homogeneous. On the other hand, it satisfies the triangle inequality and separates point properties.

The $\ell_0$-norm of a vector $\mathbf{s}$, denoted by $\|\mathbf{s}\|_0$, is given by,

$$\|\mathbf{s}\|_0 = N - \sum_{i=0}^{N-1} \mathrm{E}_0(s_i), \tag{1}$$

where $\mathrm{E}_0(s_i)$ is the indicator function on the set $\mathrm{E}_0 = \{s_i | s_i \in \mathbb{R}, s_i = 0\}$.

Having a system with finite impulse response (FIR), with length smaller than the shortest distance between two spikes in the input vector $\mathbf{s}$, it is possible to demonstrate that the $\ell_0$-norm is a constrast function. This is based on the idea that the convolution of a sparse signal with the impulse response of a linear system will result in a less sparse signal. Thus, deconvolution can be performed by adjusting the coefficients of a deconvolution filter in order to maximize the sparsity of the deconvolved signal, e.g., by minimizing its $\ell_0$-norm. Although this idea has been widely explored by several methods, a sufficient condition for that has not yet been demonstrated.

The major drawback of dealing with the $\ell_0$-norm in optimization problems, is that it is a non-continuous function. Then, the problem solution requires exhaustive or combinatorial search methods. Also, in some applications it is common to consider nearly zero-terms as zeros, and this is not taken into account in the $\ell_0$-norm. In order to solve those problems, we propose a scale invariant version of the smoothed $\ell_0$-norm [9] and derive a gradient-based algorithm for this modified criterion.

The paper is organized as follows: in Section 2 we formalize the deconvolution problem and state our notation. In Section 3 we propose a new theorem in order to set the conditions in which the $\ell_0$-norm works as a constrast function. In Section 4 we present our simulation results and compare the proposed approach with other techniques, such that Least Squares (LS) [6], Prediction Error Filter (PEF) [1], Minimum Entropy Deconvolution (MED) [2] and MED with an exponential transformation [3]. Finally, in section 5 we state our conclusions.

K. Nose-Filho thanks to CNPq process 142714/2011-9 and CAPES process 7362-13-7 for funding his doctoral research.

## 2. THE CONVOLUTION/DECONVOLUTION PROBLEM

First, let us state the problem and set our notation. The discrete-time noisy convolution model [6] is given by

$$\begin{aligned} x(n) &= h(n) * s(n) + \nu(n), \\ &= \sum_{k=0}^{L_h-1} h_k s(n-k) + \nu(n), \end{aligned} \tag{2}$$

where the symbol $*$ stands for the discrete-time convolution, $s(n)$ is the input signal, $x(n)$ is the output or observed signal, $h(n)$ is the system impulse response and $\nu(n)$ is the additive noise. In vector notation, they are represented by $\mathbf{s} = [s_0\, s_1\, \ldots\, s_{N-1}]^T$, $\mathbf{x} = [x_0\, x_1\, \ldots\, x_{N-1}]^T$, $\mathbf{h} = [h_0\, h_1\, \ldots\, h_{L_h-1}]^T$ and $\nu = [\nu_0\, \nu_1\, \ldots\, \nu_{N-1}]^T$ respectively.

Deconvolution can be carried out by applying the observed signal $x(n)$ into an inverse filter that aims to recover the original input signal:

$$\begin{aligned} y(n) &= w(n) * x(n), \\ &= \sum_{k=0}^{L_w-1} w_k x(n-k), \end{aligned} \tag{3}$$

where $w(n)$ is the impulse response of the deconvolution filter and $y(n)$ is the recovered signal. In the noiseless case, perfect deconvolution is said to be achieved when

$$y(n) = cs(n-d), \tag{4}$$

where $c$ is a constant scalar and $d$ is a discrete-time delay introduced by the deconvolution filter. In this case, the global response of the system is given by

$$g(n) = h(n) * w(n) = [0, \ldots, 0, c, 0, \ldots, 0]^T, \tag{5}$$

which is also known as the zero-forcing condition.

In the following, we present a theorem which states that, under certain conditions, the $\ell_0$-norm is a contrast function for sparse blind deconvolution.

