Minimum Measurement Deterministic Compressed Sensing based on Complex Reed Solomon Decoding

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***Abstract*—Compressed Sensing (CS) is an emerging field in communications and mathematics that is used to measure few measurements of long sparse vectors with the ability of lossless reconstruction. In this paper we use results from channel coding to design a recovery algorithm for CS with a deterministic measurement matrix by exploiting error correction schemes. In particular, we show that a generalized Reed Solomon encoding-decoding structure can be used to measure sparsely representable vectors, that are sparse in some fitting basis, down to the theoretical minimum number of measurements with the ability of guaranteed lossless reconstruction, even in the low dimensional case.**

***Index Terms*—Compressed Sensing, Reed Solomon, Deterministic, Sparsity**

I. Introduction

A. Motivation

There has been considerable interest in the emerging field of Compressed Sensing (CS). The main task of CS is to reconstruct long sparsely representable vectors out of few linear measurements , described by a measurement matrix [1]. This representability occurs in very many different disciplines like communications, electronics, medicine and physics with uncountable applications. One main challenge of CS is the design of good measurement matrices. Usually sub Gaussian random matrices are used, that are well known for good recovery in high dimensions. Sadly in many applications this is not feasible to implement or the dimension of the problem is too low for random number generators to work properly. A potential solution are deterministic measurement matrices. They are easier to implement but so far are known for decreased measurement efficiency [2]. This work was motivated by the search for optimal deterministic matrices and corresponding reconstruction algorithms, that combine the advantages of both worlds. Bosserts work on combining channel coding theory with Compressed Sensing [3] inspired us to take the same perspective and use this well studied field for solving the CS problem. Especially Complex Reed Solomon (RS) codes are well known for their high error correcting capabilities [4]. The fact that RS codes can decode **sparse** error vectors can be utilized to conquer the CS setting. Sadly the state of the art algorithms only works for vectors, that directly possess a sparse number of nonzero elements. This is due to the fact, that RS codes are used to correct *sparse* errors as they are common in communications.

To improve upon the results in [3], we go one step further by extending the idea of combining RS codes with Compressed Sensing to **sparsely representable** vectors, that can be sparse in a broader set of bases. This is done by generalizing the kind of errors that RS codes are capable of correcting and directly translating this to a CS scheme.

B. Main Contribution

Our main contribution is the derivation of the CS reconstruction algorithm RSCS with a corresponding **deterministic** measurement matrix, that is inspired by Complex Reed Solomon decoding. With $\mathbf{x} = \mathbf{\Psi c}$ and $\mathbf{c}$ being k-sparse and requirements towards $\mathbf{\Psi}$, that are fulfilled by most common representations (especially canonical and Fourier), this algorithm is guaranteed to retrieve $\mathbf{x}$ with the theoretical minimum of $2k$ measurements, even in the low dimensional case. The number of measurements is independent of the length of $\mathbf{x}$, which makes it increasingly effective for very sparse vectors. Furthermore, there is no need for tuning parameters. The algorithm includes an intrinsic estimator for the usually unknown value k which makes the algorithm easy and efficient to use.

C. Structure

The paper is structured as follows: In section II the CS problem is formally stated and the connection between CS and Reed Solomon codes is explained in the canonical case. In section III this combination is extended towards more general bases. Section IV shows numerical results comparing our contribution to state of the art algorithms. Section V concludes the paper.

II. Theoretical Background

A. Compressed Sensing

Formally spoken, the Compressed Sensing problem is the reconstruction of $\mathbf{x}$ out of few measurements

$$\mathbf{y} = \mathbf{\Phi x} \tag{1}$$

for the dense vector $\mathbf{x} \in \mathbb{C}^N$ being sparsely representable ($\exists \mathbf{\Psi} \in \mathbb{C}^{N\times N}$, s.t. $\mathbf{x} = \mathbf{\Psi c}$ with $||\mathbf{c}||_0 = k \ll N$). The measurement matrix $\mathbf{\Phi}$ is a $m \times N$-matrix with $m < N$ and