## 3. $\ell_0$-NORM DECONVOLUTION

After stating our notation, we present a sufficient condition for having the $\ell_0$-norm as a contrast function for blind deconvolution.

**Theorem 1** *Let us consider a signal* $\mathbf{s}$ *composed of a few spikes of unknown amplitude and position, separated by zero terms, and a signal* $\mathbf{g}$ *with at least one non-zero element and length* $L_g$ *smaller than the shortest distance between two spikes in* $\mathbf{s}$*. Then, the convolution of these two signals will result in a signal* $\mathbf{y}$ *with* $\|\mathbf{y}\|_0 \geq \|\mathbf{s}\|_0$*. Equality holds if, and only if,* $\mathbf{g}$ *has a single spike.*

**Proof** The assumption that the distances between the spikes in $\mathbf{s}$ are greater than $L_g$ means that the system response to each of these spikes does not overlap. Thus, the number of non-zero elements of $\mathbf{y}$ is equal to the number of non-zero elements of $\mathbf{s}$ times the number of non-zero elements of $\mathbf{g}$, i.e.,

$$\|\mathbf{y}\|_0 = \|\mathbf{g}\|_0 \|\mathbf{s}\|_0. \tag{6}$$

Since $\mathbf{g}$ has at least one non-zero element, then

$$\|\mathbf{y}\|_0 \geq \|\mathbf{s}\|_0, \tag{7}$$

and equality holds if, and only if, $\mathbf{g}$ has the form

$$\mathbf{g} = [0, \ldots, 0, c, 0, \ldots, 0], \tag{8}$$

where $c$ is a constant scalar different from zero. □

In other words, convolution implies in reduce, or at most conserve, the sparsity degree of the input signal.

In the blind deconvolution problem, we consider that $\mathbf{g}$ is the impulse response of the global convolution-deconvolution system. Let us also consider that the convolution system is a possibly non-minimum phase, linear-time invariant (LTI), FIR filter, the transfer function of which has no zeros on the unit circle [8]. Finally, the deconvolution filter is assumed to be of sufficient length, so that truncation effects can be ignored.

Hence, in the following, we present the smoothed $\ell_0$-norm and the gradient-based algorithm.

### 3.1. The Smoothed $\ell_0$-Norm

Usually, it is desirable to have continuous and smooth criteria for optimization problems. However, the $\ell_0$-norm is neither continuous nor smooth, requiring exhaustive or combinatorial search methods for its solution. Also, it does not take account that small values can be considered as null-elements, which is common in some real applications. For solving undetermined system of linear equations, a common strategy is to replace the $\ell_0$-norm by the $\ell_1$-norm [10], resulting in a continuous function which is not smooth. In order to obtain a continuous and smooth criterion, Mohimani et al. [9] proposed a smoothed version of the $\ell_0$-norm, defined by

$$F_\sigma(\mathbf{y}) = N - \sum_{i=0}^{N-1} \exp\left(\frac{-y_i^2}{2\sigma^2}\right). \tag{9}$$

This criterion, is based on an exponential transformation proposed by Ooe and Ulrych in 1979 [3], which is given by

$$f_\sigma(y_i) = 1 - \exp\left(\frac{-y_i^2}{2\sigma^2}\right). \tag{10}$$

In their work, they applied this transformation in a variant form of the varimax criterion.

In advance with the commonly $\ell_1$-norm criterion, it has a tuning parameter $\sigma$ which controls it smoothness, making it equal to the $\ell_0$-norm

$$F_\sigma(\mathbf{y}) = \|\mathbf{y}\|_0, \text{for } \sigma = 0, \tag{11}$$

and proportional to the $\ell_2$-norm

$$F_\sigma(\mathbf{y}) = \frac{\|\mathbf{y}\|^2}{2\sigma^2}, \text{for } \sigma >> y, \tag{12}$$

where $\|\mathbf{y}\| = \sqrt{\sum_{i=0}^{N-1} y_i^2}$.