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consequently $\mathbf{y} \in \mathbb{C}^m$. We define $\mathcal{I} := \{i \in \mathbb{N} | c_i \neq 0\}$ as the support of $\mathbf{c}$ and $\mathbf{c}_i$ as the corresponding i-th entry of $\mathcal{I}$, formally $\mathbf{c}_i := \mathbf{c}_{\mathcal{I}(i)}$. The equation system (1) has an infinite number of solution, so it should be solved in a way, that the sparsest solution emerges.
The joint properties of $\mathbf{A} := \mathbf{\Phi\Psi}$ determine if a solution of (1) can be found. Because many relevant problems can be cast in this form, a firm mathematical theory with requirements towards $\mathbf{A}$ is needed that guarantees perfect reconstruction of every arbitrary sparse vector $\mathbf{x}$ out of its corresponding measurements $\mathbf{y}$. To classify if a certain matrix is suited for this problem, Candes [5] introduced the now ubiquitous **Null Space Property**:

Theorem 1 (Null Space Property)**.** *Let* $\mathbf{A} \in \mathbb{R}^{m \times N}$ *be a measurement matrix and* $\mathbf{y} \in \mathbb{R}^m$ *be the corresponding measurement. Iff*

$$\textit{spark}(\mathbf{A}) > 2k \tag{2}$$

there is an unique k*-sparse* $\mathbf{x} \in \mathbb{R}^N$*, s.t.* $\mathbf{y} = \mathbf{Ax}$*. The spark is the* ***minimum*** *number of linear* ***dependent*** *columns. Formally that means*

$$\textit{spark}(\mathbf{A}) = \min_{\mathbf{x} \neq 0} ||\mathbf{x}||_0 \ \textit{s.t.}\ \mathbf{Ax} = 0 \tag{3}$$

Due to $\textit{spark}(A) \leq m + 1$*, it follows that*

$$m \geq 2k \tag{4}$$

needs to be fulfilled for unique recovery. In other words, $m = 2k$ *is the* ***theoretical minimum*** *of measurements that suffice for the guaranteed reconstruction of every* k*-sparse vector.*

Proof. Let $spark(\mathbf{A}) \leq 2k$. Then a vector $\mathbf{v}$ $\in$null$(\mathbf{A})$ exists, s.t. $||\mathbf{v}||_0 \leq 2k$. Set $\mathbf{x}_1$ and $\mathbf{x}_2$ in a way, that $||\mathbf{x}_1||_0 = ||\mathbf{x}_2||_0 = k$ and $\mathbf{v} = \mathbf{x}_1 - \mathbf{x}_2$. Then

$$0 = \mathbf{Av} = \mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2) \Rightarrow \mathbf{Ax}_1 = \mathbf{Ax}_2 \tag{5}$$

In other words, $\mathbf{x}_1$ and $\mathbf{x}_2$ can not be uniquely recovered, because they are mapped onto the same $\mathbf{y}$.
The same reasoning can be used to show that two different k-sparse vectors will never be mapped onto the same $\mathbf{y}$ as long as $spark(\mathbf{A}) > 2k$. □

Randomly chosen matrices like subgaussian fulfill the NSP with high probability, so many researchers use these matrices $\mathbf{A} = \mathbf{\Phi\Psi}$, that interprets $\mathbf{\Psi}$ as part of the sensing matrix so that the reconstruction task shifts from $\mathbf{x}$ to $\mathbf{c}$. This way, only the easier task of reconstructing the **sparse** vector $\mathbf{c}$ remains. Sadly this does not allow for an easy implementation because the reconstruction algorithm needs to be aware of the random matrix that was used at the measurement step. In this paper, to obtain an efficient hardware friendly algorithm we concentrate on **deterministic** Compressed Sensing by defining a measurement matrix $\mathbf{\Phi}$ through the theory of Complex Reed Solomon Codes. This way the deterministic matrix can be hard coded in the encoder and the decoder. With this deterministic matrix $\mathbf{\Psi}$ has to be specifically considered in the reconstruction algorithm.

B. Complex Reed Solomon Coding

To present the main reasoning more clearly, in this section $\mathbf{\Psi}$ is assumed to be the identity matrix $\mathbf{E}$. In other words $\mathbf{x} = \mathbf{c}$ is directly sparse in the canonical basis.
Complex Reed Solomon (RS-) codes belong to the family of cyclic block codes [6]. Originally, channel codes are used to protect messages against random occurring errors by adding redundancy to the information. Based on this redundancy, a decoder can find or even correct errors. Reed Solomon codes in particular are commonly used because they can correct a high number of burst errors [7]. Due to this fact, RS codes are well suited for fading channels in wireless communications. If only a few of the complex code symbols get corrupted, they can be completely restored, regardless of the actual complex value in each incorrect symbol.
While RS codes are well suited for error correction, they can be used for the CS problem of reconstructing a sparse vector from few measurements as well. Figure 1 illustrates the connection of RS decoding and CS reconstruction. The upper part describes the RS view. RS decoders with $2k$ parity symbols can correct up to k errors at any positions and amplitudes, so especially the total erasure $\mathbf{e} = -\mathbf{x}$ is a correctable error, if $\mathbf{x}$ is **sparse**. The lower part shows a CS interpretation of the problem. A sparse vector $\mathbf{x}$ is measured by a matrix $\mathbf{\Phi}$ to obtain $2k$ measurements $\mathbf{p}$ which are sufficient to reconstruct $\mathbf{x}$ even with noise.