However, blind deconvolution is an ill-posed problem and some constraints are usually necessary for its regularization. The most common is to impose that the input and the output signal have the same power [8]. This constraint can be incorporated to the cost function by making it scale invariant, which results in

$$\hat{F}_\sigma(\mathbf{y}) = N - \sum_{i=0}^{N-1} \exp\left(\frac{-y_i^2}{2\sigma^2\|\mathbf{y}\|^2}\right). \tag{13}$$

Since it is a non-convex function, initialization must be taken into account. In Section 4, we show that good results can be obtained by initializing the deconvolution filter with a spike located accordingly to the phase of the convolution system. For example, for minimum phase systems, best results are obtained by initializing the deconvolution filter with a single spike at the beginning of the system. For mixed-phase systems, it is advisable to initialize with a single spike located at the middle of the system and in the case of a maximum phase system, with a spike at the end.

### 3.2. A Gradient-Based Algorithm for Sparse Blind Deconvolution

A gradient-based algorithm for blind deconvolution can be obtained by deriving the cost function with respect to the coefficients of the deconvolution filter. In this case,

$$\frac{\partial \hat{F}_\sigma(\mathbf{y})}{\partial w_k} = -\sum_{i=0}^{N-1} \hat{f}(y_i) \frac{\partial\left(\frac{-y_i^2}{2\sigma^2\|\mathbf{y}\|2}\right)}{\partial w_k}, \tag{14}$$

where $\hat{f}(y_i) = \exp\left(\frac{-y_i^2}{2\sigma^2\|\mathbf{y}\|2}\right)$.

Making $u_i = -y_i^2$ and $v = 2\sigma^2\|\mathbf{y}\|^2$, the derivative of $u_i/v$ with respect to $w_k$ is,

$$\frac{\partial \frac{u_i}{v}}{\partial w_k} = \frac{1}{v}\frac{\partial u_i}{\partial w_k} - \frac{u_i}{v^2}\frac{\partial v}{\partial w_k}, \tag{15}$$

where

$$\frac{\partial u_i}{\partial w_k} = -2y_i x_{i-k}, \tag{16}$$

and

$$\frac{\partial v}{\partial w_k} = 4\sigma^2 \sum_{i=0}^{N-1} y_i x_{i-k}. \tag{17}$$

Finally, the gradient is given by

$$\begin{aligned} \frac{\partial \hat{F}_\sigma(\mathbf{y})}{\partial w_k} \quad = \quad & \frac{2}{v}\left(\sum_i^{N-1} \hat{f}(y_i) y_i x_{i-k}\right) - \\ & \frac{4\sigma^2}{v^2}\left(\sum_i^{N-1} y_i x_{i-k}\right)\left(\sum_i^{N-1} \hat{f}(y_i) y_i^2\right), \end{aligned} \tag{18}$$

and the adaptation of $\mathbf{w}$ by

$$\begin{aligned} \mathbf{w}^{m+1} &= \mathbf{w}^m - \mu \frac{\nabla \hat{F}_\sigma(\mathbf{y}^m)}{\|\nabla \hat{F}_\sigma(\mathbf{y}^m)\|}, \\ \mathbf{w}^{m+1} &= \frac{\mathbf{w}^{m+1}}{\|\mathbf{w}^{m+1}\|}, \end{aligned} \tag{19}$$

where normalization is performed in order to improve the convergence of the algorithm [11] and $m$ denotes the iteration index. Also, to avoid local minima, we adopt the same strategy used in [9] by adapting the parameter $\sigma$ at each iteration. We start using a big value for $\sigma^0 = \sigma_{\text{sup}}$ and then, after each iteration, we decrease its value of $\sigma^{m+1} = \sigma^m - \delta$, where $\delta = \frac{\sigma_{\text{sup}} - \sigma_{\text{inf}}}{M}$ and $\sigma_{\text{inf}}$ is the desirable value for $\sigma$ after $M$ iterations. In this case, convergence can be assured for significantly small values of $\delta$ and $\mu$ such that the minimization of $\hat{F}_{\sigma^m}(\mathbf{y})$ implies in the minimization of $\hat{F}_{\sigma^{m+1}}(\mathbf{y})$.