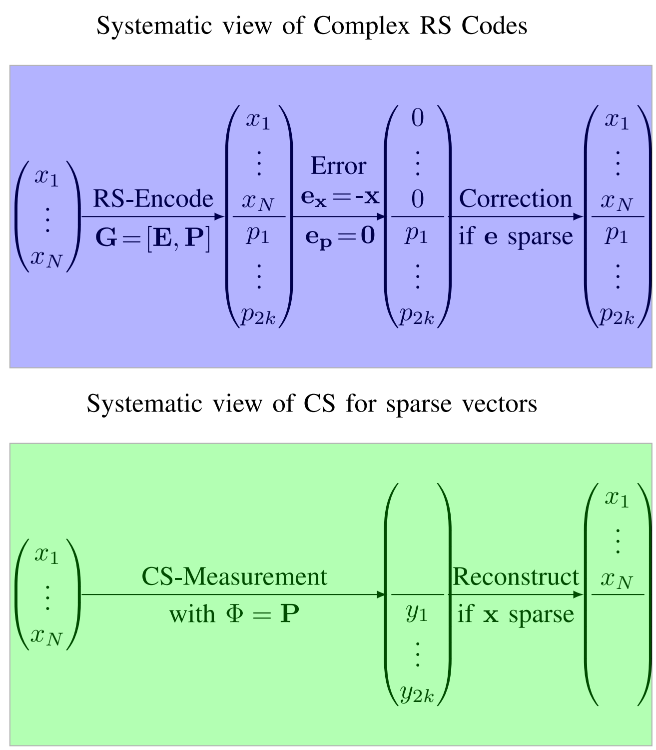

Fig. 1. Schematic for measurement and reconstruction of sparse vectors for $\mathbf{\Psi} = \mathbf{E}$

In other words, the CS algorithm can just measure the

$2k$ parity symbols of a systematic Reed Solomon encoding scheme and **guarantee** the recovery, if $\mathbf{x}$ is k-sparse. The number of parity symbols is completely independent of the underlying length of the original vector $\mathbf{x}$, so its well suited for long vectors with $k \ll N$. This scheme has a compression rate of $\frac{2k}{N}$, being increasingly effective for very sparse vectors.

More formally, we define $\mathbf{x}$ as the original sparse vector, $\alpha = \exp(2\pi j/N)$ as an element of order N, $\mathbf{G}$ as the generator matrix of a RS code and $[\mathbf{x}, \mathbf{p}] = \mathbf{G}\mathbf{x}$ as the corresponding RS code word with parity symbols $\mathbf{p}$. With $[\mathbf{f}](t)$ denoting the evaluation of the polynomial with coefficients $[\mathbf{f}]$ at t, the parity symbols need to fulfill $[\mathbf{x}, \mathbf{p}](\alpha^j) = 0 \quad \forall j = 1, \ldots, 2k$. Further defining $\mathbf{e}$ as the additive error and $\mathbf{r} = [\mathbf{x}, \mathbf{p}] + [\mathbf{e_x}, \mathbf{e_p}]$ as the received vector, we can use the standard RS syndrome decoding idea, i.e.

$$S_j = \mathbf{r}(\alpha^j) = \underbrace{[\mathbf{x}, \mathbf{p}](\alpha^j)}_{=0} + [\mathbf{e_x}, \mathbf{e_p}](\alpha^j) \text{ for } j = 1, \ldots, 2k \tag{6}$$

to reconstruct the error vector. In terms of CS only the parity part is known, meaning that the systematic part can be seen as totally erased by an error vector $[\mathbf{e_x}, \mathbf{e_p}] = [-\mathbf{c}, \mathbf{0}]$. Thus, the decoder simply has to find $\mathbf{e}$ to reconstruct $\mathbf{x}$. Consequently, the decoding is actually only based on this erasure hypothesis, so the syndromes can be directly computed at the CS measurement step. Formally, this results in

$$\begin{pmatrix} S_0 \\ \vdots \\ S_{2k-1} \end{pmatrix} = \begin{pmatrix} -\mathbf{x}(\alpha^0) \\ \vdots \\ -\mathbf{x}(\alpha^{2k-1}) \end{pmatrix} = - \begin{pmatrix} \alpha^0 & \cdots & \alpha^{0N} \\ \vdots & \ddots & \vdots \\ \alpha^{2k-1} & \cdots & \alpha^{(2k-1)N} \end{pmatrix} \mathbf{x}. \tag{7}$$

In this way, the complete encoding step is cast in the same way as in equation (1), the standard CS problem. In other words, we can interpret the computation of syndromes as a CS compatible sensing matrix $\mathbf{\Phi}$ with N columns and $2k \ll N$ rows. Namely $\mathbf{\Phi}$ is the Fourier matrix

$$\mathbf{\Phi} = \begin{pmatrix} \alpha^0 & \cdots & \alpha^{0N} \\ \vdots & \ddots & \vdots \\ \alpha^{2k-1} & \cdots & \alpha^{(2k-1)N} \end{pmatrix} \tag{8}$$

with known high RIP qualities [8] in comparison to other deterministic matrices. Because of the equivalence of both views, this sensing matrix still allows us to use known algorithms like the Berlekamp Massey algorithm [4] for the reconstruction of the original sparse vector, effectively combining the two theories. The number of measurements $m = 2k$ is the theoretical minimum for guaranteed recovery and independent of the length of the original vector $\mathbf{x}$.

To really combine CS with RS, one crucial detail needs to be clarified. The previous derivations all assumed $\mathbf{x}$ to be sparse, but CS generally includes the vast group of sparsely representable vectors of the form

$$\mathbf{x} = \mathbf{\Psi}\mathbf{c}, \; \mathbf{c} \text{ sparse.} \tag{9}$$

In general, there are not exactly k nonzero elements in $\mathbf{x}$ but k columns of $\mathbf{\Psi}$ are active, e.g. k frequencies. For the algorithm to still work in this more general case, it needs to be extended as presented in the next section.

III. Complex Reed Solomon Reconstruction of Sparsely Representable Vectors

A. Recap of RS decoding

To explain the extension of the reconstruction algorithm in [3] we again start with the canonical basis. That is, at first $\mathbf{x}$ has the form

$$\mathbf{x} = \sum_{i=1}^{k} \mathbf{c}_i \mathbf{E}_{\mathcal{I}(i)} \tag{10}$$

with $\mathbf{E_i}$ being the columns of the identity matrix. In other words, $\mathbf{x}$ is sparse in the canonical basis. The first main step of the standard Complex Reed Solomon decoder is the calculation of the error locator polynomial (ELP) of the form

$$\Lambda(t) = \prod_{i=1}^{k} (t - \alpha^{\mathcal{I}(i)}) = t^k + \Lambda_1 t^{k-1} + \Lambda_2 t^{k-2} + \cdots + \Lambda_k \tag{11}$$

with $t = \alpha^{\mathcal{I}(i)}$ as roots for the active set $\mathcal{I}$ of nonzero entries. Inserting the roots leads to k equations for the parameters $\Lambda_1, \ldots, \Lambda_k$:

$$\begin{pmatrix} \alpha^{\mathcal{I}(1)\cdot(k-1)} & \cdots & \alpha^{\mathcal{I}(1)\cdot 0} \\ \vdots & \ddots & \vdots \\ \alpha^{\mathcal{I}(k)\cdot(k-1)} & \cdots & \alpha^{\mathcal{I}(k)\cdot 0} \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \vdots \\ \Lambda_k \end{pmatrix} = \begin{pmatrix} -\alpha^{\mathcal{I}(1)\cdot k} \\ \vdots \\ -\alpha^{\mathcal{I}(k)\cdot k} \end{pmatrix} \tag{12}$$