In the next section, we are going to present some results obtained with the proposed method.

## 4. RESULTS

The proposed method, which we are going to refer as S$\ell_0$, is going to be compared with several algorithms, such as:

- Least-Squares (LS) Filter [6]. It is a supervised technique which is used in order to illustrate the best deconvolution filter (in terms of least-squares) that is possible to be obtained if the input sequence is available.
- Prediction Error Filter (PEF) or Wiener filtering [1]. It is the classical method for blind deconvolution if the input sequence is white and the convolution system is minimum phase. For non-minimum phase systems it is quite useless.
- Minimum Entropy Deconvolution (MED) [2]. It is a classical method for blind deconvolution of sparse signals based on the maximization of the varimax criterion.
- Minimum Entropy Deconvolution with an exponential transformation (MED-ex) [3]. It is a method derived from MED that uses the exponential transformation, given by (10), in the varimax criterion. In this case, we used $\sigma = \max\{\mathbf{y}(n)\}/2$, as proposed in [3].

The major advantage of the MED and the MED-ex algorithms are their fast convergence, which can be achieved around 20 to 50 iterations [2, 3].

Results are presented in two different scenarios. First, a scenario where the sufficient conditions, for having the $\ell_0$-norm as a contrast function, are respected. Second, a scenario where these conditions are relaxed.

For both scenarios, we considered a non-minimum phase system, which impulse response is illustrated by the second signal, from top to botton, in Figure 1. Results are compared in terms of the Pearson correlation coefficient, given by

$$\frac{\sum_{i=0}^{N-1} \tilde{y}_i s_i}{\|\tilde{\mathbf{y}}\| \|\mathbf{s}\|}, \tag{20}$$

where $\tilde{\mathbf{y}}$ is the estimated signal with corrected lag delay.

### 4.1. Scenario 1

In the first scenario, the input signal is composed of three spikes, equally separated by 60 null samples. The convolution system is mixed-phase of length $L_h = 21$ and for all methods we considered a deconvolution filter of length $L_w = 20$. For the proposed method, we considered $\mu = 0.01$, $M = 5000$, $\sigma_{\text{sup}} = 4$ and three different values for $\sigma_{\text{inf}}$, given by 0.5, 2 and 4. Since it is known a-priori that the convolution system is mixed phase, the initialization was made with a single spike at the middle of the deconvolution filter, more specifically, in the eleventh sample. In Figure 1 we illustrate the results obtained with the different methods. From top to bottom: the input signal, the impulse response of the system, the convolved signal, the deconvolved signal obtained by: LS, PEF, MED, MED-ex and S$\ell_0$ with $\sigma_{\text{inf}} = 0.5$, 2, 4. The Pearson correlation coefficients are given by Table 1.

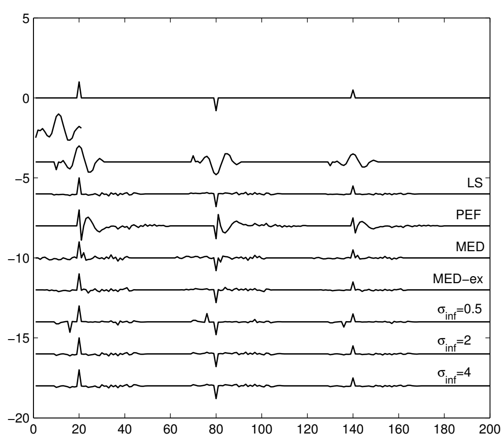


**Fig. 1**. Results obtained for the first scenario. For description, see the text.

Given Figure 1 and Table 1 it is possible to observe that the results obtained with S$\ell_0$ with $\sigma_{\text{inf}} = 2$ and 4 and with MED-ex, which are unsupervised methods, are comparable to the LS approach, which is a supervised method. Also, these

**Table 1**. Pearson correlation coefficient for the first scenario.

| LS | PEF | MED | MED-ex, $\sigma = \max\{\mathbf{y}(n)\}/2$ | S$\ell_0$, $\sigma_{\text{inf}} =$ 0.5 | 2 | 4 |
|---|---|---|---|---|---|---|
| 0.94 | 0.58 | 0.86 | 0.93 | 0.81 | 0.94 | 0.94 |

results are better than the PEF and MED. In this case, where truncation effects are considerable high, it is possible to see that small values for $\sigma_{\text{inf}}$ it is not so good, since it becomes too polarized by the $\ell_0$-norm, and, consequently, does not distinguish between small and big amplitudes.