The syndromes (7) that can be used to evaluate these equations are of the form

$$S_j = \mathbf{\Phi}_j \mathbf{x} = \begin{pmatrix} \alpha^j & \cdots & \alpha^{jN} \end{pmatrix} \left(\sum_{i=1}^{k} \mathbf{c}_i \mathbf{E}_{I(i)} \right) = \sum_{i=1}^{k} \mathbf{c}_i \alpha^{I(i)\cdot j} \tag{13}$$

so multiplying by $\mathbf{c}_i$ and different powers of α before adding the rows in (12) leads to

$$\begin{pmatrix} S_{k-1}k & \cdots & S_0 \\ \vdots & \ddots & \vdots \\ S_{2k-2} & \cdots & S_{k-1} \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \vdots \\ \Lambda_k \end{pmatrix} = \begin{pmatrix} -S_k \\ \vdots \\ -S_{2k-1} \end{pmatrix} \tag{14}$$

The rank of this system is the number of nonzero elements of $\mathbf{x}$. If this number is less than the maximum $k = m/2$, the matrix has to be transformed appropriately to get a solvable full rank system. With Λ_i given the generalized ELP can be computed. The roots of the ELP now describe the active set $\mathcal{I}$. With $\mathcal{I}$ known, the corresponding $\mathbf{c}_i$ can be easily derived by L_2 minimization.

B. Modification to sparsely representable vectors

Now we extend this algorithm to $\mathbf{x}$ of the form

$$\mathbf{x} = \sum_{i=1}^{k} \mathbf{c}_i \mathbf{\Psi}_{\mathcal{I}(i)} \tag{15}$$

for arbitrary $\mathbf{\Psi}$. We propose a modification of the error locator polynomial to take this change in account. With the definition $\mathbf{A}_\alpha = \mathrm{diag}(\alpha, \alpha^2, \ldots, \alpha^N)$ we propose the generalization

$$\Lambda_{\mathbf{\Psi}}(\mathbf{t}) = \mathbf{1} \left(1 \cdot \mathbf{A}_\alpha^k + \Lambda_1 \mathbf{A}_\alpha^{k-1} + \ldots + \Lambda_k \right) \mathbf{t} \tag{16}$$

with $\mathbf{t} = \mathbf{\Psi}_{\mathcal{I}(i)}$ as root for the active set $\mathcal{I}$. Because $\mathbf{t} \in \mathbb{C}^{N\times 1}$ it follows that

$$\Lambda_{\mathbf{\Psi}}(\mathbf{t}) : \mathbb{C}^{N\times 1} \rightarrow \mathbb{C}^1 \tag{17}$$

Interpreting the condition for the roots as a linear equation system with

$$\mathbf{\Psi}_{\mathcal{I}(i)}(\alpha^j) = \begin{pmatrix}\alpha^{1\cdot j} & \cdots & \alpha^{N\cdot j}\end{pmatrix}\begin{pmatrix}\mathbf{\Psi}_{\mathcal{I}(i),1}\\ \vdots \\ \mathbf{\Psi}_{\mathcal{I}(i),N}\end{pmatrix} \tag{18}$$

leads to

$$\begin{pmatrix}\mathbf{\Psi}_{\mathcal{I}(1)}(\alpha^{k-1}) & \cdots & \mathbf{\Psi}_{\mathcal{I}(1)}(\alpha^0)\\ \vdots & \ddots & \vdots\\ \mathbf{\Psi}_{\mathcal{I}(k)}(\alpha^{k-1}) & \cdots & \mathbf{\Psi}_{\mathcal{I}(k)}(\alpha^0)\end{pmatrix}\begin{pmatrix}\Lambda_1\\ \vdots\\ \Lambda_k\end{pmatrix} = \begin{pmatrix}-\mathbf{\Psi}_{\mathcal{I}(1)}(\alpha^k)\\ \vdots\\ -\mathbf{\Psi}_{\mathcal{I}(k)}(\alpha^k)\end{pmatrix} \tag{19}$$

This extension is consistent, because if $\mathbf{\Psi}$ is the identity matrix, the modified equation system and the corresponding error locator polynomial lead to the exact same result as the original version. Namely

$$\Lambda_{\mathbf{E}}(e_i) = \Lambda(\alpha^i) \tag{20}$$

But if $\mathbf{\Psi}$ is not equal to $\mathbf{E}$ as is the case in the general setting, this modification can still be used to utilize the syndromes, if $\mathbf{\Psi}$ meets one requirement. Multiplying the equations by $\mathbf{c}_i$ before adding them leads to the corresponding syndrome equation