In the following, in order to verify the robustness of the proposed method, we present some results in more realistic scenarios.

### 4.2. Scenario 2

In the second scenario, the input signal is modeled by a zero mean, unit variance Bernoulli-Gaussian random variable, with probability $P_x$ of non-zero occurrence, plus a Gaussian random variable with zero mean and variance $\sigma_x^2$. The addition of the Gaussian random variable is made in order to produce a more realistic signal, in which the spikes are separated by nearly zero-terms and not exactly zero-terms. Also, we considered the presence of additive white Gaussian noise (AWGN).

For the input signal, it was considered three levels of sparsity (1) a signal with $P_x = 0.1$ and $\sigma_x^2 = 0.01$; (2) a signal with $P_x = 0.2$ and $\sigma_x^2 = 0.02$; and (3) a signal with $P_x = 0.3$ and $\sigma_x^2 = 0.03$. Observe that, $P_x$ controls the degree of sparsity of the signals. The bigger is its value, less sparse is the signal. Also, $\sigma_x^2$ controls the variance of the small coefficients that are between the spikes given by the Bernoulli Gaussian distribution. For the AWGN, it was considered four different signal-to-noise ratios (SNR).

In order to obtain the average performance curves, in terms of the Pearson correlation coefficient, for each algorithm, it was performed 100 Monte Carlo simulations (MC). These curves are illustrated by Figure 2. To improve the visualization, we are going to omit, from this figure, the results obtained with S$\ell_0$ with $\sigma_{\text{inf}} = 0.5$ and 4 and the results obtained with the PEF. In addition, we are going to include the Pearson correlation coefficient of the observed signal $\mathbf{x}(n)$, which illustrates the performance if no deconvolution was performed.

In Figure 3 we present the results of one realization of the MC simulations, for $P_x = 0.3$, $\sigma_x^2 = 0.03$ and SNR = 12dB. From top to bottom: the input signal, the impulse response of the system, the convolved signal, the deconvolved signal obtained by: LS, PEF, MED, MED-ex and S$\ell_0$ with $\sigma_{\text{inf}} = 0.5$, 2, 4.

Finally, the Pearson correlation coefficients for one realization of the MC simulations are given by Table 2.

Given Figure 2 it is possible to observe that, the decrease

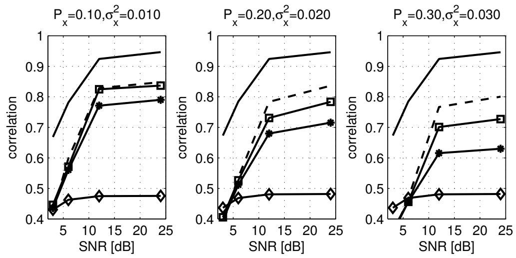


**Fig. 2**. Performance curves for the second scenario: (solid) LS, (dashed) proposed method for $\sigma_{\text{inf}} = 2$, (asterix) MED, (square) MED-ex and (diamond) the convolved signal.

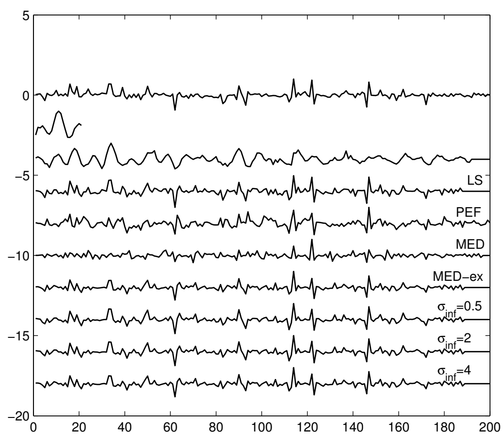


**Fig. 3**. Results obtained for only one realization of the second scenario. For description, see the text.

of the sparsity of the signal jeopardizes the results of the blind algorithms, also, AWGN has a destructive effect, even for the case where the input sequence is known (LS). Actually, this is to be expected, since the input signal is being estimated by a deconvolution filter. In the average, it is possible to see that, the proposed algorithm, performs significantly better than the classical MED and the MED-ex for the less sparse cases. From Table 2 it is possible to see the difference between the methods considering only one realization of the MC simulations and from Figure 3 it is possible to see the similarity between the real input and the input estimated by the proposed method.

**Table 2**. Pearson correlation coefficient for one realization of the MC simulations of the second scenario, with $P_x = 0.3$, $\sigma_x^2 = 0.03$ and SNR = 12dB.

| LS | PEF | MED | MED-ex, $\sigma =$ | $S\ell_0$, $\sigma_{\text{inf}} =$ | | |
|---|---|---|---|---|---|---|
| | | | $\max\{\mathbf{y}(n)\}/2$ | 0.5 | 2 | 4 |
| 0.90 | 0.64 | 0.59 | 0.85 | 0.86 | 0.87 | 0.86 |

## 5. CONCLUSIONS

In this work, we explored the problem of blind deconvolution of sparse signals, based on the assumption that the convolution of a sparse signal with the impulse response of a linear system results in a less sparse signal. In this context, we proposed a theorem that states a sufficient condition for that, which is also sufficient for the $\ell_0$-norm to be a contrast function. To perform deconvolution, we proposed a gradient-based algorithm for the minimization of a scale invariant version of the smoothed $\ell_0$-norm.

Extensive results with synthetic data demonstrates the superiority of the proposed method, if compared with conventional ones, and also indicates that the conditions imposed about the signals are sufficient but not strictly necessary.

## REFERENCES


[1] E. A. Robinson, *Predictive decomposition of time series with applications to seismic exploration*, Ph.D. thesis, Massachusetts Institute of Technology, Dept. of Geology, 1954.

[2] R. A. Wiggins, "Minimum entropy deconvolution," *Geoexploration*, vol. 16, no. 1,2, pp. 21 – 35, 1978.

[3] M. Ooe and T. J. Ulrych, "Minimum entropy deconvolution with an exponential transformation," *Geophysical Prospecting*, vol. 27, pp. 458–473, 1979.

[4] O. Rosec, M. Boucher, B. Nsiri, and T. Chonavel, "Blind marine seismic deconvolution using statistical mcmc methods," *IEEE Journal of Oceanic Engineering*, vol. 28, pp. 502–512, 2003.

[5] A. K. Nandi, D. Mampel, and B. Roscher, "Blind deconvolution of ultrasonic signal in nondestructive testing applications," *IEEE Transactions on Signal Processing*, vol. 45, pp. 1382–1390, 1997.

[6] J. M. T. Romano, R. Attux, C. C. Cavalcante, and R. Suyama, *Unsupervised Signal Processing: Channel Equalization and Source Separation*, CRC Press, 2011.

[7] D. Donoho, "On minimum entropy deconvolution," *Academic Press*, pp. 656–608, 1981.

[8] O. Shalvi and E. Weinstein, "New criteria for blind deconvolution of nonminimun phase systems (channels)," *IEEE Transactions on Information Theory*, vol. 36, pp. 312–321, 1990.

[9] H. Mohimani, M. Babaie-Zadeh, and C. Jutten, "A fast approach for overcomplete decomposition based on smoothed $\ell^0$ norm," *IEEE Transactions on Signal Processing*, vol. 57, pp. 289–301, 2009.

[10] D. L. Donoho, "For most large undetermined systems of equations, the minimal $\ell^1$-norm near-solution approximates the sparsest near-solution," *Communications on Pure and Applied Mathematics*, vol. 59, pp. 907–934, 2006.

[11] S. C. Douglas, S. I. Amari, and S. Y. Kung, "On gradient adaptation with unit-norm constraints," *IEEE Transactions on Signal processing*, vol. 48, pp. 1843–1847, 2000.