$$S_j = \mathbf{\Phi}_j\mathbf{x} = \begin{pmatrix}\alpha^j \cdots \alpha^{jN}\end{pmatrix}\left(\sum_{i=1}^{k} c_i\mathbf{\Psi}_{\mathcal{I}(i)}\right) \tag{21}$$

$$= \sum_{i=1}^{k} c_i\mathbf{\Psi}_{\mathcal{I}(i)}(\alpha^j) \tag{22}$$

as in equation (13), so

$$\begin{pmatrix}S_{k-1} & \cdots & S_0\end{pmatrix}\begin{pmatrix}\Lambda_1\\ \vdots\\ \Lambda_k\end{pmatrix} = \begin{pmatrix}S_k\end{pmatrix} \tag{23}$$

emerges. If for all $\mathbf{\Psi}_i$ the condition

$$\mathbf{\Psi}_{\mathcal{I}(i)}(\alpha^j) = \mathbf{\Psi}_{\mathcal{I}(i)}(\alpha)^j, \quad j = 1,\ldots,m \tag{24}$$

is fulfilled, the same idea of multiplying by $\mathbf{c}_i$ and different powers of $\mathbf{\Psi}_{\mathcal{I}(i)}(\alpha)$ can be used to also get the higher syndrome equations, so that the same system as in (14) results. In other words, $2k$ measurements are sufficient for the perfect reconstruction of the original vector by solving the equation system for Λ_i and testing the columns of $\mathbf{\Psi}$ in the modified ELP. This again also effectively solves the problem of finding the unknown sparsity parameter k. Analog to the normal RS decoder, the rank of the linear equation system of syndromes determines the sparsity, so **the sparsity does not need to be known at the decoder entrance**.

C. *Making use of the condition*

The condition

$$\mathbf{\Psi}_{\mathcal{I}(i)}(\alpha^j) = \mathbf{\Psi}_{\mathcal{I}(i)}(\alpha)^j, \quad j = 1,\ldots,m \tag{25}$$

seems odd at first, but can be transformed into a design criterion

$$\mathbf{\Psi}_{\mathcal{I}(i)} = \mathrm{IFFT}\begin{pmatrix}\beta\\ \vdots\\ \beta^m\end{pmatrix} \tag{26}$$

for any $\beta \in \mathbb{C}$ as the IFFT matrix is the Vandermonde matrix built from $x_j = \alpha^j$. Due to this fact, $\mathbf{\Psi}_{\mathcal{I}(i)}(\alpha^j)$ can be expressed as the j-th row of FFT($\mathbf{\Psi}_{\mathcal{I}(i)}$) (see (18)). In other words, any $\mathbf{\Psi}_{\mathcal{I}(i)}$ that fulfills (26) fulfills condition (25). The identity matrix $\mathbf{\Psi} = \mathbf{E}$ can be built naturally by setting $\beta = \alpha^i$ for each column of $\mathbf{E}_i$. Setting β on the complex unity sphere leads to periodic functions that form Fourier bases of different frequencies.
As there are numerous situations where researchers want to use CS and have control over the design of $\mathbf{\Psi}$, this condition can be easily fulfilled (e.g. the CS-MUD setting [9]). The next section now compares the RS algorithm with fitting $\mathbf{\Psi}$ as described above against existing algorithms.

IV. Comparison to Other CS algorithms

We compare our RS algorithm with the greedy Orthogonal Matching Pursuit (OMP, [10]) and the more sophisticated Smoothed L0 (SL0, [11]) algorithm. To show a fair comparison, the dictionary $\mathbf{\Psi}$ is created as described in (26) for random β and tested for the RS algorithm, but other algorithms will perform on their preferred sub Gaussian sensing matrices. Figure 2 shows the phase diagram for $N = 100$.

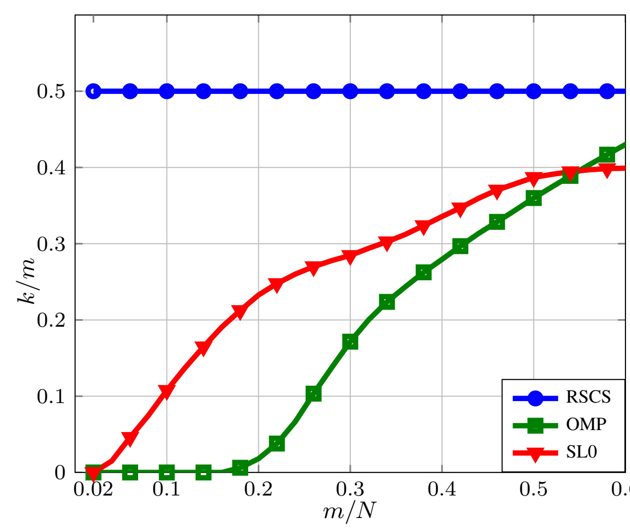

Fig. 2. Phase diagram for RS, SL0 and OMP, $N = 100$ for even numbers of m.

In figure 2, for every relative number of measurements m/N the relative sparsity k/M is plotted that could be perfectly reconstructed in more than 50% of the time. For odd numbers, due to $m/2 \notin \mathbb{N}$, the optimal reconstruction line is $\lfloor \frac{m}{2} \rfloor / m$. As this values are disturbing the $\frac{m}{2}/m = \frac{1}{2}$ line of even numbers, only the results for even numbers are plotted. The RS algorithm easily outperforms the other algorithms in this setting. The theoretical maximum of $k/m = 0.5 \equiv m = 2k$ (as stated in theorem 1) is achieved for every m/N value. In comparison the OMP and SL0 algorithms are increasing in performance with an increased number of measurements. While the SL0 algorithm outperforms the OMP by a vast margin for few measurements, both are getting equally performant in the high m/N region. But still even at their peak, both algorithms fall short in comparison to our new contribution. Note, that an extreme case like $N = 10000, m = 2, k = 1$ without noise can still be solved by our new approach, which makes it increasingly performant in very sparse settings. But as mentioned in the theoretical derivation before, this huge increase in reconstruction capabilities comes with the cost of being well tailored to the dictionary condition $\mathbf{\Psi}_{\mathcal{I}(i)}(\alpha^j) = \mathbf{\Psi}_{\mathcal{I}(i)}(\alpha)^j$, while the effect of the dictionary for SL0 and OMP are nearly neglectable.

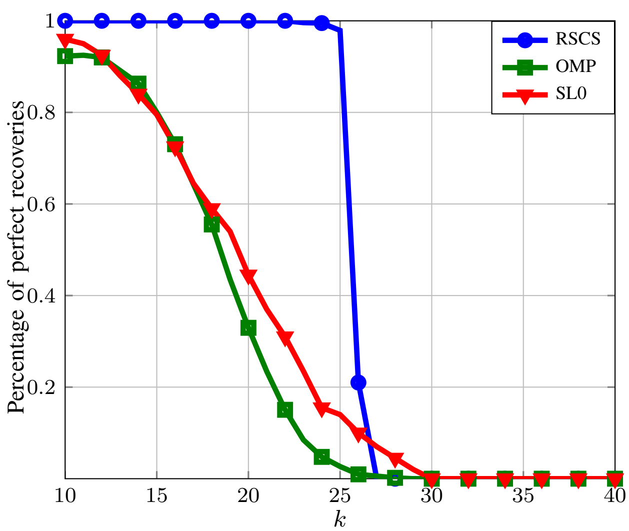

Fig. 3. Number of perfect recoveries for $N = 100, m = 50$

Figure 3 shows a bisection of figure 2 at $m = 50$. That means, for $N = 100$ and $m = 50$, the percentage of perfect recoveries is depicted against the sparsity k. Here the steep transition at $k = 25 = m/2$ is very present. In CS the steepness of the phase transition depicted in figure 3 will usually increase with $N \to \infty$ keeping the value at 0.5 constant. As can be easily seen, our approach shows a sudden transition from perfect reconstruction to failure already at low N. Also, this transition always occurs at $m = 2k$ for even m. OMP and SL0 show the expected behaviour at much lower k and need $m \geq 5k$ for equivalent reconstruction performance. Especially for low dimension scenarios this is a huge advantage. Random matrices can only be guaranteed to be incoherent enough for $N \to \infty$, so there are not well suited for small N. This shows that combined with our algorithms, **deterministic** Fourier matrices outperform random matrices by a fair margin, if $\mathbf{\Psi}$ can be cast in a suitable form (see (25)).

V. Conclusion

We showed that a generalization of the Reed Solomon decoding algorithm can be used as a powerful CS algorithm that only needs the theoretical minimum number of $2k$ measurements for guaranteed reconstruction and even works for sparsely representable vectors that are sparse in a fitting basis. Future work will lower the requirement towards $\mathbf{\Psi}$ even further to transform it into an universal CS algorithm.

VI. Acknowledgment

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